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MAGNETOHYDRODYNAMIC COSMOLOGIES

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ABSTRACT

We analyse a class of cosmological models in magnetohydrodynamic regime extending and completing the results of a previous paper. The material content of the models is a perfect fluid plus electromagnetic fields. The fluid is neutral in average but admits an electrical current which satisfies Ohm's law. All models fulfil the physical requirements of near equilibrium thermodynamics and can be favourably used as a more realistic description of the interior of a collapsing star in a magnetohydrodynamic regime with or without a magnetic field.

Key-words: Cosmology; Magnetohydrodynamic; Einstein equation.

1 INTRODUCTION

The importance of the study of magnetohydrodynamics[1] cosmologies lies in the possible presence of a primordial intergalactic magnetic field[2]. The presence of such field in the era prior to recombination (when the Universe temperature was between 10^9K and 10^3K) would induce a cosmological electric current, since the material content in this era was ionized hydrogen and radiation[3]. Analogous reasoning can be applied to the collapse of a self-gravitating bounded fluid with a magnetic field, which - for temperatures above 10^3K - would enter a regime of magnetohydrodynamics. The study of such large-scale electric current has not been undertaken yet. A first step toward this direction is the study of magnetohydrodynamics cosmological models which is the aim of this paper.

In a previous paper[4] we have analysed a class of Kantowski-Sachs models in magnetohydrodynamic regime having a Bertotti-Robinson-like solution as limiting configuration. Here we extend our analysis to include also Bianchi type I and III models. The material content is, as before, a perfect fluid plus electromagnetic fields. The fluid is electrically neutral in average but admits a spacelike electrical current which satisfies Ohm's law. Under some restrictions on the metrical coefficients the problem of solving Einstein-Maxwell equations is reduced then to the analysis of an equivalent one-dimensional Hamiltonian system, and all physically admissible solutions are obtained. The thermodynamics of the models is also examined, due to dissipative effects arising from the presence of a large-scale electric field satisfying Ohm's law.

In the paper we basically state results. The enormous algebraic manipulations are straightforward and were purposely omitted. Our calculations were checked with the algebraic computer system Reduce 3.3. In section II we treat the dynamics of the models and we present the equations necessary to analyse them. We discuss also the thermodynamics of the models and show that the increase of entropy is guaranteed by the positivity of the electrical conductivity. This exam was undertaken for configurations near the thermodynamic equilibrium and a relation among the sign of electric conductivity, the increase of the entropy and the expansion parameter of the models is established. In section III, we list all physical solutions of Einstein-Maxwell equations. They are described by two parameters and the complete characterization of the physical domain of the parameters (including previous results in the literature) is given. We conclude by discussing the possible applications of these models to describe internal configurations of self gravitating collapsing bodies in a magnetohydrodynamic regime.

2 THE DYNAMICS OF THE MODELS

The geometry of the models is described by the line element

$$ds^2 = dt^2 - A^2(t)d\chi^2 - B^2(t)k^2(\theta)(d\theta^2+\sin^2\theta d\phi^2) \qquad (2.1)$$

where the function $k(\theta)$ obeys the differential equation

$$\frac{d^2k}{d\theta^2} + \cot g\theta \frac{dk}{d\theta} - \frac{1}{k}\left(\frac{dk}{d\theta}\right)^2 - k = \varepsilon k^3$$

According to the sign of ε we have spatially homogeneous models of Bianchi type I (ε=0, $k=1/\sin\theta$), Bianchi type III (ε=1, $k=1/\cos\theta$) and Kantowski-Sachs (ε=-1, k=1).

As in our previous paper[4], the matter content of the models is a perfect fluid plus electromagnetic fields. We assume that observers comoving with the fluid have four-velocity $u = \partial/\partial t$, and we denote by ρ and p the matter-energy density and pressure, respectively, as measured locally by the comoving observers. The equation of state for the fluid is assumed[1] to be:

$$p = \lambda\rho, \qquad -\frac{1}{3} \le \lambda \le 1 \quad . \qquad (2.2)$$

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[1] Although the standard physical range of λ is $0 \le \lambda \le 1$, the extension to negative values is considered here for completeness, since some authors have discussed extreme matter configurations in which a net negative scalar pressure appears[11].

