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SU(4) PROPERTIES OF THE DIRAC-KÄHLER EQUATION

by

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Abstract

We use the Dirac–Kähler formalism in the space of differential forms (endowed with a Clifford product) to study the $SU(4)$ symmetry related to the description of spin-1/2 particles found previously in the usual matrix treatment. We show that differential forms may be taken as the generators spanning the algebra of the SU(4) group and how the operations of this group can be related to a change of frame of reference in the algebra.

We demonstrate that minimal left ideals of the algebra constitute irreducible representations for spin-1/2 particles for Clifford operation from the left, and exhibit how these ideals are related via space inversion, time reversal and their product.

We also consider the dual space of minimal right ideals and show how the Dirac–Kähler differential operator acts from the right, leaving the minimal right ideals invariant. This allows the introduction of an adjoint form and through the definition of a suitable scalar product, of conserved currents. We emphasize the relevance of all these features to the problem of proliferation of fermion species in the continuum limit of the lattice formalism.

Key-words: Dirac-Kähler equation; Kähler-Atiyah algebras; Relativistic spin 1/2 particles.

1 Introduction

The differential-form approach to the description of fermions, the so-called Dirac–Kähler equation, was introduced in the early sixties by the well-known mathematician Erich Kähler [1] and its relation to the Dirac spinorial formalism was later established by Graf [2]. They are sometimes referred to in the literature as "geometric fermions," a term coined by Benn and Tucker [3]. However, it was mainly after the detailed work of Becher and Joos [4] that several of the features of the Dirac–Kähler equation were properly understood.

Becher and Joos [4] pointed out that, mathematically, the Dirac–Kähler equation takes into account the full Clifford algebra introduced on differential forms through an additional symmetric product operation. It is well known, however, in the mathematical literature (see, for instance, [5]) that Dirac spinors are related not to the full Clifford algebra of gamma matrices, but instead to (any) one of four (in a four-dimensional spacetime) invariant subspaces of the Clifford algebra, which are its minimal left ideals. Therefore, the Dirac–Kähler equation describes four Dirac spinors, corresponding to each minimal left ideal.

The present authors have studied the minimal left ideals of the Clifford algebra in the spinorial formalism [6,7,8,9]. We displayed an $SU(4)$ symmetry of the Dirac equation (and, of course, of the Dirac hamiltonian) which arises from the Lie algebra spanned by all fifteen independent Dirac gamma matrices and their products (once properly turned hermitian), identified as the Lie algebra of the generators of the $SU(4)$ group. We also showed that the $SU(4)$ discrete subgroup $Z_2 \times Z_2$, generated by parity and time reversal transformations, forms the "reduction group", as many authors [4,10,11] named it, out of which projectors onto the minimal left ideals are constructed. Hence, this provides a physical interpretation for each fermion. In this work, we consider this problem for the description of fermions by the Dirac–Kähler equation for differential forms.

Our initial motivation for studying the Dirac–Kähler equation was the clarification of the properties of fermions when they are introduced on a lattice. As is well known, the naive transposition of the Dirac equation to discretized spacetime [12] introduces a 16-fold multiplicity of particles on the fermionic spectrum. It can be eliminated by the addition of several extra terms, with zero continuum limit, to the action [13]. Nonetheless, these terms break chiral invariance (for zero-mass fermions) and the resulting theory is unable of treating particles with a well-defined chirality such as the neutrino. Nielsen and Ninomiya [14] later demonstrated under very general conditions that it is impossible to have, on the lattice, chiral symmetry without a proliferation of the fermionic spectrum.

One popular way of avoiding the Nielsen–Ninomiya theorem is the Kogut–Susskind formalism [15] of lattice fermions in which each Dirac spinor component is alternately assigned to different lattice sites. The spectral degeneracy is then reduced to four and one can define a discrete version of chiral symmetry. The point is that, as Becher and Joos proved [4], the lattice Dirac–Kähler equation, in which differential forms are replaced by cochains, seems again to incorporate the full Clifford algebra, decomposed also in four minimal left ideals. It has the same number of fermion species as the continuum Dirac–Kähler equation, with no further proliferation of the spectrum. Thus the discretized Dirac–Kähler equation has the same degree of spectral degeneracy as the Kogut–Susskind scheme and, in fact, they are equivalent. This suggests that the Kogut–Susskind copies can be associated with the minimal left ideals representing fermions on the lattice. The existence of parity and time reversal relationships among these fermion species has been shown explicitly [6,8,9].

In this article, we intend to exhibit $SU(4)$ features of fermions described in the differential-form scheme, that is, for the Dirac–Kähler equation in the continuum.[1] In section 2 we review the basic formalism, developed by Kähler and Graf, of differential forms endowed with the Clifford algebra, as is suited to the problem; next, we comment on the elementary construction formulated in a paper by Rabin [16] of the Dirac–Kähler equation and the more elaborate developments by Becher and Joos [4].

In section 3 we show specifically how the $SU(4)$ symmetry arises in the differential-form formalism. As in the matrix case, we establish the relationship between the minimal left ideals in the space of forms and the (Clifford) commuting generators that are taken to build a Cartan subalgebra. These generators are combined in projectors for each minimal left ideal. We emphasize the different interpretations that the Dirac–Kähler equation admits in the (naive) vector space of all differential forms and on the subspaces spanned by the minimal left ideals.

Section 4 is devoted to the analysis of what could be taken as the transpose of a minimal left ideal. The need to find a differential operator on forms acting irreducibly on minimal right ideals is discussed. The correct one is found to be the same as that appearing in the Dirac–Kähler equation, but acting from the right on differential forms. This allows us to introduce the adjoint minimal right ideal, whose components satisfy the correct adjoint Dirac equation. We find a necessity for another notion of adjointness than the usual one in differential forms. This we satisfy later in section 7. In section 5 we define the discrete transformations of charge conjugation, parity and time reversal on forms and show how they relate the minimal left ideals; a differential-form version of the CPT theorem is also provided.

Another construction, introduced in a previous article [17], which will be much needed later, is then made in section 6: that of the scalar value of a differential form. This construction incorporates the same algebraic features of the trace of products of Dirac gamma matrices. We have used it previously to show that the chiral abelian anomaly for the gauge-covariant Dirac–Kähler operator in the continuum is the correct limit of the one obtained through the Kogut–Susskind and Becher–Joos lattice formalism, in agreement with Göckeler [18]. It also allows one to introduce conserved currents, which is the subject of the last section.

We leave section 8 for our final comments.

The material contained in this article must be taken as introductory to what we think are interesting possibilities for theoretical physics to incorporate a formalism with definite geometrical meaning in the description of particles of spin-1/2. We have tried to emphasize as much as possible the features which are common to the familiar treatment in terms of Dirac matrices and spinors, but we have also incorporated new aspects of the formalism of differential forms that may be useful for the description of those particles in various frameworks of theoretical physics.

2 The Kähler–Graf formalism

Let M be a 4-dimensional flat spacetime and consider its cotangent bundle T^*M. We can construct the exterior-algebra bundle over M, $\Lambda^*(M) \equiv \bigoplus_{k=0}^4 \Lambda^k T^*M$, where $\Lambda^k T^*M$ is the space of all differential k-forms over M, generated by the exterior product $\wedge$ of k 1-forms. In a coordinate frame $\{x^\mu\}$, $\mu = 0, \ldots, 3$, a basis for $\Lambda^*(M)$ may be chosen as

$$\begin{aligned}&\{1, dx^\mu, dx^\mu \wedge dx^\nu\ (\mu < \nu), dx^\mu \wedge dx^\nu \wedge dx^\rho\ (\mu < \nu < \rho), \varepsilon\},\\ &\qquad \text{for } \mu, \nu, \rho = 0, \ldots, 3,\end{aligned} \tag{1}$$

where ε is the volume element of M, $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$.

The first step in the description of fermions through differential forms is the observation made by Graf [2] that we can define an isomorphism between the set of 16 forms that constitute the basis (1) for $\Lambda^*(M)$ and the set of the 16 independent 4×4 Dirac matrices and their products, together with the identity matrix:

$$\{I, \gamma^\mu, \sigma^{\mu\nu} = \tfrac{1}{2}[\gamma^\mu, \gamma^\nu], \gamma^5\gamma^\mu, \gamma^5\}. \tag{2}$$

However, this isomorphism will only be well defined if the dx^μ obey the same algebra as the γ^μ do. As is well known, the gamma matrices obey a Clifford algebra,

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{3}$$

where the $g^{\mu\nu}$ are the components of the spacetime flat metric with negative Minkowski signature. It is, then, necessary to implement such an algebra among differential forms.

