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THE QUANTIZATION OF QUADRATIC FRICTION REVISITED*

by

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Abstract

The quantization of a system subject to a friction force quadratic in the velocity and position dependent is carried out in the Feynman path integral framework. The resulting Hamiltonian coincides with the one obtained by using the Weyl-ordering canonical prescription.

Key-words: Friction; Path integral quantization; Dissipaty systems; Weil-ordering.

1 INTRODUCTION

The problem posed by the quantization of a particle subject to a velocity-dependent force has recently received particular attention[1,2]. Its interest comes from the fact that for some special cases of this kind of nonconservative systems an appropriate Lagrangian can be defined[3]. The simplest example is that of a particle subject to a friction force which is quadratic in the velocity[4]. This class of interactions, at the microscopic level, could be of some relevance in non-relativistic nuclear physics systems where dissipation is described by a velocity-dependence more general than the linear one. From a more theoretical point of view, the interest on these systems is related to the ambiguities appearing in the ordering of the operators $\hat{x}$ and $\hat{p}$ in the Hamiltonian, when the canonical quantization procedure is carried out. This comes from the fact that the classical Hamiltonian for a particle in a viscous field involves an x-dependent kinetic term. It is precisely here that the so-called Weyl-ordering method applies[5]. This procedure was introduced in order to cope with the arbitrariness in the Hamiltonian operator definition giving the correct symmetrization between operators $\hat{x}$ and $\hat{p}$.

On the other hand, inside the path integral quantization framework[6], the Weyl-ordering result comes out naturally, when the Hamiltonian operator is extracted from the propagator definition. Nevertheless, this result is only guaranteed when the path integral is correctly defined in terms of the Hamiltonian, through $p\dot{q} - H$, instead of being in terms of the Lagrangian. This clearly implies a detailed and careful treatment of the p-functional integration.

In this paper we present an exhaustive analysis of the path-integral quantization of a system subject to a force quadratic in the velocity and x-dependent. In this way we end with the correct Weyl-ordered Hamiltonian operator for the system.

In section II we summarize the classical and quantum canonical treatment of the large class of v^2- dependent friction forces considered. In section III, the path integral approach is presented in some detail together with the results obtained. Section IV is devoted to our conclusions and final remarks. Here we include our critique to a recent treatment[1] of the present problem where the above mentioned p-integration was not properly considered.

II CLASSICAL AND QUANTUM CANONICAL TREATMENT

The classical equation of motion for a particle (m=1) in a field of force quadratic in the velocity reads:

$$\ddot{x} = -\frac{1}{2} \gamma(x) \dot{x}^2 - \frac{dV(x)}{dx} \qquad (2.1)$$

Where $\gamma(x)$ drives the friction $(\dot{x} > 0)$ or anti-friction $(\dot{x} < 0)$ force and $V(x)$ is the potencial energy. As it is well known eq. (2.1) can not be derived directly from a Lagrangian. Nevertheless if we multiply eq (2.1) by the integrating factor

$$f(x) = \exp \int_0^x ds\, \gamma(s) \qquad (2.2)$$

we obtain

$$\ddot{x} f(x) = -\frac{1}{2} f'(x) \dot{x}^2 - f(x) \frac{dV}{ds}, \qquad (2.3)$$

which is equivalent to eq.(2.1). and can now be derived from the following Lagrangian.

$$L = \frac{1}{2} \dot{x}^2 f(x) - \int_0^x ds\, f(s) \frac{dV}{ds}. \qquad (2.4)$$

In order to proceed to the Hamiltonian formalism, we first calculate the canonical momentum

$$p = \dot{x} f(x) \qquad (2.5)$$

and then write

$$H = \frac{p^2}{2} f^{-1}(x) + W(x)$$

$$\text{with } W(x) = \int_0^x ds\, f(s) \frac{dV}{ds} \qquad (2.6)$$

Let us note that in this case the Hamiltonian is not the total energy of the system, as it should be expected because of the presence of non-conservative forces.

Once the Hamiltonian has been obtained we can quantize the system by imposing the canonical commutation relation

$$[\hat{x}, \hat{p}] = i \qquad (\hbar=1) \qquad (2.7)$$

If no further conditions are imposed, the quantum Hamiltonian will not be uniquely defined because different ways of arranging the ordering of p's and q's lead to the same classical limit.

Among the different proposals to cope with this difficulty[5], the Weyl-ordering prescription provides us a systematic way to solve the ambiguity. In the present case it gives

$$H_W = \frac{1}{8}(p^2 f^{-1}(x) + 2pf^{-1}(x)p + f^{-1}(x)\, p^2) + W(x) \qquad (2.8)$$

Note that H_W is self-adjoint and implies the validity of Ehrenfest's theorems. On the other hand, the use of the Weyl-ordering avoids the trial and error approach in building the correct Hamiltonian that satisfies these theorems.

III PATH INTEGRAL APPROACH

In this section we consider the path integral quantization method for a system driven by a v^2- dependent force. The quantum Hamiltonian operator is derived from the corresponding Feynman propagator and it is shown that it coincides with the previously introduced Weyl-ordered one, given in eq. (2.8).