Electromagnetic fields satisfy Maxwell's equations with source. From spatial homogeneity and the existence of a preferred spatial direction determined by the Killing field $\partial/\partial\chi$, we restrict the electromagnetic tensor to

$$F_{01} = AE(t) \ ,$$

$$F_{23} = B^2 \sin(\theta) H(t) \ ; \tag{2.3}$$

all other components zero. This is actually the unique possibility, as can be shown from purely algebraic considerations in Einstein-Maxwell equations for the geometry (2.1).

The electric four-currrent is parallel to the electric field, as follows from Maxwell equations for (2.1) and (2.3), and it can therefore be given in the covariant form of Ohm's law (Bekenstein and Oron 1978):

$$j^\alpha = \sigma E^\alpha = \sigma F^{\lambda\alpha} u_\lambda \ . \tag{2.4}$$

The spacelike character of the four-current (2.4) implies that the density of electric charge of the fluid must be zero. In other words, the fluid is said to be electrically neutral on the average and said to be in a magnetohydrodynamic regime with a conductivity current. Since the four-current is parallel to the electric field, the scalar σ is identified as the electric conductivity of the fluid. As we have mentioned earlier, σ depends on the cosmological time, and this dependence must be prescribed by Maxwell equations. To specify the dynamics completely we assume here a relation between A and B and restrict ourselves to a class of models such that

$$A = B^{r} \tag{2.5}$$

with r a real parameter.

Einstein-Maxwell equations[2] for (2.1)-(2.5) yield then

$$\sigma = - \frac{d}{dt} \ln (EB^{2}) \tag{2.6}$$

$$H = H_{o}/B^{2} \tag{2.7}$$

$$E^{2} = (1-r) \frac{\ddot{B}}{B} + (1-r^{2}) \left(\frac{\dot{B}}{B}\right)^{2} - \frac{\varepsilon}{B^{2}} - \frac{H_{o}^{2}}{B^{4}} \tag{2.8}$$

[2] We use units such that $8\pi G=c=1$.

$$\rho = \left(\frac{r-1}{2}\right) \frac{\ddot{B}}{B} + \frac{r^2+4r+1}{2} \left(\frac{\dot{B}}{B}\right)^2 - \frac{\varepsilon}{2B^2} - \Lambda \tag{2.9}$$

$$\frac{(1+\lambda)r+3-\lambda}{2} \frac{\ddot{B}}{B} + \frac{(1+\lambda)r^2+4\lambda r+1+\lambda}{2} \left(\frac{\dot{B}}{B}\right)^2 - \frac{\varepsilon(1+\lambda)}{2B^2} - \Lambda(1+\lambda) = 0 \tag{2.10}$$

where H_o is a constant and Λ is the cosmological constant. Equations (2.6)-(2.8) are considered as defining the electric conductivity σ, the matter-energy density ρ and the square of the electric field (in a local Lorentz frame) E^2. In the domain of solutions of equation (2.10), we must then guarantee the positiveness of σ, ρ, and E^2 in order to have physically admissible solutions.

Introducing a new time variable η defined by $d\eta=B^{-\alpha}dt$, where[3]

$$\alpha = \frac{(1+\lambda)r^2+4\lambda r+1+\lambda}{(1+\lambda)(r-r_o)} \tag{2.11}$$

and $r_0 = (\lambda-3)/(\lambda+1)$, equation (2.10) can be reexpressed as

[3]The case $r=r_0$ can be integrated without change of variable, but it gives no physical solutions.

$$B'' - \left(\frac{\varepsilon+2\Lambda B^2}{r-r_o}\right) B^{2\alpha-1} = 0 \tag{2.12}$$

Here a prime denotes derivative with respect to η.