So, following Kähler [1], we define the Clifford product between forms, denoted by the symbol $\vee$:

$$\begin{aligned} dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu &= 2g^{\mu\nu} \cdot 1 && (4) \\ 1 \vee 1 &= 1 && (5) \\ 1 \vee dx^\mu = dx^\mu \vee 1 &= dx^\mu, && (6) \end{aligned}$$

for $\mu, \nu = 0, \ldots, 3$, which is related to the exterior product,

$$dx^\mu \vee dx^\nu = dx^\mu \wedge dx^\nu + g^{\mu\nu}, \tag{7}$$

so that, for 1-forms α, β,

$$\alpha \vee \beta = \alpha \wedge \beta + \alpha \cdot \beta. \tag{8}$$

The dot indicates scalar product between forms.[2]

The space $\Lambda^*(M)$, together with the operations of $\wedge$, $\vee$ and $\cdot$, determines a fibre bundle over M whose cross-sections satisfy both Grassmann and Clifford algebras, which are related by eq. (8). This structure is called a Kähler–Atiyah algebra and the fibre bundle is sometimes referred to as a Kähler–Atiyah bundle [2]. In the following, we shall call their sections Kähler–Atiyah (KA) differential forms. Therefore, the isomorphism between Dirac matrices and differential 1-forms is to be understood as

$$\gamma^\mu \to dx^\mu \vee . \tag{9}$$

The elements of the space $\Lambda^*(M)$ are known as general differential forms, that is,

$$\phi = \varphi(x)1 + \varphi_\mu(x)dx^\mu + \frac{1}{2!}\varphi_{\mu\nu}(x)dx^\mu \wedge dx^\nu$$

$$
\begin{aligned}
&+\frac{1}{3!}\varphi_{\mu\nu\rho}(x)dx^\mu \wedge dx^\nu \wedge dx^\rho + \varphi_5(x)\varepsilon && (10)\\
\equiv\ & \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \phi^{(3)} + \phi^{(4)}. && (11)
\end{aligned}
$$

In this equation, we have introduced the notation $\phi^{(p)}$, by which we denote the p-form part of a general differential form.

It is interesting to define two automorphisms of $\Lambda^*(M)$: the so-called main automorphism, $\mathcal{A}$:

$$
\begin{aligned}
\mathcal{A}\phi &= (-1)^k\phi && (12)\\
&= \phi^{(0)} - \phi^{(1)} + \phi^{(2)} - \phi^{(3)} + \phi^{(4)} && (13)
\end{aligned}
$$

and the main anti-automorphism, $\mathcal{B}$, also known as "reversion" in the literature,

$$
\begin{aligned}
\mathcal{B}\phi &= (-1)^{\left[\frac{k}{2}\right]}\phi && (14)\\
&= \phi^{(0)} + \phi^{(1)} - \phi^{(2)} - \phi^{(3)} + \phi^{(4)}, && (15)
\end{aligned}
$$

where $\left[\frac{k}{2}\right]$ means the integral part of $k/2$, with the properties:

$$
\begin{aligned}
\mathcal{A}^2 &= \mathcal{B}^2 = 1 && (16)\\
\mathcal{A}\mathcal{B} &= \mathcal{B}\mathcal{A} && (17)\\
\mathcal{A}(\alpha\wedge\beta) &= (\mathcal{A}\alpha)\wedge(\mathcal{A}\beta) && (18)\\
\mathcal{B}(\alpha\wedge\beta) &= (\mathcal{B}\alpha)\wedge(\mathcal{B}\beta), \qquad \alpha,\beta\in\Lambda^*(M). && (19)
\end{aligned}
$$

It is also convenient to define a contraction operation between a vector of the tangent bundle TM over M, $X = x^\mu e_\mu$, where $\{e_\mu\}$, $\mu = 0,\ldots,3$, is a basis for TM, and a differential p-form $\phi^{(p)}$:

$$
\begin{aligned}
i_X\phi^{(p)} \equiv X\rfloor\phi^{(p)} &= (x^\mu e_\mu)\rfloor\left(\frac{1}{p!}\varphi_{\nu_1\cdots\nu_p}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_p}\right)\\
&= \frac{1}{(p-1)!}x^\mu\varphi_{\mu\nu_2\cdots\nu_p}dx^{\nu_2}\wedge\cdots\wedge dx^{\nu_p}. && (20)
\end{aligned}
$$

Kähler [1] extended this notation to a kind of "inner product" between forms, that can also be considered a "differentiation" of forms (in the sense that it satisfies Leibniz's rules). Taking $e^\mu = g^{\mu\nu}e_\nu$, let

$$
\begin{aligned}
e^\mu\rfloor dx^\nu &= g^{\mu\nu} && (21)\\
e^\mu\rfloor 1 &= 0, && (22)
\end{aligned}
$$

so that, according to (20),

$$\begin{aligned} e^{\mu}\rfloor(\phi+\omega) &= e^{\mu}\rfloor\phi + e^{\mu}\rfloor\omega && (23)\\ e^{\mu}\rfloor(\phi\wedge\omega) &= (e^{\mu}\rfloor\phi)\wedge\omega + (\mathcal{A}\phi)\wedge e^{\mu}\rfloor\omega && (24)\end{aligned}$$

$$(e^{\mu}\rfloor e^{\nu}\rfloor + e^{\nu}\rfloor e^{\mu}\rfloor)\phi = 0, \tag{25}$$

for $\phi,\omega\in\Lambda^*(M)$. Eq. (7) may then be rewritten as

$$dx^{\mu}\vee dx^{\nu} = dx^{\mu}\wedge dx^{\nu} + e^{\mu}\rfloor dx^{\nu} \tag{26}$$

and generalized to

$$\begin{aligned} dx^{\mu}\vee\phi &= dx^{\mu}\wedge\phi + e^{\mu}\rfloor\phi && (27)\\ \phi\vee dx^{\mu} &= \phi\wedge dx^{\mu} - e^{\mu}\rfloor\mathcal{A}\phi. && (28)\end{aligned}$$

Finally, the Clifford product between two general differential forms ϕ,ω in $\Lambda^*(M)$ is written as

$$\phi\vee\omega = \sum_p \frac{1}{p!}(-1)^{[\frac{p}{2}]}\mathcal{A}^p(e_{\mu_1}\rfloor\cdots\rfloor e_{\mu_p}\rfloor\phi)\wedge(e^{\mu_1}\rfloor\cdots\rfloor e^{\mu_p}\rfloor\omega). \tag{29}$$

From this, we derive the properties

$$\begin{aligned} e^{\mu}\rfloor(\phi\vee\omega) &= (e^{\mu}\rfloor\phi)\vee\omega + (\mathcal{A}\phi)\vee e^{\mu}\rfloor\omega && (30)\\ \mathcal{A}(\phi\vee\omega) &= (\mathcal{A}\phi)\vee(\mathcal{A}\omega) && (31)\\ \mathcal{B}(\phi\vee\omega) &= (\mathcal{B}\omega)\vee(\mathcal{B}\phi). && (32)\end{aligned}$$

The exterior calculus of forms is naturally extended to $\Lambda^*(M)$. The exterior differentiation operator, acting on $\phi\in\Lambda^*(M)$, is simply

$$d\phi = dx^{\mu}\wedge\partial_{\mu}\phi. \tag{33}$$

With the Hodge star operator, $*$, which takes a p-form into its dual $(4-p)$-form, that is,

$$*(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}) = \frac{1}{(4-p)!}g^{\mu_1\nu_1}\cdots g^{\mu_p\nu_p}\epsilon_{\nu_1\cdots\nu_p\nu_{p+1}\cdots\nu_4}dx^{\nu_{p+1}}\wedge\cdots\wedge dx^{\nu_4}, \tag{34}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Cività antisymmetric symbol, one then defines the codifferential operator[3] $\delta = -*d*$. Acting on $\phi \in \Lambda^p(M)$, it may be written as

$$\begin{aligned}\delta\phi &= \frac{1}{(p-1)!}\partial_\mu \varphi^\mu{}_{\nu_2\cdots\nu_p} dx^{\nu_2}\wedge\cdots\wedge dx^{\nu_p} \\ &= e^\mu \rfloor \partial_\mu(\varphi_{\nu_1\cdots\nu_p}) dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_p} \\ &= e^\mu \rfloor \partial_\mu \phi, \end{aligned} \tag{35}$$

also trivially extended to $\Lambda^*(M)$.

