CNPq

CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Notas de Física

CBPF-NF-006/92

SELF-DUALITY CONDITION AND CRITICAL POTENTIALS

by

C. CRONSTRÖM, Prem P. SRIVASTAVA and K. TANAKA

Abstract

It is shown that the self-duality constraint on the scalar field (combined with the equations of motion) by itself leads to the critical forms for the potential that minimizes the energy functional in the Chern-Simons Higgs system.

In the case, we have only the Chern-Simons term in the SL(2,R) gauge group one obtains a formalism that yields the equations of motion of a variety of non-linear models in two dimensions when the curvature is set equal to zero.

Key-words: Chern-Simons Higgs; Gauge theory; Self-dual solutions.

The abelian Higgs system with both the Maxwell and Chern-Simons (CS) terms in 2+1 dimensions has drawn much interest recently [1,2,3]. In the earlier papers the critical potential leading to self-dual solutions, which minimizes the energy functional, in the Chern-Simons-Higgs (CSH) system without the Maxwell term, was obtained [4,5]. It was also pointed out [6] that the self-duality constraint on the scalar field combined with the equations of motion by itself leads to the critical forms for the potentials in the case when only the CS or only the Maxwell term is present. The notion of self-duality was also extended to scalar superfield and the critical superpotential obtained [6].

The Lagrangian for the *bosonic Chern-Simons Higgs* system is ($\hbar$=c=1)

$$\mathcal{L} = (\tilde{\mathcal{D}}^\mu a^*)(\mathcal{D}_\mu a) - V(|a|^2) + \frac{\kappa}{4}\epsilon^{\mu\nu\rho}\, v_\mu f_{\nu\rho} - \frac{1}{4} f_{\mu\nu} f^{\mu\nu}, \tag{1}$$

where $\mathcal{D}_\mu = \partial_\mu - iev_\mu$, $\tilde{\mathcal{D}}_\mu = \partial_\mu + iev_\mu$, $\mu = 0,1,2$ are the spacetime indices and a is the scalar field.

It was also noted in [4,5] that in the Chern-Simons-Higgs system, the energy functional obeys a Bogomol'nyi-type [7] lower bound when a special choice of the Higgs potential is imposed. The bound is achieved if the scalar field a satisfies the following first order self-duality condition (i=1,2 and $\epsilon^{12} = 1$)

$$\mathcal{D}_1 a = -i\mathcal{D}_2 a, \quad or \qquad \mathcal{D}_i a = -i\epsilon^{ij}\mathcal{D}_j a. \tag{2}$$

which may be regarded as the two dimensional analogue of the self-dual gauge field strength in four dimensional space-time.

The general result we find for the potential with the aid of eq.(2) and the static condition is [8]

$$V'(|a|^2) = e^2 {v_0}^2 + \frac{e}{\kappa}(2e^2|a|^2 - \partial_i^2)v_0, \tag{3}$$

where v_0 is given by

$$[\frac{1}{\kappa^2}(2e^2|a|^2 - \partial_i^2) + 1]v_0 = \frac{e}{\kappa}(|a|^2 - C^2). \tag{4}$$

In the limit $\kappa \to 0$ (no Chern-Simons term), we find

$$V = (e^2/2)(|a|^2 - C^2)^2, \tag{5}$$

and in the limit $e \to \infty$, $\kappa \to \infty$ such that $(e^2/\kappa) \to$ finite the terms originating from the Maxwell term in the eqs. of motion drop out. We find

$$V(|a|^2) = (e^2/\kappa)^2(|a|^2 - C^2)^2|a|^2 \tag{6}$$

Both (5) and (6) agree with the previously known results.

Our procedure can be extended also to the *scalar Superfield* [6]

$$\Phi(x,\theta) = a(x) + i\bar{\theta}\psi(x) + i\bar{\theta}\theta f(x). \tag{7}$$

Here $a(x)$ is a complex scalar, $\psi^{\alpha}(x)$ its complex superpartner and $f(x)$ an auxiliary complex scalar. The gauge covariant spinorial derivatives may be defined to be

$$\nabla^{\alpha}\Phi = (D^{\alpha} + e\Gamma^{\alpha})\Phi, \qquad \bar{\nabla}^{\alpha}\Phi^{*} = (D^{\alpha} - e\Gamma^{\alpha})\Phi^{*}. \quad , \tag{8}$$

where α is spinorial index, D^{α} is the covariant spinorial derivative and Γ^{α} is a Majorana spinor connection spinor superfield.

