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QUANTUM INSTABILITY OF MINKOWSKI SPACETIME

by

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ABSTRACT

Taking into account vacuum fluctuations through a quadratic curvature term in the gravitational action, a minisuperspace model is constructed where the stability of Minkowski spacetime is analized. We find it to be unstable independently of the quadratic term coupling constant.

Key-words: Quantum gravity; Quantum field theory; General Relativity; Canonical quantization of gravitation.

1 INTRODUCTION

The stability of Minkowski solution in classical general relativity is a well-known result. On the other hand, in the semiclassical approach where quantum fields are considered on a classical background, the same problem has been studied and Minkowski spacetime was shown to be unstable under different conditions [1] due to renormalization effects. More recently [2], it has been verified that flat spacetime with quantum fields is always unstable under a certain type of perturbations. One should expect the same result when a full quantum theory of gravity is used. In this paper we will analize the problem in the context of the canonical quantum theory of gravity [3].

We will consider a $R + \epsilon R^2$ model which may be viewed as an effective theory that takes into account the first short-distance corrections to classical relativity. These corrections must be necessarily included when quantum fields are present in order to renormalize the theory, since divergences proportional to R^2 do appear [4]. They may also come from the low energy limit of a more fundamental theory (as for instance superstrings) [5]. It is then of interest to consider such type of modifications of general relativity. The R^2 gravity has been analized in the context of quantum cosmology by several authors [6-8]. In particular, the wave function of the universe has been computed within different approximations using Hartle-Hawking [6] and Vilenkin boundary conditions [7,8]. Our interest here is to study the evolution of a wave packet initially peaked around Minkowski spacetime.

The paper is organized as follows. In Section II we write the classical action of R^2 gravity for flat Robertson Walker spacetimes. We derive the Wheeler-DeWitt equation which has only two degrees of freedom: the scale factor and the scalar curvature (we are thus reducing superspace to a two dimensional minisuperspace). In Section III we find a solution of the Wheeler-DeWitt equation that shows that Minkowski spacetime is unstable for all values of the constant ϵ. Some final remarks are included in Section IV.

II THE MODEL

We will study a model with gravitational action given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}(R - \epsilon R^2) \qquad (2.1)$$

where G is the gravitational constant, R is the scalar curvature and g is the determinant of the spacetime metric. The signature is (-+++) and we work with natural units in which $h = c = 1$.

We will consider a flat Robertson Walker spacetime with metric

$$ds^2 = -dt^2 + a^2(t)\{dx^2 + dy^2 + dz^2\} \qquad (2.2)$$

where a(t) is the scale factor. The Ricci scalar is given by

$$R = 6(\dot{H} + 2H^2) \quad , \quad H = \frac{\dot{a}}{a} \tag{2.3}$$

so the action (2.1) becomes a functional of $a(t)$, $\dot{a}(t)$ and $\ddot{a}(t)$. In order to quantize canonically this theory with higher derivatives, one must consider $a(t)$ and $R(t)$ as independent coordinates [6-9]. The Wheeler-DeWitt equation is obtained as usual by replacing the momenta by derivatives with respect to the conjugate variables in the classical Hamiltonian constraint, which is the generalization of the tt-Einstein equation.

It is useful to write the action (2.1) in terms of the independent variables $a(t)$ and $\phi(t) = \ell n(1+2\varepsilon R)^{1/2}$, which is then given by

$$S = -\frac{3V}{8\pi G}\int dt \left\{ a e^{2\phi}\dot{a}^2 + 2a^2 e^{2\phi}\dot{\phi}\dot{a} + \frac{a^3}{24\varepsilon}\left(e^{2\phi}-1\right)^2 \right\} \tag{2.4}$$

where V is the spatial volume, that we will assume to be a finite constant The conjugate momenta are

$$\pi_a = -\, 2Ke^{2\phi}(a\dot{a} + a^2\dot{\phi}) \tag{2.5a}$$

$$\pi_\phi = -\, 2ke^{2\phi}a^2\dot{a} \tag{2.5b}$$

where $k = 3V/8\pi G$. The classical constraint which comes from the time reparametrization invariance of the theory is

$$\mathcal{H} = -\frac{e^{-2\phi}}{2a^2k}\pi_a\pi_\phi + \frac{e^{-2\phi}}{4a^3k}\pi_\phi^2 + \frac{ka^3}{24\varepsilon}\left[e^{+2\phi}-1\right]^2 = 0 \tag{2.6}$$