For the Kantowski-Sachs case (ε=-1), the point

$$B_o = \sqrt{\frac{-\varepsilon}{2\Lambda}} \quad , \qquad B' = 0 \tag{2.13}$$

is a solution of (2.6)-(2.10) with ρ=0, σ=0 and

$$E^2 + H^2 = 2\Lambda$$

This is a limiting case of the solutions found by Bertotti[5] and Robinson[6] (see also ref. [7]). In general, a first integral of (2.12) is given by

$$(B')^2 + V(B) = C \tag{2.14}$$

where C is an integration constant, and

$$V(B) = -\left[\frac{\varepsilon}{\alpha} + \frac{2\Lambda B^2}{(\alpha+1)}\right] \frac{B^{2\alpha}}{(r-r_o)} \tag{2.15}$$

In the general case equation (2.14-15) is much complicated to be integrated. However we can interpret this equation as the conserved Hamiltonian of a one-dimensional system with the effective potential given by (2.15). This procedure allows us to analyse qualitatively the dynamics of the models once the value of the "energy" C is given.

Physically acceptable solutions must be restricted to have the energy density ρ and E^2 positive during the time evolution of the models. We shall also impose the positivity of the electric conductivity σ, this condition being justified by the thermodynamical theory of fluids *near equilibrium*. In the case of perfect fluids plus electromagnetic fields with electrical current[8] the theory gives the following equation for the flux vector of entropy density S^α, for our models,

$$S^\alpha{}_{;\alpha} = \sigma E^2 \tag{2.16}$$

Therefore in order to satisfy the second law of thermodynamics ($S^\alpha{}_{;\alpha} \geq 0$) we must impose the positivity of σ. We correct here an assertion made in the conclusion of our previous paper[4]. The total entropy of all models considered there increases due to the presence of a positive electric conductivity, namely the conductivity is positive or negative if and only if the entropy increases or decreases, respectively.

To analyse the positiveness of ρ, E^2 and σ along the evolution we need to express these variables as functions of B only. Using equations (2.11), (2.12), (2.14) and (2.15) we have:

$$\rho = \frac{2\Lambda}{(1+\lambda)(r-r_o)} \left(\frac{2(1-r)}{\alpha+1} + \frac{\varepsilon r}{\alpha \Lambda B^2} + \frac{C(r+1)(2r+1)}{\Lambda B^{2\alpha+2}} \right) \tag{2.17}$$

$$E^2 = \frac{2\Lambda}{r-r_o} \left(\frac{(r-1)(r+2)}{\alpha+1} - \frac{\varepsilon r(r-r_1)}{\alpha \Lambda B^2} + \frac{C(2r+1)(1-r)(1-\lambda)}{(1+\lambda)\Lambda B^{2\alpha+2}} \right) - \frac{H_o^2}{B^4} \tag{2.18}$$

$$\sigma = \frac{2\Lambda}{E^2 B^{\alpha+1}(r-r_o)} \left(\frac{2(1-r)(r+2)}{\alpha+1} + \frac{\varepsilon r(r-r_1)}{\alpha \Lambda B^2} + \frac{C(\alpha-1)(2r+1)(1-r)(1-\lambda)}{(1+\lambda)\Lambda B^{2\alpha+2}} \right) \frac{dB}{d\eta} \tag{2.19}$$

The *physical* domain of B is bounded by the following curves C(B) for which ρ, E^2 and σ are null, namely

$$C_{\rho=0}(B) = \left(\frac{-\varepsilon r}{\alpha(2r+1)(r+1)} + \frac{2(r-1)\Lambda B^2}{(\alpha+1)(2r+1)(r+1)} \right) B^{2\alpha} \tag{2.20}$$

$$C_{E^2=0}(B) = \frac{1+\lambda}{1-\lambda}\left(\frac{(r-r_0)H_o{}^2B^{-2}}{2(1-r)(2r+1)} + \frac{\varepsilon r(r-r_1)}{\alpha(1-r)(2r+1)} + \frac{(r+2)\Lambda B^2}{(\alpha+1)(2r+1)}\right) B^{2\alpha} \tag{2.21}$$

$$C_{\sigma=0}(B) = \frac{1+\lambda}{1-\lambda}\left(\frac{\varepsilon r(r-r_1)}{\alpha(1-\alpha)(2r+1)(1-r)} + \frac{2(r+2)\Lambda B^2}{(1-\alpha^2)(2r+1)}\right) B^{2\alpha} \tag{2.22}$$

where $r_1 = -2\lambda/(\lambda+1)$.