We may now introduce, by analogy with eq. (33), a new differential operator on $\Lambda^*(M)$, roughly speaking, in such a way as to give a meaning to the substitution of the wedge product by the Clifford product in the right-hand side of eq. (33). This should respect the Kähler–Atiyah algebra, generating what Kähler calls the "inner calculus" of forms [1]. With this in mind, through eq. (27) we define

$$dx^\mu \vee \partial_\mu = (dx^\mu \wedge + e^\mu \rfloor)\partial_\mu \tag{36}$$

and, comparing with (33) and (35), we can also write

$$dx^\mu \vee \partial_\mu = d + \delta. \tag{37}$$

Taking into account the isomorphism (9), we could naively say that the Kähler operator $d + \delta$ is equivalent to the Dirac operator $\gamma^\mu\partial_\mu$. We would then write the Dirac–Kähler equation as

$$i(d+\delta)\phi = m\phi. \tag{38}$$

This correspondence is, however, to be taken in the sense that, as Graf [2] pointed out, one has to be aware of the fact that the Clifford algebra is naturally decomposed in its minimal left ideals. In the next section, we study carefully these peculiarities of the formalism.

To end this section, we would like to point out that, whereas the left-hand side of eq. (37) is written in terms of a coordinate representation, its right-hand side is independent of coordinates, having the intrinsic content peculiar to the formulation of physics through differential forms, whence its interest for studying topological properties of physical systems, or for the study of gravitation.

3 Dirac equation and SU(4) structure of the space of KA differential forms

The connection between the Dirac–Kähler formalism and the usual Dirac framework with gamma matrices and spinors may be presented following the work of Rabin [16] as the initial step.

Consider the Dirac equation for a massive particle,

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{39}$$

As usual, ψ is represented by a column matrix (a spinor):

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{40}$$

The same set of equations results for the components ψ_i, $i = 1, \ldots, 4$, if they are placed in the first column of a 4×4 matrix, while all other elements are taken to be zero:

$$\Psi^{[1]} = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}. \tag{41}$$

As any 4×4 matrix, $\Psi^{[1]}$ can be expanded in the basis (2) (see, for instance, Ref. [19], p. 118):

$$\Psi^{[1]} = fI + f_\mu\gamma^\mu + \frac{1}{2!}f_{\mu\nu}\gamma^\mu\gamma^\nu + \frac{1}{3!}f_{\mu\nu\rho}\gamma^\mu\gamma^\nu\gamma^\rho + f_5\gamma_5. \tag{42}$$

The coefficients are determined in terms of the ψ_i and result from taking traces:

$$f_A = \frac{1}{4}\mathrm{tr}(\gamma^A\Psi^{[1]}), \tag{43}$$

where the index $A = 1, \ldots, 16$ stands for any of the (sub)superscripts in the decomposition above, therefore depending on the picture chosen for the Dirac matrices.[4]

One could have taken as well any other 4×4 matrix with only one column filled with the spinor components ψ_i and all other elements zero. Let us denote each of them by $\Psi^{[b]}$, $b = 1,\ldots,4$. Again, applying the differential operator on any of these matrices, one would obtain the same set of coupled equations as for the original Dirac spinor. The point is that the coefficients in terms of products of γ matrices in an expansion like (42) would change.

Any one of these matrices constitutes a minimal left ideal of the Clifford algebra (or, better, of the Clifford ring[5]) of Dirac matrices. By definition, they are left ideals because each is an invariant subspace under left multiplication by any member of the ring. They are minimal since one cannot find a smaller invariant subspace.

The equivalence between spinors and minimal left ideals of the related Clifford algebra is well known in the mathematical literature on the subject [5]. Some authors even claim that it is the best framework to formulate the theory [21]. We shall provide some indications that this merger of vector-space concepts in the spinorial formalism and the algebraic view in terms of left-invariant minimal ideals is a necessity for the treatment of spin-1/2 particles.

Following Rabin [16], let us try to see how similar results are obtained within the framework of differential forms. According to Graf [2], the recipe is simple: just take an expansion in terms of differential forms with the *same* coefficients as the ones used for Dirac matrices:

$$\phi^{[1]} = f1 + f_\mu dx^\mu + \frac{1}{2!} f_{\mu\nu} dx^\mu \wedge dx^\nu + \frac{1}{3!} f_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho + f_5 \epsilon. \quad (44)$$

Analogously to the matrix case, the general differential form above may be reexpressed as the sum of four components, which transform onto themselves under left Clifford multiplication by another general differential form. That is, they can be taken again as members of an invariant minimal left ideal, now related to the Clifford algebra introduced in the space of differential forms.

Summing up for the moment, we can rewrite $\Psi^{[1]}$ in eq. (42) as

$$\Psi^{[1]} = \psi_1 L_1^{[1]} + \psi_2 L_2^{[1]} + \psi_3 L_3^{[1]} + \psi_4 L_4^{[1]}, \quad (45)$$

with $L_1^{[1]},\ldots,L_4^{[1]}$ the basis elements of the 'first' minimal left ideal:

$$L_1^{[1]} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ldots, L_4^{[1]} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (46)$$

Each member of the basis can be written as a linear combination of Dirac matrices. The coefficients can be extracted from the original decomposition [eq. (42)].

In terms of differential forms, we write

$$\phi^{[1]} = \psi_1 E_1^{[1]} + \psi_2 E_2^{[1]} + \psi_3 E_3^{[1]} + \psi_4 E_4^{[1]}, \tag{47}$$

where, now, $E_1^{[1]}, \ldots, E_4^{[1]}$ correspond exactly to $L_1^{[1]}, \ldots, L_4^{[1]}$ written in terms of differential forms through the application of the Graf isomorphism (9).

To complete the discussion, if one applies the Dirac–Kähler operator (37) to $\phi^{[1]}$, one recovers the *same* set of coupled differential equations for the ψ_i, $i = 1, \ldots, 4$, as the original Dirac equation for the spinor ψ, eq. (39).

We shall indicate later how to compute the coefficients of the expansion for $\phi^{[1]}$ in the framework of differential forms with Clifford product. The argument has used so far the analogy coming from the Graf isomorphism between Dirac matrices and KA differential forms.

Let us now come back to the original problem. It seems that the choice of one among four possible minimal left ideals is completely arbitrary. Moreover, once the choice is made, the relevance of the other minimal left ideals becomes a question.

In order to answer this, we need more insight in the algebraic structure underlying the formalism. The fact that we emphasize the differential-form approach in this article comes from its intrinsic geometrical meaning, which lacks correspondence in the matrix construction. The next point we shall analyze, however, is of algebraic origin, and for that purpose, it is better expressed (at least, customarily) in terms of matrices.

Besides forming a Clifford algebra, the fifteen independent gamma matrices γ^K may also be taken as the generators of a Lie algebra, after converting all of them into hermitian matrices by suitably multiplicating by i those matrices whose square equals minus the identity. To see this, one considers the commutator

$$[\gamma^K, \gamma^L] = \gamma^K \gamma^L - \gamma^L \gamma^K, \qquad K, L = 1, \ldots, 15. \tag{48}$$

As we have shown in previous work [6,7,8] by straightforward computation, this algebra closes under commutation, that is,

$$[\gamma^K, \gamma^L] = C^{KL}{}_M \gamma^M. \tag{49}$$

All Lie algebras were classified and related to continuous Lie groups of transformations by Cartan at the beginning of the century [22]. From the set of commutators, eq. (49), we can always find a subset of three simultaneously commuting matrices. These can be taken to form the Cartan subalgebra of the Lie algebra. Gathering all information, we find that the Dirac matrices span the Lie algebra $su(4)$ [6,7,8].

Two important consequences follow:

- the choice of a subset for the Cartan subalgebra is in fact equivalent to singling out a particular picture for the gamma matrices;
- the minimal invariant left ideals under multiplication can be obtained through appropriate projection operators made out of linear combinations of the Cartan subalgebra generators and the identity.

The Dirac equation is implemented to be form invariant under the choice of picture for gamma matrices. The unitary matrices needed to perform the similarity transformation linking two pictures are seen, from the above, to be $SU(4)$ transformations on the algebra.

Minimal left ideals, when acted on by a similarity transformation, can be seen to go into another set of minimal left ideals in the transformed picture. Acted upon by left multiplication, elements of a minimal left ideal just transform as spinors do, but the ideal remains fixed.

This last property, however, is limited to transformations not involving space inversion, time reversal, or their product. As we shall show below, the latter transformations can be realized in terms of right multiplications, so that consistency for the Dirac equation allows for the minimal left ideals to be transformed as matrices.

For differential forms, one proceeds by analogy. This means that "hermitian" forms also satisfy a commutator algebra like (49) with the *same* constants ${C^{KL}}_M$ as the matrices. The operator now defining the algebra is the Clifford commutator introduced in the work by Becher and Joos [4]:

$$[dx^K, dx^L]_\vee = dx^K \vee dx^L - dx^L \vee dx^K. \tag{50}$$

In terms of this operation, we have the analogy for forms of eq. (49):

$$[dx^K, dx^L]_\vee = {C^{KL}}_M dx^M. \tag{51}$$

Since the algebra of gamma matrices is $su(4)$, we have the same algebra, through Clifford multiplication, for differential forms. Actually, the differential forms reproduce the algebra of $su(4)$ generators, and the γ's are its matrix representation. This shows that, aside from its original geometrical content in spacetime, the Clifford algebra of differential forms possesses an intrinsic geometrical content in terms of group theory.