In terms of R and $H = \dot{a}/a$ Eq. (2.6) reads as follows

$$\dot{R} = \frac{R^2}{12H} - \frac{H}{2\epsilon} - RH \tag{2.7}$$

It is worth noting that this constraint and the Euler Lagrange equations derived from the action Eq. (2.4) considering a(t) and $\phi(t)$ as independent variables

$$\ddot{\phi} + \frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 + 2\,\frac{\dot{a}}{a}\,\dot{\phi} + 2\dot{\phi}^2 = \frac{1}{16\varepsilon}\, e^{-2\phi}(e^{2\phi}-1)^2 \tag{2.8a}$$

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \frac{(e^{2\phi}-1)}{12\varepsilon} \tag{2.8b}$$

are equivalent to the definition of R (Eq.(2.3)) and to

$$R_{\mu\nu} - \frac{1}{2}\, Rg_{\mu\nu} + \varepsilon\, {}^{(1)}H_{\mu\nu} = 0 \tag{2.9}$$

where the tensor ${}^{(1)}H_{\mu\nu}$ is defined through

$${}^{(1)}H_{\mu\nu} = \frac{1}{\sqrt{-g}}\,\frac{\delta}{\delta g^{\mu\nu}}\int\sqrt{-g}\; R^2 d^4x$$

In the quantum version the classical constraint given above becomes the Wheeler-DeWitt equation

$$\left\{-\frac{\partial^2}{\partial\phi^2} + 2a\,\frac{\partial^2}{\partial a\partial\phi} + \frac{k^2a^6e^{6\phi}}{6\varepsilon}\left[e^{-2\phi}-1\right]^2\right\}\psi(a,\phi) = 0$$

in the minisuperspace of the variables a and ϕ. We have chosen a factor ordering such that the differential operator in Eq. (2.10) is the D'alambertian operator in our minisuperspace.

III THE SOLUTION

In order to study the stability of Minkowski spacetime we must analize the evolution of a wave packet centered on the classical solution, that is, $\phi = 0$, which means zero curvature. We will then look for a solution of the Wheeler-DeWitt equation. As a matter of fact, it will be sufficient to restrict our analisis to the case where the wave function depends much more strongly on the scale factor than on the curvature, that is

$$\frac{\partial\phi}{\partial\phi} << a \frac{\partial\psi}{\partial a} \tag{3.1}$$

and the Eq. (2.10) reduces to

$$\left\{\frac{\partial^2}{\partial a\partial\phi} + \frac{k^2 a^5 e^{6\phi}}{12\varepsilon}\left[e^{-2\phi} - 1\right]^2\right\}\psi(a,\phi) = 0 \tag{3.2}$$

This equation can be solved by separation of variables and admits a general solution of the form

$$\Psi(a,\phi) = \int d\lambda\, A(\lambda)\, e^{\frac{i}{\hbar}\left[\frac{\lambda a^6}{6} + \frac{M(\phi)}{\lambda}\right]} \tag{3.3}$$

where λ is the separation constant and

$$M(\phi) = \frac{k^2}{24\varepsilon}\left[e^{2\phi} - e^{4\phi} + \frac{1}{3}e^{6\phi} - \frac{1}{3}\right] \tag{3.4}$$

(see figure 1). From Eq. (3.3) it can be immediately seen that the validity of approximation (3.1) depends on the range of values of λ. Therefore we choose the arbitrary function $A(\lambda)$ such that it has a peak in $\lambda = \lambda_o$ with λ_o large enough.

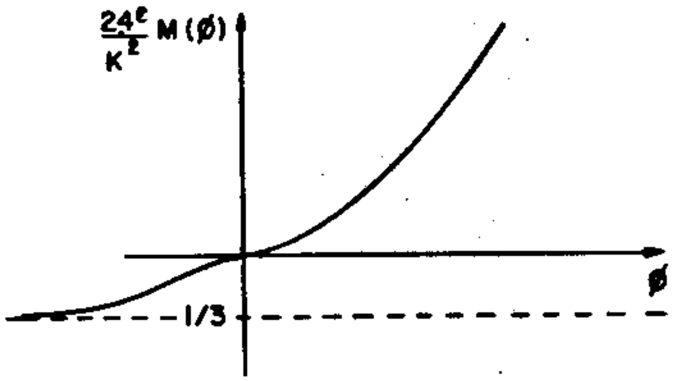

Figure 1 - A graph of the function M(ϕ) defined by relation (3.4).