By plotting the various types of curves (2.15), (2.20-22) which depend on the parameters r, λ, ε, H_o and Λ, the sign of ρ, E^2 and σ can be analised.

We list now the physical solutions[9]:

i) Bianchi I

For the Bianchi I case, we must impose that $\Lambda=H_o=0$ and $C>0$. The physical quantities are given as function of time as

$$B = \left[\sqrt{C}\,(1+\alpha)\right]^{\frac{1}{1+\alpha}} |t|^{\frac{1}{1+\alpha}} \tag{2.23}$$

$$\rho = \frac{2(r+1)(2r+1)}{(1+\lambda)(r-r_0)(1+\alpha)^2}\ t^{-2} \tag{2.24}$$

$$E^2 = \frac{2(2r+1)(1-r)(1-\lambda)}{(1+\lambda)(r-r_o)(1+\alpha)^2}\ t^{-2} \tag{2.25}$$

$$\sigma = \left(\frac{\alpha-1}{\alpha+1}\right)\ t^{-1} \tag{2.26}$$

The parameter r must be restricted to the interval (-1/2,1) in order to guarantee the positivity of ρ and E^2. Using eq. (2.11) we can show that $0<\alpha<2$ if $-1/3\leq \lambda<1$ and $-1/2<r<1$ (see Fig. 1).

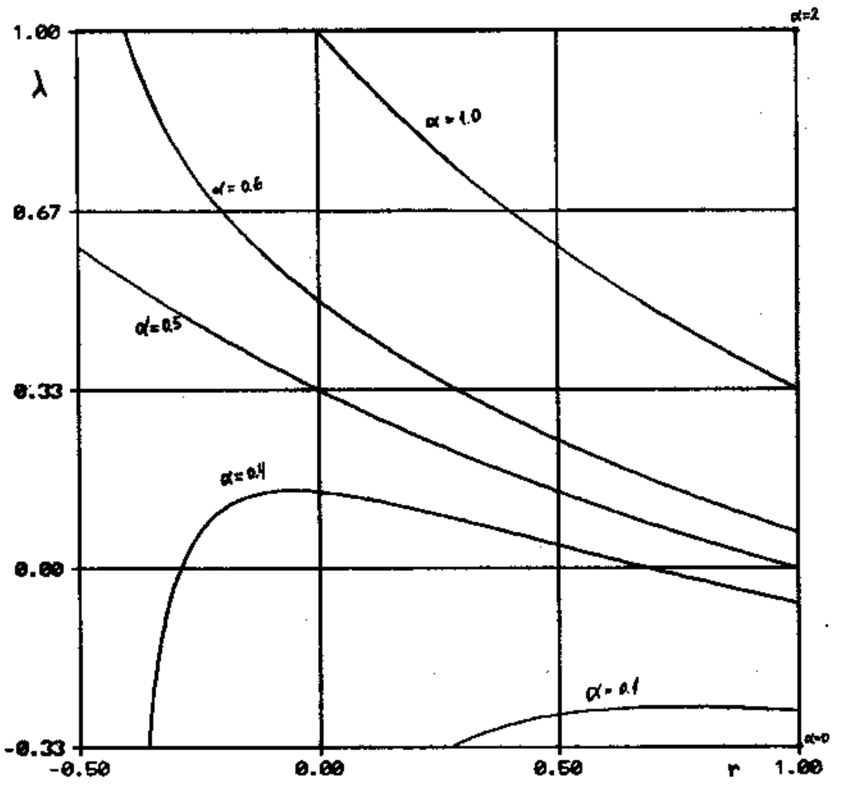

Fig. 1 Curves α=constant. The curve α=1/2 is a divide between the curves that cross the line λ=1 and those that cross the line λ=-1/3.

Concerning the sign of σ, two kinds of solutions can be considered: The expanding ones where t is in the interval $(0,\infty)$ and the contracting ones where t is in the interval $(-\infty,0)$. In the first case we must have $1<\alpha<2$. These solutions were derived by Dunn and Tupper[10]. The parameters a and b used by the authors are related to ours through the equations:

$$A = t^{a}$$
$$B = t^{b}$$

The region of solutions is described by the dotted triangle in the plane (axb) of figure 2.