Let us start by considering the evolution of the system, described by the wave function ψ, from (x', t') up to (x, t). This evolution can be written as

$$\psi\ (x,\ t) = \int_{-\infty}^{\infty} dx'\ K\,(x,\ t;\ x',t')\ \psi\ (x',\ t') \tag{3.1}$$

where the Feynman propagator K is given by the following phase space path integral

$$K\ (x,\ t;\ x',\ t') = \int\ Dx\,\frac{Dp}{2\pi}\ \exp\ [iS(x,\ t;\ x't'] \tag{3.2}$$

and the action S is

$$S\ (x,\ t;\ x',\ t') = \int_{t'}^{t} d\tau\ \left(p\,\frac{dx}{d\tau}\ -\ H\ \right) \tag{3.3}$$

including the classical Hamiltonian (2.6)

It should be noticed that our definition of the propagator K explicitely includes the path integration over the momentum p. In the present case this integration is not trivial because the kinetic term in the classical Hamiltonian (eq. (2.6)) depends on the coordinate x.

In order to carry out the path integral one starts by doing a suitable discretization.

We consider the standard procedure[6] where in each interval $[x_{i-1}, x_i]$ one takes the value of the functions involved in the mean point of the interval $(x_{i-1} + x_i)/2$. Notice the difference with the choice in Ref.1. When the p-integration is so performed[7], one ends with

$$K(x,t;\, x',t') = (2\pi i\varepsilon)^{1/2} \int \prod_{i=1}^{n} \frac{dx_i}{(2\pi i\varepsilon)^{1/2}}\, \delta(x_1 - x')\, \delta(x_n - x)\,.$$

$$.\ \exp\left\{ i \sum_{i=2}^{n} \left[\frac{\varepsilon}{2} \left(\frac{x_i - x_{i-1}}{\varepsilon}\right)^2 .\ f\left(\frac{x_i + x_{i-1}}{2}\right) - \varepsilon V\left(\frac{x_i + x_{i-1}}{2}\right)\right]\right\}$$

$$.\ \prod_{i=2}^{n} \left[f\left(\frac{x_i + x_{i-1}}{2}\right)\right]^{1/2} \qquad (3.4)$$

The last factor in this expression comes precisely from the above mentioned p-integration.

In order to evaluate the quantum Hamiltonian we consider the propagation of the system in a very small time interval ε and we compare the result with the Schrödinger equation. To this end, let us come back to eq. (3.1) under the mentioned conditions, to have

$$\psi(x,t+\varepsilon) = \int_{-\infty}^{\infty} d\eta\, K(x,t+\varepsilon\,;\ x-\eta,t)\, \psi(x-\eta,t) \qquad (3.5)$$

where, due to the smallness of ε , the propagator K can be approximated by

$$K(x,t+\varepsilon;\ x-\eta,t) \doteq (2\pi\varepsilon)^{-1/2}\exp i\left[\frac{\eta}{2}f\left(x-\frac{\eta}{2}\right)-W\left(x-\frac{\eta}{2}\right)\right] \cdot \left[f\left(x-\frac{\eta}{2}\right)\right]^{1/2} \qquad (3.6)$$

when ε is very small, the propagator K gets contributions only for η in a region of the order $(\varepsilon)^{1/2}$. Then the integrand in eq.(3.5) can be expanded around $\eta=0$ in order to obtain the contributions up to the order ε . In so doing one gets

$$\psi(x,t+\varepsilon) = \psi(x,t) + i\varepsilon\left\{\frac{1}{2}f^{-1}(x)\frac{\partial^2}{\partial x^2}-\frac{1}{2}f^{-2}(x)\,f'(x)\frac{\partial}{\partial x} + \frac{1}{4}f^{-3}(x)f'(x)^2-\frac{1}{8}f^{-2}(x)f''(x) - W(x)\right\}\psi(x,t) \qquad (3.7)$$

As usual[6] , we have taken the simultaneous limit $\varepsilon \to 0$ both in the time interval for the wave functions and in the computation of the propagator K.

Comparing now the expression (3.7) with the Schrödinger equation and using the x - representation, $\hat{p}=-i\frac{\partial}{\partial x}$ for the momentum operator we finally obtain

$$\hat{H} = \frac{1}{8}\left[\hat{p}^2 f^{-1}(x) + 2\hat{p}\, f^{-1}(x)\hat{p} + f^{-1}(x)\hat{p}^2\right] + W(x) , \qquad (3.8)$$

which is identical to the Weyl-ordered Hamiltonian given in eq. (2.8).

IV CONCLUSION

We have carried out the quantization of a system subject to a viscous force quadratic in the velocity. We were able to consider a large class of x-dependent friction forces as long as γ (x) does not induce strong singularities.

The ambiguities related to the undefined ordering of the non-commutating $\hat{x}$ and $\hat{p}$ operators were avoided by using the Weyl-ordering procedure in the canonical quantization approach. Afterwards we completed the Feynman path integral quantization of the system. In so doing we have employed the usual prescription for the discretization of the functional integral, obtaining exactly the same Weyl-ordered quantum Hamiltonian as before.

It should be notice that the recent result of Ref.1, where a parameter β (between 0 and 1) was intro - duced for fixing the point in the interval $[x_{i-1}, x_i]$ where the functions are computed after discretization, is fortuitous. In fact, as they started from the path -integral defined in the configuration space and not in the phase space, they missed the already mentioned factor that appears in the Feynman propagator when the p-functional integration is performed. For this reason they were led to the unexpected value $\beta = 1/3$. Moreover, they could find the Weyl-ordered Hamiltonian because they treated the special case $f(x) = \exp(\gamma x)$, with γ constant.

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