It is also important to stress that, having a set of generators for $su(4)$ given by the differential forms on spacetime, we have a representation for the elements of the group in terms of forms. Specifically, for a given element $g \in SU(4)$, we have a decomposition of the type

$$\begin{aligned} g &= e^{i a_{\mu\nu} dx^\mu \vee dx^\nu} \\ &= \cos(n^\mu n^\nu a_{\mu\nu}) + i\, a_{\mu\nu} dx^\mu \vee dx^\nu \sin(n^\mu n^\nu a_{\mu\nu}), \end{aligned} \tag{52}$$

where n^μ, n^ν are unit vectors in the spacetime directions μ, ν. In general, any element of $SU(4)$ can be represented analogously through differential forms, as any 4×4 matrix can be written as in eq. (42) in terms of gamma matrices.

We recall also the existence of the same sets of commuting elements of the algebra as we have found for the Dirac matrices [8]. Again, we can define a Cartan subalgebra of dimension three, made out of some three mutually commuting differential forms.

Analogous to the Dirac matrices, we can find a set of four minimal left ideals under Clifford multiplication. The relationship between the Cartan subalgebra and minimal left ideals can be used to define the notion of picture for differential forms. In a given picture, the corresponding minimal left ideals can be chosen as left eigenvectors of the forms belonging to the respective Cartan subalgebra.

Now, as with matrices, minimal left ideals can be associated to spinors, i.e., to irreducible representation vectors of the proper Lorentz group. The operator acting on the minimal left ideals that reproduces the set of coupled differential equations of the Dirac equation for a spinor is the Dirac–Kähler operator (37) acting on a suitable minimal left ideal.

This is an all-important difference with the point of view of Rabin [16] and several authors. They prefer to consider the Dirac–Kähler equation as a set of five coupled equations for the components of differential forms, written as in eq. (44):

$$i(d\phi^{(p-1)} + \delta\phi^{(p+1)}) = m\phi^{(p)}, \qquad p = 0, \ldots, 4. \tag{53}$$

When decomposed in the basis of each subspace of forms of a given degree, we find at the end four copies of the Dirac coupled equations. This happens for each ideal when written in the decomposition proposed by Rabin.

Notice that we have worked entirely in terms of differential forms. In their work, Becher and Joos introduced minimal left ideals for differential forms, but devised a mixed approach using also Dirac matrices which allowed them to elegantly single out the minimal left ideals for forms.

To be specific, let us consider the more common pictures for the Dirac gamma matrices; they are the Dirac–Pauli (in which γ^0 is diagonal) and Kramers–Weyl or chiral (in which γ^5 is diagonal) ones.[6] In the framework of differential forms, the respective Cartan subalgebras consist of

$$\begin{array}{ll} dx^0,\ idx^1\wedge dx^2,\ idx^0\wedge dx^1\wedge dx^2 & \text{(Dirac–Pauli)} \\ dx^0\wedge dx^3,\ idx^1\wedge dx^2,\ i\varepsilon & \text{(Kramers–Weyl).} \end{array}$$

Minimal left ideals can be constructed as eigenvectors, by right multiplication, of the projection operators made out of these differential forms and the identity. We have

$$\mathcal{P}^{[1]} = \frac{1}{4}(1+dx^0+idx^1\wedge dx^2+idx^0\wedge dx^1\wedge dx^2) \tag{54}$$

$$\mathcal{P}^{[2]} = \frac{1}{4}(1+dx^0-idx^1\wedge dx^2-idx^0\wedge dx^1\wedge dx^2) \tag{55}$$

$$\mathcal{P}^{[3]} = \frac{1}{4}(1-dx^0+idx^1\wedge dx^2-idx^0\wedge dx^1\wedge dx^2) \tag{56}$$

$$\mathcal{P}^{[4]} = \frac{1}{4}(1-dx^0-idx^1\wedge dx^2+idx^0\wedge dx^1\wedge dx^2), \tag{57}$$

for the Dirac–Pauli picture and similar ones for the Kramers–Weyl picture used by Becher and Joos [4]. We observe that each of them may be generated by an abelian "reduction group", $\mathcal{R}$, isomorphic to $Z_2\times Z_2$:

$$\mathcal{R}_{\text{Dirac-Pauli}} = \{1,\tau,\sigma,\tau\vee\sigma\} \tag{58}$$

$$\mathcal{R}_{\text{Kramers-Weyl}} = \{1,\tau,\varepsilon,\tau\vee\varepsilon\}, \tag{59}$$

where $\tau = idx^1\wedge dx^2$ and $\sigma = dx^0$.

4 The adjoint Dirac–Kähler equation

We shall now examine the analogue of the adjoint spinor in the formalism of differential forms and the equation of motion it satisfies.

The Dirac–Kähler operator decomposes the linear space of differential forms into a coherent sum of contributions from minimal left ideals. This is also true for the space of matrices, in the Rabin construction, but as the latter is not so familiar, this fact escaped the attention of most people. Recalling that $\Lambda^*(M)$ denotes the graded ring of all differential forms of the manifold M, one would write this statement as

$$\Lambda^*(M) = \bigoplus_n L^{(n)}. \tag{60}$$

This is reflected in the study of topological properties based on $\Lambda^*(M)$. As we have shown [7,17], from the Atiyah–Singer theorem for the gauge-invariant Dirac–Kähler differential operator (which in the mathematical literature is known as the twisted signature operator [24]) the index comes out to be the sum of the indices obtained when each minimal left ideal is interpreted as a separate spinor for a Dirac gauge-invariant differential operator.

Moreover, the transformations of the proper Lorentz group act irreducibly on elements of each minimal left ideal as operators from the left.

The emphasis here on minimal left ideals comes mainly from their rôle as analogues to spinors. But KA differential forms possess intrinsically a similar structure for operation by Clifford multiplication on the right. That is, it is possible to find irreducible subspaces under Clifford multiplication on the right, which are the right ideals. Minimal right ideals, as was the case for left ideals, are the smallest among these subspaces. This translates directly, by virtue of the Graf isomorphism, to the usual formalism. It is easy to see, then, that minimal right ideals in terms of 4×4 matrices can be chosen as matrices having all elements null, except those of a single row.

The considerations above induce one to look for a similar decomposition when the minimal right ideals of the algebra generated by the differential forms are properly taken into account [6,8]. That is, we would expect that it is possible to decompose the linear spaces on the right so as to be in correspondence with the structure on the left. This kind of mirror symmetry between objects that are acted upon on the right or on the left by the same operators seem essential to any Grassmann-algebra approach to the description of fermions. Moreover, this is needed to build quantities of physical relevance, namely, the analogues of the bilinear covariants (currents, for instance) in the usual Dirac theory.

Let us recall some elementary concepts. The adjoint spinor is customarily defined by

$$\overline{\psi} = \psi^\dagger \gamma^0, \tag{61}$$

which is a row spinor; that is, in the Kramers–Weyl picture, for example,

$$\overline{\psi} = -\left(\psi_3^* \quad \psi_4^* \quad \psi_1^* \quad \psi_2^* \right). \tag{62}$$

To be specific, let us follow the steps leading to the adjoint equation. First, let us take the hermitian adjoint of the Dirac equation. One obtains

$$i(\partial_\mu \psi^\dagger)\gamma^{\mu\dagger} + m\psi^\dagger = 0. \tag{63}$$

Use now the properties

$$(\gamma^0)^2 = 1, \qquad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{64}$$

to get

$$i(\partial_\mu \psi^\dagger)\gamma^0\gamma^\mu + m\psi^\dagger\gamma^0 = 0. \tag{65}$$

This is the adjoint equation, which we further rewrite as

$$i\left(\partial_\mu(\gamma^0\psi)^\dagger\right)\gamma^\mu + m(\gamma^0\psi)^\dagger = 0. \tag{66}$$

This apparent complication of writing $\overline{\psi} = (\gamma^0\psi)^\dagger$ will become useful in what follows.

Let us recall Rabin's procedure for constructing the general differential form which corresponds to a spinor. The usual Dirac spinor was identified with one of the irreducible vector spaces, the minimal left ideals, which decompose the Clifford algebra of Dirac gamma matrices. These were chosen to have a matrix representation in which the spinor components fill one of the columns of that matrix, the rest of the elements being zero. This matrix was then expanded in a basis composed of the identity and the fifteen independent γ^K. The sought-after differential form was finally obtained by use of the Graf isomorphism.