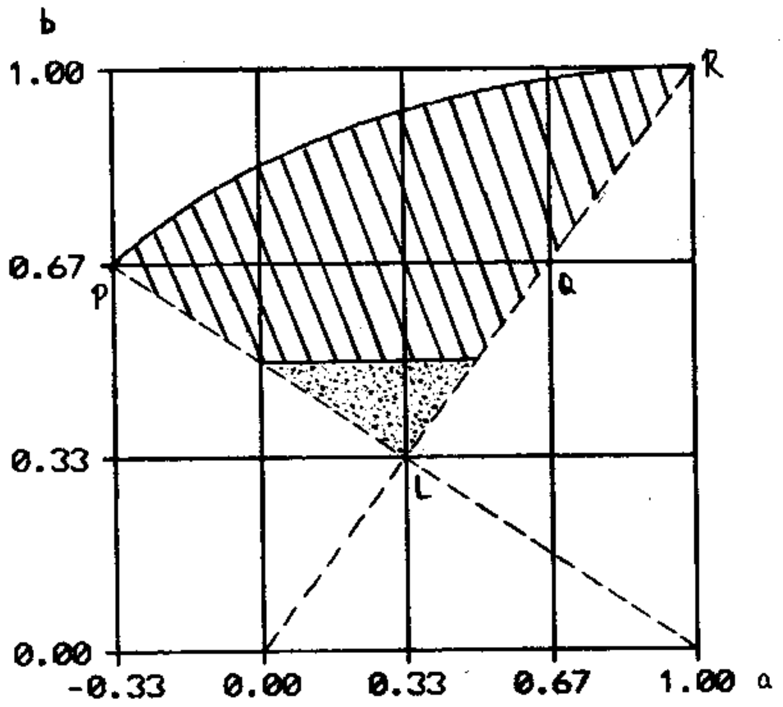

Fig. 2 Region of solutions of Bianchi I models. The dotted region represents the expanding physical solutions[10] and the hatched region represents new contracting physical solutions. The dashed lines are excluded. The line b=1/2 corresponds to solutions where σ = 0. The point P is a Kasner solution.

In our parametrization, the dotted region corresponds exactly to the specification $\alpha>1$, $r<1$ and $\lambda<1$. To see this, note from (2.23) and (2.5) that

$$b = \frac{1}{1+\alpha} \tag{2.27}$$

and

$$a = br \tag{2.28}$$

For $1<\alpha<2$ we have from (2.28) that $1/3<b<1/2$. Now let us analyse the domain of a. By fig. 1 we see that the values of r varies from 1 until the intersection value of the curve α=constant with the line $\lambda=1$ $(r=\alpha-1)$. Hence for a given value of b we have $b(\alpha-1)<\alpha<b$, and using (2.27) we obtain

$$1-2b<a<b$$

The lower value of a is bounded by the line $b=(1-a)/2$ (line PL of fig. 2) and its upper value is bounded by the line $b=a$ (line LR of fig. 2). This reproduces the region described by Dunn & Tupper[10].

To include the second case, we must have $0<\alpha<2$. It is easy to see that the triangle PLQ of fig. 2 corresponds to the interval $1/2<\alpha<2$. When $\alpha<1/2$ the curves α=constant cross the line $\lambda=-1/3$, then the lower value of a changes and are given by the curve PR whose equation is

$$b = \frac{a+5 - \sqrt{25+34a-23a^2}}{12} \qquad (2.29)$$

The hatched region of fig. 2, including the curve PR without the end points, represents therefore new contracting physical solutions of Bianchi type I with non-null electrical conductivity.

ii) Bianchi III

For Bianchi III we must impose that $\Lambda=H_o=0$ and $C\geq0$. The parameters λ and r must be in the following intervals: $-1/3\leq\lambda\leq0$ and $0\leq r\leq r_1$ in order to have $\rho>0$, $\sigma>0$ and $E^2>0$. For C=0 the physical quantities are given by:

$$B = - \frac{t}{\sqrt{\alpha(r-r_o)}} \qquad (2.30)$$

$$\rho = \frac{2r}{(1+\lambda)t^2} \qquad (2.31)$$

$$E = - \frac{\sqrt{2r(r_1-r)}}{t} \qquad (2.32)$$

$$\sigma = -\frac{1}{t} \tag{2.33}$$

where $t\epsilon(-\infty,0)$. Note from (2.30) that r cannot be null and this implies that $\lambda\neq0$. All solutions with C=0 have then negative presssure.