The maximum of the wave function (3.3) can be obtained using the stationary phase approximation. If we write

$$A(\lambda) = |A(x)| e^{2\beta(\lambda)}$$

then the maximum of $\psi(a,\phi)$ is on the curve $\phi(a)$ defined by

$$M(\phi) = \lambda_o^2 \left[\frac{d\beta}{d\lambda} (\lambda_o) + \frac{a^6}{6} \right] \qquad (3.5)$$

Now we can choose the phase $\beta(\lambda)$ in such a way that $\phi = 0$ is a solution of Eq. (3.5) for a given value of $a = a_o$, that is

$$M(\phi) = \frac{\lambda_o^2}{6} (a^6 - a_o^6) \qquad (3.6)$$

The wave packet (3.3) is now peaked around $\phi = 0$ for $a = a_o$.

In addition, for λ_o large enough, $\pi_\phi \sim 0(\lambda_o^{-1})$ is small. This means that the Hubble constant, as can be seen from Eq. (2.5b), is also small. From these results we can show the instability of Minkowski spacetime. Indeed, for $a = a_o + \delta$ where δ is infinitesimal, the maximum of the wave packet will be shifted to

$$M(\phi_{Max}) = \delta\lambda_o^2 a_o^5 \tag{3.7}$$

In spite of the fact that δ is small, it is obvious that the arbitrariness of λ_o allow us to have ϕ_{Max} as different from $\phi = 0$ as wished. This explicitly demonstrates the unstable character of Minkowski spacetime in quantum gravity.

It is worth noting that an inspection of figure 1 shows that Eq. (3.7) has a solution only for

$$\frac{24\varepsilon}{k^2}\,\delta\lambda^2{}_o\delta^5{}_o \geq -\frac{1}{3}$$

otherwise the wave packet is smeared out.

IV FINAL REMARKS

In order to have a better understanding of the meaning of the parameter λ_o, it is useful to approximate the derivatives of the wave function (3.3) with the classical expressions for the momenta. We have that

$$-i\,\frac{\partial\psi}{\partial\phi} = \pi_\phi\psi \approx \frac{V(\phi)}{\lambda_o}\,\Psi \tag{4.1a}$$

$$- i \left.\frac{\partial\psi}{\partial a}\right|_{a=ao} = \pi_a\psi = \lambda_o a_o^5 \Psi \qquad (4.1b)$$

where $V(\phi) = \frac{k^2}{12\varepsilon} e^{2\phi}(e^{2\phi}-1)^2$ and we used the fact that $A(\lambda)$ is sharply peaked at $\lambda = \lambda_o$. From Eqs. (2.5) and remembering that λ_o is very large we find

$$H = -\frac{\varepsilon R^2 k}{6\lambda_o a_o^3} \qquad (4.2a)$$

$$\dot{R} = -\frac{\lambda_o a_o^3}{2\varepsilon k} \qquad (4.2b)$$

where ϕ is written in terms of the scalar curvature. In consequence, we see that the relevant parameter in our model is $\tilde{\lambda} = \lambda_o a_o^3/\varepsilon k$, which is proportional to the ratio R^2/H. Besides, if we couple Eqs. (4.2) we obtain

$$\dot{R} \approx \frac{R^2}{12H} \qquad (4.3)$$

which is the classical Hamiltonian constraint where the first term is dominant when $\tilde{\lambda}$ is large. A simple inspection of Eqs. (4.2) shows that indeed high values of $\tilde{\lambda}$ mean both small H and large $\dot{R}$.

Finally, we would like to point out that it should be possible to derive the $R + \varepsilon R^2$ model considered in this paper beginning with the Einstein Hilbert gravitational action plus a non homogeneous matter field. The Wheeler-DeWitt equation will contain in this case the energy density associated to the matter field, which turns out to be infinite. A covariant re normalization of this divergence should give rise to terms quadratic in the curvature tensor, as is well known in the framework of Quantum Field Theory in Curved Spacetime. As we showed here, these terms will produce the instability of Minkowski spacetime. We will address the issue of renormalization in the Wheeler-DeWitt equation elsewhere.

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