If one wishes to adhere to Rabin's praxis in order to construct a form corresponding to the adjoint spinor, one should then introduce a 4×4 matrix with only one of its rows filled with the components of $\overline{\psi}$. Again, there are four possibilities, one of them being

$$\overline{\Psi}^{[1]} = -\begin{pmatrix} \psi_3^* & \psi_4^* & \psi_1^* & \psi_2^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{67}$$

Of course, the above matrix does not belong to a minimal left ideal, and the Rabin procedure in this case is not so straightforward. The matrix $\overline{\Psi}^{[1]}$ belongs instead to a minimal *right* ideal of the same Clifford algebra. These are objects with an analogous algebraic defining property as their left counterparts, with respect, now, to Clifford multiplication on the right.

In the same way as for the minimal left ideals, the minimal right ideals are characterized by projection operators, this time acting to the right. They turn out to be precisely the *same* combinations of the elements of the Cartan subalgebra and the identity as were introduced for the minimal left ideals, such as the ones exemplified for differential forms in eqs. (54)–(57). Clearly, to any minimal left ideal one can make correspond a unique minimal right ideal, which is given in matrices by simple transposition.

To illustrate possible alternative approaches to this matter, let us be consistent with the Rabin scheme and write

$$\overline{\Psi} = \overline{f}I + \overline{f}_\mu\gamma^\mu + \frac{1}{2!}\overline{f}_{\mu\nu}\gamma^\mu\gamma^\nu + \frac{1}{3!}\overline{f}_{\mu\nu\rho}\gamma^\mu\gamma^\nu\gamma^\rho + \overline{f}_5\gamma_5, \tag{68}$$

where the quantities $\overline{f}_K$ are combinations of the components of the row. In order to agree with the adjoint equation, when passing to the formalism of differential forms, the corresponding form must be (notice the signs!)

$$\overline{\phi} = \overline{f}1 + \overline{f}_\mu dx^\mu - \frac{1}{2!}\overline{f}_{\mu\nu}dx^\mu \wedge dx^\nu - \frac{1}{3!}\overline{f}_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho + \overline{f}_5\varepsilon. \tag{69}$$

That is, the form corresponding to the adjoint spinor does not follow directly from the Dirac matrices. Instead, the main anti-automorphism $\mathcal{B}$, defined in eq. (14), must be used. Acting on this form, the Dirac–Kähler operator gives now,

$$i(d+\delta)\overline{\phi} = -m\overline{\phi}. \tag{70}$$

In each ideal, this equation reproduces the correct adjoint equations for the ψ_i^*.

Yet one must be cautious. The minimal right ideal is not the original one, since the operator $\mathcal{B}$ transforms minimal left ideals onto minimal right ideals (and vice-versa), corresponding to different projection operators. This method of obtaining the adjoint form thus does not match the considerations made at the beginning of this section.

Let us remark that

- the $\mathcal{B}$ anti-automorphism maps a minimal left ideal into a minimal right one (and vice versa); however, the minimal left ideal it produces is not the one corresponding to the minimal left ideal one started with; this may be corrected through

- the operations of parity and inversion along a given axis, which exchange minimal left ideals or minimal right ideals.

For the Dirac–Pauli and Kramers–Weyl pictures, we then arrive at the following form for the operation that turns minimal left ideals into the minimal right ideals corresponding to the same projection, thereby obtaining a "transposed" general differential form ϕ^t:

$$P\mathcal{I}_2\mathcal{B}\phi = \phi^t. \tag{71}$$

Here $\mathcal{I}_2$ is the inversion of the 2-axis and P is the parity transformation $x^i \mapsto -x^i$ that will be studied in section 5. With this at hand, we can define the proper adjoint minimal right ideal of a given minimal left ideal as

$$\overline{\phi} = P\mathcal{I}_2\mathcal{B}(dx^0 \vee \widetilde{\phi}), \tag{72}$$

where $\widetilde{\phi}$ denotes the minimal-left-ideal forms in which the spinor components ψ_i are replaced by their complex conjugates, ψ_i^*, that is,

$$\widetilde{\phi} = \sum_{i=1}^{4} \psi_i^* E_i. \tag{73}$$

We can write alternatively

$$\overline{\phi} = P\mathcal{B}(dx^0 \vee \phi^*), \tag{74}$$

where now ϕ^* incorporates also a complex conjugation of the ideal basis, due to the property

$$\mathcal{I}_2\widetilde{\phi} = \phi^*. \tag{75}$$

(The operations $P, \mathcal{I}_2$ and $\mathcal{B}$ all commute among each other.)

It would be interesting to investigate further the geometrical meaning of this compound operator, $P\mathcal{I}_2\mathcal{B}$. The operators involved have clearly a geometrical meaning when acting on the forms of a given manifold, and it could illustrate better the possible relationship between geometry and the spin-1/2 representations available for the manifold.

In general, however, the operator that transposes a given ideal is picture dependent. To proceed further, recall that we need a differential operator that implements the decomposition of the space of differential forms $\Lambda^*(M)$

in terms of minimal right ideals. Fortunately, the solution to this problem is straightforward.

Remember that the definition of the action of the exterior differential, eq. (33), involves an explicit positioning of the new differential element to the left of the form on which it is acting. It is equally possible to adopt an equivalent criterium and define an exterior differential operator acting from the right:

$$\omega \overleftarrow{d} = (\partial_\mu \omega) \wedge dx^\mu. \tag{76}$$

We define as well (in four dimensions) the codifferential acting from the right:

$$\overleftarrow{\delta} = - * \overleftarrow{d} *. \tag{77}$$

It is thus easily found that the same Dirac–Kähler operator acts irreducibly from the right on the minimal right ideals. We have, therefore, for the Dirac equation to be satisfied by a minimal right ideal,

$$\omega i(\overleftarrow{d} + \overleftarrow{\delta}) = -m\omega. \tag{78}$$

The question of the adjoint Dirac–Kähler equation is then readily solved. When a minimal right ideal ω is written in terms of components, ω_i, $i = 1, \ldots, 4$, it is easy to see that the equations satisfied are the same as the Dirac ones, except for the sign of the time derivative. In the usual setting, this is the analogue of taking the hermitian adjoint to the Schrödinger equation

$$H\psi = i\hbar \frac{\partial \psi}{\partial t}, \tag{79}$$

where we find

$$\psi^\dagger H^\dagger = -i\hbar \frac{\partial \psi^\dagger}{\partial t}. \tag{80}$$

Thus we can state that

$$-i(\overleftarrow{d} + \overleftarrow{\delta}) = [i(d+\delta)]^\dagger. \tag{81}$$

The heuristic definition of the differential operator in eq. (78) provides the required properties to deal with the adjoint forms as minimal right ideals. At this point, we should ask ourselves with respect to which inner product the operator $-i(\overleftarrow{d} + \overleftarrow{\delta})$ is the hermitian adjoint of $i(d+\delta)$. This will be defined in section 7.

The construction of the adjoint minimal right ideal seems rather cumbersome, but seems to us necessary; more on this matter will be also discussed in Section 7. Compare, for instance, with the definition of an adjoint differential form as given by Becher and Joos [4], $\overline{\phi}_{BJ} = \mathcal{A}\phi^{\dagger}$. This makes the adjoint form correspond to a minimal left ideal in the Kramers–Weyl picture and the differential operator is therefore the usual one, $i(d+\delta)-m$. It is not easy to interpret it in the Dirac–Pauli picture, since the automorphism $\mathcal{A}$ in the latter transforms minimal left ideals into right ones.

Having introduced a satisfactory candidate for the adjoint minimal right ideal, we are ready to discuss the lagrangian of the theory, the currents, normalization properties, etc. Before doing this, however, let us complete the kinematical framework developed here by considering the transformation properties of the ideals under discrete symmetries.