For C>0 the unique solution with non-negative pressure is $r=\lambda=0$ and $\alpha=1/3$. In this case the physical quantities are given by

$$B = \sqrt{t^2 - 2\sqrt{C}\ t} \tag{2.34}$$

$$\rho = E^2 = \frac{2C}{3(t^2-2\sqrt{C}\,t)^{4/3}} \tag{2.35}$$

$$\sigma = \frac{2}{3}\,\frac{\sqrt{C} - t}{t^2 - 2\sqrt{C}\ t} \tag{2.36}$$

where $t\epsilon(-\infty,0)$.

iii) Kantowski-Sachs

For Kantowski-Sachs, the positivity of ρ, E^2 and σ implies that all solutions must have the Bertotti-Robinson-like models as a limiting configuration. In this case, the constant C must have

the value $V(B_o)$ and r must be in the interval $-1/2 \le r \le 1$. These solutions were discussed in details in ref. [4].

We remark that the above models (i)-(iii) are not symmetric with respect to $t \to -t$ because this transformation changes the sign of σ. For the cases (ii)-(iii) this is a direct consequence of eq. (2.19) for the physical solutions, which implies that the sign of σ is opposite to the sign of the expansion θ of the model. We note that the expansion parameter is given by $\theta = (2+r)/B^{\alpha+1}dB/d\eta$, for the models considered here. The physical region of time for these models is $(-\infty,0)$ where $\sigma>0$. For the case (i) , the relation between the sign of σ and θ depends on the parameter α. If α is in the interval $1<\alpha<2$ (Dunn & Tupper solutions) the expansion has the same sign of the conductivity σ; for $0<\alpha<1$, the expansion and the conductivity have opposite sign.

FINAL CONCLUSIONS

We have made an exhaustive study of the dynamics of a class of cosmological models in magnetohydrodynamic regime. We found new exact solutions of Einstein-Maxwell equations in the case of a perfect fluid plus electromagnetic fields, extending and completing previous results in the literature. The fluid admits the presence of electric charges but is neutral in average. We

imposed the equation of state $p=\lambda\rho$, $-1/3\leq\lambda\leq1$, and also the validity of Ohm's law $j^{\alpha}=\sigma E^{\alpha}$ relating the electrical current and the electrical field. The conductivity σ is in our case a decreasing (increasing) function of time, what is physically expected since the expansion (or contraction) of the models increases (decreases) the mean free path of the electric charges through the fluid, decreasing (increasing) the conductivity. As also expected they are not symmetric with respect to $t \to -t$ because this transformation changes the sign of σ, which is opposite to the sign of the expansion parameter θ of the models. The positivy of σ is a strong restriction on the set of physically admissible solutions. This set could be enlarged (to include also expanding configurations) if matter and electromagnetic fields distributions with $\sigma<0$ are also allowed. However we know that matter configurations with $\sigma<0$ are far from thermodynamical equilibrium[12] and highly unstable, and such configurations are only preserved during a phase of adiabatic evolution of the model ($\theta\ll|\sigma|$ in suitable units).

The models fulfil the physical requirements of near equilibrium thermodynamics, and they can be used as a more accurate description of a collapsing distribution of matter for temperatures above 10^3 °K. In fact during collapse as the temperature raises above 10^3 °K-matter would enter a regime of magnetohydrodynamics with or without magnetic field. We recall

that only class (iii) admits a non-zero magnetic field. The matter distribution inside any finite proper radius contracts toward the singularity where all physical quantities go to infinity. The singularity is point-like if the parameter $r>0$ or cigar-like if $r\leq 0$. The electric conductivity is positive warranting the validity of the second law of thermodynamics.

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