5 Discrete symmetries

We develop in this section a formalism for defining the general differential forms that may be interpreted as the charge-conjugate, parity- and time-reversal-transformed forms, in the sense that these satisfy the appropriately modified Dirac–Kähler equation for each of them, thereby giving for each minimal left ideal the corresponding set of Dirac coupled equations in each case.[7]

To be explicit, let us work within a definite picture of the Dirac matrices. We choose the Kramers–Weyl one [23], in which eq. (39) corresponds to the following Dirac coupled equations for the ψ_i:

$$\left.\begin{array}{rcl} -i\partial_0\psi_3 + i\partial_3\psi_3 + i\partial_1\psi_4 + \partial_2\psi_4 & = & m\psi_1 \\ i\partial_1\psi_3 - \partial_2\psi_3 - i\partial_0\psi_4 - i\partial_3\psi_4 & = & m\psi_2 \\ -i\partial_0\psi_1 - i\partial_3\psi_1 - i\partial_1\psi_2 - \partial_2\psi_2 & = & m\psi_3 \\ -i\partial_1\psi_1 + \partial_2\psi_1 - i\partial_0\psi_2 + i\partial_3\psi_2 & = & m\psi_4 \end{array}\right\}. \tag{82}$$

The Kramers–Weyl general differential forms ϕ which, upon action of the Dirac–Kähler operator $i(d+\delta)-m$, reproduce the above set of coupled equations are written as follows, after combining all forms with a common coefficient (the superscript inside square brackets indicates the minimal left ideal; their numbering is taken by analogy with the matricial ones, in which $[b]$ refers to the matrix $\Psi^{[b]}$):

$$\begin{array}{rcl} \phi^{[1]}(x) & = & \frac{1}{4}[\psi_1(1 + dx^{03} + idx^{12} + i\varepsilon) \\ & & +\psi_2(dx^{01} - idx^{02} - dx^{13} + idx^{23}) \\ & & +\psi_3(-dx^0 - dx^3 - idx^{012} - idx^{123}) \end{array}$$

$$
\begin{aligned}
& +\psi_4(-dx^1 + idx^2 + dx^{013} - idx^{023})](x); && (83)\\
\phi^{[2]}(x) = & \tfrac{1}{4}[\psi_1(dx^{01} + idx^{02} + dx^{13} + idx^{23}) \\
& +\psi_2(1 - dx^{03} - idx^{12} + i\varepsilon) \\
& +\psi_3(-dx^1 - idx^2 - dx^{013} - idx^{023}) \\
& +\psi_4(-dx^0 + dx^3 + idx^{012} - idx^{123})](x); && (84)\\
\phi^{[3]}(x) = & \tfrac{1}{4}[\psi_1(-dx^0 + dx^3 - idx^{012} + idx^{123}) \\
& +\psi_2(dx^1 - idx^2 + dx^{013} - idx^{023}) \\
& +\psi_3(1 - dx^{03} + idx^{12} - i\varepsilon) \\
& +\psi_4(-dx^{01} + idx^{02} - dx^{13} + idx^{23})](x); && (85)\\
\phi^{[4]}(x) = & \tfrac{1}{4}[\psi_1(dx^1 + idx^2 - dx^{013} - idx^{023}) \\
& +\psi_2(-dx^0 - dx^3 + idx^{012} + idx^{123}) \\
& +\psi_3(-dx^{01} - idx^{02} + dx^{13} + idx^{23}) \\
& +\psi_4(1 + dx^{03} - idx^{12} - i\varepsilon)](x). && (86)
\end{aligned}
$$

In the above and in the following equations, we adopt the simplifying notation $dx^{\mu\nu} \equiv dx^\mu \wedge dx^\nu$, $dx^{\mu\nu\rho} \equiv dx^\mu \wedge dx^\nu \wedge dx^\rho$.

5.1 Charge conjugation

In the spinorial formalism, the charge-conjugate spinor to (40) is defined as

$$\psi^C = i \begin{pmatrix} -\psi_4^* \\ \psi_3^* \\ \psi_2^* \\ -\psi_1^* \end{pmatrix}, \tag{87}$$

which satisfies

$$-i(\gamma^\mu)^* \partial_\mu \psi^C = m\psi^C. \tag{88}$$

In the Kramers–Weyl picture, this implies the following set of equations:

$$\left.\begin{aligned}
i\partial_0\psi_2^* - i\partial_3\psi_2^* + i\partial_1\psi_1^* - \partial_2\psi_1^* &= -m\psi_4^* \\
-i\partial_1\psi_2^* - \partial_2\psi_2^* - i\partial_0\psi_1^* - i\partial_3\psi_1^* &= m\psi_3^* \\
-i\partial_0\psi_4^* - i\partial_3\psi_4^* + i\partial_1\psi_3^* - \partial_2\psi_3^* &= m\psi_2^* \\
-i\partial_1\psi_4^* - \partial_2\psi_4^* + i\partial_0\psi_3^* - i\partial_3\psi_3^* &= -m\psi_1^*
\end{aligned}\right\}, \tag{89}$$

which, comparing with the original set (82), are just the usual Dirac equations with the change $\psi_i \to \psi_i^*$, for all i.

In order to incorporate a similar procedure in the Dirac–Kähler scheme, we have to find a "charge-conjugate general differential form," ϕ^C, satisfying

$$-i(d+\delta)\phi^C = m\phi^C. \tag{90}$$

This is obtained from the original general differential forms by taking the complex conjugation of the form basis and by substituting the coefficients according to (87). We exemplify by giving the result for the first minimal left ideal in the Kramers–Weyl picture:

$$\begin{aligned}\phi^{C[1]} &= \tfrac{1}{4}[\psi_1^*(dx^1 + idx^2 - dx^{013} - idx^{023}) \\ &\quad +\psi_2^*(-dx^0 - dx^3 + idx^{012} + idx^{123}) \\ &\quad +\psi_3^*(dx^{01} + idx^{02} - dx^{13} - idx^{23}) \\ &\quad +\psi_4^*(-1 - dx^{03} + idx^{12} + i\varepsilon)].\end{aligned} \tag{91}$$

By substituting this in (90), one recovers the coupled equations (89).

We notice that, from comparison with eqs. (83)–(86), we have

$$\phi^{C[b]}(x) = -A\widetilde{\phi}^{[b]}(x) \vee dx^2. \tag{92}$$

This reproduces a result of Becher and Joos [4].

We also observe that the C-operation produces a form belonging to another minimal left ideal. In the example shown, the form (91) belongs to the fourth minimal left ideal, although the operation was performed on a first-ideal form.

5.2 Parity

The parity-transformed spinor is defined (Kramers–Weyl picture) by

$$\psi^P(t,\boldsymbol{x}) = \gamma^0\psi(t,-\boldsymbol{x}) = -\begin{pmatrix}\psi_3(t,-\boldsymbol{x})\\ \psi_4(t,-\boldsymbol{x})\\ \psi_1(t,-\boldsymbol{x})\\ \psi_2(t,-\boldsymbol{x})\end{pmatrix}. \tag{93}$$

Under parity transformation the Dirac equation is written in a transformed Lorentz reference frame but keeps its form unchanged:

$$i\gamma^\mu\partial'_\mu\psi' = m\psi', \tag{94}$$

where ψ' incorporates the changes in (93) and

$$\partial'_0 = \partial_0; \qquad \partial'_i = -\partial_i, \tag{95}$$

so that one gets in the transformed system the same set of coupled Dirac equations,

$$\left.\begin{array}{rcl} -i\partial'_0\psi'_1 - i\partial'_3\psi'_1 - i\partial'_1\psi'_2 - \partial'_2\psi'_2 &=& m\psi'_3 \\ -i\partial'_1\psi'_1 + \partial'_2\psi'_1 - i\partial'_0\psi'_2 + i\partial'_3\psi'_2 &=& m\psi'_4 \\ -i\partial'_0\psi'_3 + i\partial'_3\psi'_3 + i\partial'_1\psi'_4 + \partial'_2\psi'_4 &=& m\psi'_1 \\ i\partial'_1\psi'_2 - \partial'_2\psi'_3 - i\partial'_0\psi'_4 - i\partial'_3\psi'_4 &=& m\psi'_2 \end{array}\right\}. \tag{96}$$

Analogously, the Dirac–Kähler equation should read

$$i(d' + \delta')\phi^P = m\phi^P, \tag{97}$$

with

$$d' = dx^{\mu'} \wedge \partial'_\mu. \tag{98}$$

Performing the same changes as above, in (93) and (95), in the general differential forms of eqs. (83)–(86), one defines their parity transform. We again exemplify with the first minimal left ideal in the Kramers–Weyl picture:

$$\begin{aligned} \phi^{P[1]} \;=\; & -\tfrac{1}{4}[\psi'_1(-dx^{0'} + dx^{3'} - idx^{0'1'2'} + idx^{1'2'3'}) \\ & +\psi'_2(dx^{1'} - idx^{2'} + dx^{0'1'3'} - idx^{0'2'3'}) \\ & +\psi'_3(1 - dx^{0'3'} + idx^{1'2'} - i\epsilon') \\ & +\psi'_4(-dx^{0'1'} + idx^{0'2'} - dx^{1'3'} + idx^{2'3'})](t, -\boldsymbol{x}), \end{aligned} \tag{99}$$

which correctly satisfies (97), as it reproduces (96). This is also the case for all other $\phi^{P[b]}$.

We may notice that parity-transformed forms are related to the original ones, (83)–(86), but are in other minimal left ideals:

$$\phi^{P[1]}(t, \boldsymbol{x}) = -\phi^{[3]}(t, -\boldsymbol{x}) \tag{100}$$

$$\phi^{P[2]}(t, \boldsymbol{x}) = -\phi^{[4]}(t, -\boldsymbol{x}), \tag{101}$$

etc. However, the precise relation is picture dependent. Another observation, linked to the one above, is that

$$\phi^{P[1]}(t, \boldsymbol{x}) = \phi^{[1]}(t, -\boldsymbol{x}) \vee dx^0 \tag{102}$$

with similar relations for the remaining minimal left ideals, which may be taken as a differential-form definition of parity transformation. As is well known [4], the right Clifford product results in a change of minimal left ideals, which is confirmed by the relations (100), (101). In view of the Graf isomorphism, this transformation is reminiscent of the similar one for spinors, eq. (93).

5.3 Time reversal

One proceeds in a manner similar to that of the previous transformations. The time reversal of a spinor in the Kramers–Weyl picture is defined as[8]

$$\psi^T(t,x) = i\gamma^1\gamma^3\psi^*(-t,x) = i\begin{pmatrix} \psi_2^*(-t,x) \\ -\psi_1^*(-t,x) \\ \psi_4^*(-t,x) \\ -\psi_3^*(-t,x) \end{pmatrix}. \tag{103}$$

Again, this corresponds to a change of reference frame in which

$$\partial_0' = -\partial_0, \qquad \partial_i' = \partial_i. \tag{104}$$

The time-reversed Dirac equation also includes complex conjugation of the differential operator, so that ψ^T satisfies

$$-i(\gamma^\mu)^*\partial_\mu'\psi^T = m\psi^T. \tag{105}$$

In the present case, the explicit coupled equations are

$$\left.\begin{array}{rcr} (-i\partial_0' - i\partial_3')\psi_4^* + (i\partial_1' - \partial_2')\psi_3^* &=& m\psi_2^* \\ (-i\partial_1' - \partial_2')\psi_4^* + (i\partial_0' - i\partial_3')\psi_3^* &=& -m\psi_1^* \\ (-i\partial_0' + i\partial_3')\psi_2^* + (-i\partial_1' + \partial_2')\psi_1^* &=& m\psi_4^* \\ (i\partial_1' + \partial_2')\psi_2^* + (i\partial_0' + i\partial_3')\psi_1^* &=& -m\psi_3^* \end{array}\right\}. \tag{106}$$

As for the Dirac–Kähler equation, under time reversal transformation it is written in a similar manner as

$$-i(d' + \delta')\phi^T = m\phi^T. \tag{107}$$

To obtain its solutions in the various minimal left ideals, one incorporates in the forms (83)–(86) the changes in the ψ_i given in (103) together with the appropriate change of frame, that is, $dx^0 \to -dx^{0'}$, $dx^i \to dx^{i'}$, and performs a complex conjugation of the bases of the ideals. For the Kramers–Weyl picture we once more exemplify by giving the result of the first minimal left ideal:

$$
\begin{aligned}
\phi^{T[1]} &= \tfrac{1}{4}[\psi_1^*(dx^{0'1'} + idx^{0'2'} + dx^{1'3'} + idx^{2'3'}) \\
&\quad +\psi_2^*(1 - dx^{0'3'} - idx^{1'2'} + i\varepsilon') \\
&\quad +\psi_3^*(dx^{1'} + idx^{2'} + dx^{0'1'3'} + idx^{0'2'3'}) \\
&\quad +\psi_4^*(dx^{0'} - dx^{3'} - idx^{0'1'2'} + idx^{1'2'3'})](-t, x). \qquad (108)
\end{aligned}
$$

This form, as well as the corresponding ones for the remaining minimal left ideals, when inserted in (107), correctly give the time-reversed set (106).

In this case as well, we notice a relation between real transformed forms and the original ones in different ideals; for instance, in the Kramers–Weyl picture,

$$\phi^{T[1]}(t, x) = i\mathcal{A}\phi^{[2]}(-t, x). \qquad (109)$$

We are then led to a differential-form picture-independent definition of the time-reversal transformation through the action of a right Clifford product:

$$\phi^{T[b]}(t, x) = i\mathcal{A}\widetilde{\phi}^{[b]}(-t, x) \vee dx^{13}, \qquad (110)$$

in which the form $\widetilde{\phi}$ is written in terms of the components of the conjugate spinor, ψ_i^*, as in eq. (73). This is again reminiscent of the time reversal of a spinor, eq. (103), in the sense of the Graf isomorphism. Notice that

$$
\begin{aligned}
\left[\phi^{T[b]}\right]^T &= i\mathcal{A}\left[i\mathcal{A}\widetilde{\phi^{[b]} \vee dx^{13}}\right] \vee dx^{13} \\
&= -\phi(t, x), \qquad (111)
\end{aligned}
$$

in agreement with the result for spinors. This curious property can be taken as another piece of evidence for the $SU(4)$ characterization of the formalism for spin-1/2 particles.

5.4 Combined transformations and the CPT theorem

The combined parity–time reversal transformation, PT, is defined on forms as

$$
\begin{aligned}
\phi^{PT} = (\phi^P)^T &= (\phi \vee dx^0)^T = i\mathcal{A}(\widetilde{\phi} \vee dx^0) \vee dx^{13} \\
&= -i\mathcal{A}\widetilde{\phi} \vee dx^{013}, \qquad (112)
\end{aligned}
$$

which may be directly verified by using the explicit expressions in a given picture. Observe that

$$\begin{aligned} \phi^{TP} = (\phi^T)^P &= (i\mathcal{A}\tilde{\phi} \vee dx^{13})^P = i\mathcal{A}\tilde{\phi} \vee dx^{013} \\ &= -\phi^{PT}. \end{aligned} \tag{113}$$

Let us now perform the above PT transformation on the form ϕ^C. We obtain, from (92),

$$\begin{aligned} \phi^{CPT} &= (\phi^C)^{PT} = (-\mathcal{A}\tilde{\phi} \vee dx^2)^{PT} = i\mathcal{A}(\mathcal{A}\phi \vee dx^2) \vee dx^{013} \\ &= -i\phi \vee \varepsilon, \end{aligned} \tag{114}$$

where we have used the properties (31) and (16). This result may be expressed in terms of the "modified Hodge operator" [4],

$$\star = *\mathcal{B}, \tag{115}$$

which has the property

$$\star\phi = \phi \vee \varepsilon. \tag{116}$$

Therefore,

$$\phi^{CPT[b]} = -i \star \phi^{[b]}. \tag{117}$$

Keeping the discussion within the Kramers–Weyl picture, in which the general differential forms $\phi^{[b]}$ are antiselfdual for $b = 1, 2$ and selfdual for $b = 3, 4$ under $\star$, we have, apart from a phase factor,

$$\star\phi^{[b]} = \begin{cases} -i\phi^{[b]} & \text{for } b = 1,2 \\ +i\phi^{[b]} & \text{for } b = 3,4. \end{cases} \tag{118}$$

We see that the combined operation of CPT reproduces the form of the original minimal left ideal:

$$\phi^{CPT[b]} = \begin{cases} -\phi^{[b]} & \text{for } b = 1,2 \\ +\phi^{[b]} & \text{for } b = 3,4. \end{cases} \tag{119}$$

Again, this can be checked directly using eqs. (83)–(86).

The usual cyclic property of the change of the order in which the C, P and T transformations are applied is also valid in this formalism:

$$\phi^{CPT} = \phi^{PTC} = \phi^{TCP}. \tag{120}$$

6 The scalar value of differential forms

In this section we introduce a new operation in the inner calculus of differential forms. In order to have a more complete analogy between the manipulations of forms and gamma matrices, induced by the isomorphism (9), we wish to find an operation on forms which corresponds to the trace of (Clifford) products of γ's. This will be called the scalar value of a differential form.

We shall denote by the symbol \$ the operation on a differential form which gives its scalar value. It is defined such that, when applied to the Clifford product of two forms, one obtains their contraction (see eq. (20)), taking into account that each minimal left ideal provides the same contribution. In a spacetime with n (even) dimensions, the number of minimal left ideals is $d(n) = 2^{n/2}$, and equals the dimension of the spinorial representation in that spacetime. Therefore, we define

$$\begin{aligned} \$(1) &\equiv d(n) && (121)\\ \$(dx^\mu) &\equiv \$(dx^\mu \vee 1) \equiv e^\mu \rfloor 1 = 0 && (122)\\ \$(dx^\mu \vee dx^\nu) &\equiv e^\mu \rfloor dx^\nu \cdot \$(1) = d(n) g^{\mu\nu}, && (123)\end{aligned}$$

where $g^{\mu\nu}$ is the Minkowski flat metric and eqs. (21) and (22) have been used. As a consequence we have, through repeated application of eq. (30),

$$\begin{aligned} &\$(dx^\mu \vee dx^\nu \vee dx^\rho \vee dx^\sigma)\\ &= \$(e^\mu \rfloor (dx^\nu \vee dx^\rho \vee dx^\sigma))\\ &= \$(g^{\mu\nu} dx^\rho \vee dx^\sigma - g^{\mu\rho} dx^\nu \vee dx^\sigma + g^{\mu\sigma} dx^\nu \vee dx^\rho)\\ &= d(n)(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}). && (124)\end{aligned}$$

In the particular case in which $\mu \neq \nu \neq \rho \neq \sigma$, in 4 dimensions, eq. (124) becomes

$$\begin{aligned} \$(\varepsilon) &= \$(dx^0 \vee dx^1 \vee dx^2 \vee dx^3)\\ &= 4(g^{01} g^{23} - g^{02} g^{13} + g^{03} g^{12}) = 0. && (125)\end{aligned}$$

The application of the \$ operator to the Clifford product of an odd number, m, of differentials always reduces to an application of \$ to a product of $m-2$ (again odd, of course) differentials, and so on, until we have \$ applied to a single differential, which is zero, due to (122). Thus the action of \$ on any odd number of Clifford products of differentials always vanishes. In particular,

$$\$(dx^\mu \vee \varepsilon) = 0. \tag{126}$$

One may easily show that the scalar value

$$\$(dx^\mu \vee dx^\nu \vee \varepsilon) = 0 \tag{127}$$

holds in four dimensions; however, for $n = 2$ (with the convention $\epsilon^{01} = +1$):

$$\begin{aligned}
\$(dx^\mu \vee dx^\nu \vee \varepsilon) &= \$(e^\mu \rfloor (dx^\nu \vee dx^0 \vee dx^1)) \\
&= \$(g^{\mu\nu}\varepsilon - g^{\mu 0} dx^\nu \vee dx^1 + g^{\mu 1} dx^\nu \vee dx^0) \\
&= 2(-g^{\mu 0} g^{\nu 1} + g^{\mu 1} g^{\nu 0}) \\
&= -2\epsilon^{\mu\nu}.
\end{aligned} \tag{128}$$

The analogous property for $n = 4$ is:

$$\begin{aligned}
&\$(dx^\mu \vee dx^\nu \vee dx^\rho \vee dx^\sigma \vee \varepsilon) \\
&= \$(g^{\mu\nu} dx^\rho \vee dx^\sigma \vee \varepsilon - g^{\mu\rho} dx^\nu \vee dx^\sigma \vee \varepsilon + g^{\nu\rho} dx^\mu \vee dx^\sigma \vee \varepsilon \\
&\quad -\epsilon_\beta{}^{\mu\nu\rho} dx^\beta \vee dx^\sigma \vee \varepsilon \vee \varepsilon) \\
&= -\epsilon_\beta{}^{\mu\nu\rho} \$(dx^\beta \vee dx^\sigma) \\
&= 4\epsilon^{\mu\nu\rho\sigma},
\end{aligned} \tag{129}$$

where we have used the "Chisholm relation" [25]

$$dx^\mu \vee dx^\nu \vee dx^\rho = g^{\mu\nu} dx^\rho - g^{\mu\rho} dx^\nu + g^{\nu\rho} dx^\mu + \epsilon_\beta{}^{\mu\nu\rho} dx^\beta \vee \varepsilon. \tag{130}$$

One can see, through the examples shown above, that the scalar value operation on the Clifford product of forms has the same formal algebraic properties as the trace of products of Dirac gamma matrices. Many of these properties here described for forms were also verified for matrices by Veltman [26].

7 Inner product and currents

To complete the introduction of physical quantities, analogues to the usual bilinear covariants of spinor theory should be constructed within the Dirac–Kähler approach.

To emphasize the fact that the whole space of KA differential forms is to be decomposed into the direct sum of four equivalent (except under discrete symmetries) minimal ideals (either left or right); hence, it is necessary to devise an inner product of forms that operates separately on an ideal of a given class and the corresponding one in the other (that is, the one projected by the same idempotent). For this, we have introduced in the preceding sections the adjoint of a given minimal left ideal, which should be a minimal right ideal, and the scalar value for differential forms.

We define a complex-valued inner product of two members of a given minimal left ideal subspace as the Lorentz scalar given by

$$(\phi, \psi) = \$(\overline{\phi} \vee \psi). \tag{131}$$

This fixes uniquely the adjoint form and, besides, with this definition, the adjoint of an operator acting from the left on minimal left ideals recovers its usual meaning of being an operator acting from the right on the corresponding minimal right ideal. The inner product does not mix ideals in this sense. Two members of different minimal left ideals are always orthogonal.

With the inner product so defined, we can now introduce the objects analogous, in this description, to the usual bilinear covariants in spinors. They are in general of the form

$$\mathcal{O}^K = (\psi, dx^K \vee \psi), \tag{132}$$

where dx^K is any of the sixteen independent differential forms in four dimensions. The current operators are

$$j^\mu = (\psi, dx^\mu \vee \psi) \tag{133}$$

for the vector current, and

$$j_5^\mu = (\psi, dx^\mu \vee i\varepsilon \vee \psi) \tag{134}$$

for the axial-vector one. As a consequence, all formal results valid for the spinor description translate simply in this language. The spinor notion is replaced by that of the minimal left ideal.[9]

The lagrangian density that provides through functional minimization the Dirac–Kähler equation reads

$$\mathcal{L} = \frac{1}{2}\left(\phi, (i(d+\delta) - m)\phi\right) - \frac{1}{2}\left((i(d+\delta) + m)\phi, \phi\right). \tag{135}$$

For a superposition of minimal left ideals, we would get a sum of separate contributions.

8 Conclusions

In this work we have analyzed in detail the application of Kähler–Atiyah differential forms to the study of spin-one-half particles. We have made clear how the formalism developed by Kähler and lately applied to field theory by Becher and Joos can be strictly interpreted in terms of independent minimal left ideals which play the rôle of inequivalent spinors (in the sense that they transform among themselves under the discrete operations of space and time reversal and their product).

To exhibit completely the physical content of the theory, we needed to introduce several new ingredients: the scalar value of Kähler–Atiyah differential forms allowed the definition of a new inner product, which consistently realizes the decomposition of the space of KA differential forms in separate contributions from each of the minimal left ideals; the notion of the adjoint of a minimal left ideal and of the adjoint of the Dirac–Kähler differential operator followes again in a consistent way. Finally, the currents and the lagrangian description fit nicely in the formalism.

At the foundation of the entire work is the formal property that KA differential forms can represent the generators of the $su(4)$ algebra. In this sense, though isomorphic to the usual ring of Dirac gamma matrices, they play a certain parental rôle with respect to the latter. The Dirac gamma matrices can, in fact, be considered a possible matrix representation for the algebra of differential forms. There is, moreover, the intrinsic geometric content of differential forms which we have not fully exploited here but which appears promising.

The notion of picture, which is somewhat accesory in the language of matrices, gets in KA differential forms a complete formal characterization. It is directly linked to the $su(4)$ Cartan subalgebra.

The fact that the discrete symmetries of Minkowski spacetime do really matter in this description points to several possible new developments in the study of symmetries in theories of elementary particles. (Further consideration of this problem is being currently completed.) In particular, this fact will be reflected in the translation of the theory from the continuum into the lattice, which we have performed in the framework of the description by Dirac matrices in the accompanying article [8].

We believe that this study will stimulate further developments, particularly in the light of the current growing interest on the formulation of field theories from a geometric (and topological) point of view.

Footnotes

1. We intend to analyze the properties of the lattice Dirac–Kähler equation in a future article.

2. This is analogous to the expression for γ matrices,

$$\gamma^{\mu}\gamma^{\nu} = g^{\mu\nu} + \frac{1}{2}[\gamma^{\mu}, \gamma^{\nu}],$$

or, for Pauli matrices,

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot \vec{A} \wedge \vec{B}.$$

3. This definition only applies to even dimensions. For genéral dimension n, it is $\delta = (-1)^{np+n+1} * d*$, where p is the degree of the form on which the operator acts.

4. We prefer to introduce the term *picture* for what is commonly referred to as a *representation* of a Dirac matrix, since the latter can be misleading; this happens most trivially when comparing the differential-form and matrix formalisms, as the latter is strictly a representation of the former, in the group-theoretic sense.

5. in the terminology of Corson [20]. See a thorough analysis in our preceding article [8].

6. For conventions, see the book by Itzykson and Zuber [23].

7. The procedures introduced in this section are not unique. Other, alternate possibilities will be thoroughly exposed in a forthcoming article.

8. This transformation is identical to the corresponding one in the Dirac–Pauli picture.

9. We could make a superposition of minimal left ideals. In this case, using the properties mentioned at the beginning, the current operators would appear as a sum of the contributions from the single ideals separately.

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