The self-duality constraint on the matter superfield now takes the form [6]

$$\nabla^{\alpha}\Phi = i(\gamma^{0}\nabla)^{\alpha}\Phi, \qquad \bar{\nabla}^{\alpha}\Phi^{*} = -i(\gamma^{0}\bar{\nabla})^{\alpha}\Phi^{*}. \tag{9}$$

The specific (critical) superpotential $V(|\Phi|^2)$ can be obtained and shown to contain the results of the purely bosonic theory without invoking any explicit N=2 supersymmetry of the action [9].

Consider next the following *gauged non-abelian* Lagrangian with the Chern-Simons term [10,11] ,($\hbar$=c=e=1),

$$\mathcal{L} = i\psi^{\dagger}\mathcal{D}_t\psi + \frac{1}{2m}(\mathcal{D}^i\psi)^{\dagger}(\mathcal{D}_i\psi) + V + \frac{\kappa}{2}\,\epsilon^{\mu\nu\rho}\,(A_{\mu}{}^{a}\partial_{\nu}A_{\rho a} + \frac{1}{3}f^{a}{}_{bc}A_{\mu a}A_{\nu}{}^{b}A_{\rho}{}^{c}) - \frac{1}{4}F_{\mu\nu}{}^{a}F^{\mu\nu}{}_{a}, \tag{10}$$

where ψ is a multiplet of matter fields, $\mu = 0,1,2$ or (t,x,y), $A_{\mu} = A_{\mu}{}^{a}X_a$, X_a being the Lie algebra generators, $\mathcal{D}_{\mu} = (\partial_{\mu} + A_{\mu})$ and V is the potential to be determined when we impose the static and self-duality conditions. The eqs. of motion resulting from (10) are

$$i\mathcal{D}_t\psi = -\frac{1}{2m}\mathcal{D}_i\mathcal{D}_i\psi - \frac{\delta V(\psi,\psi^{*})}{\delta\psi*}, \tag{11}$$

$$\mathcal{D}_{\mu}F^{\mu\nu} + \frac{\kappa}{2}\,\epsilon^{\nu\rho\mu}\,F_{\rho\mu} = J^{\nu}, \tag{12}$$

where $J^{\mu a} = -\partial\mathcal{L}_{\mathrm{matter}}/\partial A_{\mu a}$.

On adding to them the self-dual equations $\mathcal{D}_i\psi = -i\epsilon^{ij}\mathcal{D}_j\psi$ and assuming the static configuration, eqs.(11) and (12) simplify very much. For example, in the absence of the kinetic term for the gauge field, we find, following the procedure described for the abelian case, that A_0 may not vanish, and the critical potential is determined to be [12] ($J^{0a} \equiv \rho^{a}$)

$$V = (\frac{-1}{2m\kappa})(\rho^* \rho_a) + const, \tag{13}$$

which was assumed at the beginning in Ref.[10]. In the presence of the kinetic term, it is possible to choose A_0 to vanish and we find $V = (-1/4m\kappa)(\rho^* \rho_a) + const.$ while $\kappa = 2m$ is required for consistency.

Finally, we make the following remarks for the case when $\kappa \to \infty$ in eq.(12). We find

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \tag{14}$$

On the other hand we have, in two dimensions, the curvature two-form [13,14]

$$\Omega = d\Gamma + \Gamma \wedge \Gamma, \qquad \Gamma = \theta_a X_a, \qquad a = 1, 2, 3, \tag{15}$$

where θ_a are 1-forms. Explicitly

$$\Omega \equiv \frac{1}{2}\{\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu]\} dx^\mu \wedge dx^\nu. \tag{16}$$

Let us consider the X_a to be the generators of SL(2,R), i.e. $X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Equation (14) is then seen as a zero curvature condition $\Omega = 0$, if we impose, for example, y independence, viz, $\partial_y(\,) = 0$ and make the identification $\mu = x, \nu = t, \Gamma_\mu = A_\mu = A_x, \Gamma_\nu = A_\nu = A_t$, and $\Gamma_y = A_y = 0$. Making various choices of A_t and A_x we obtain the non-linear equations in two dimensions like sine- Gordan, modified Korteweg-de Vries (MKdV) , non-linear Schrodinger model, KdV, and Liouville equations.

For the case of $\partial_t(\,) = 0$ and the identifications $\mu = x, \nu = y, \Gamma_t = A_t = 0$,

$$\Gamma_\mu = A_x = \begin{pmatrix} -\eta & \frac{1}{2}u_x \\ -\frac{1}{2}u_x & \eta \end{pmatrix}, \qquad \Gamma_\nu = A_y = \frac{1}{4\eta}\begin{pmatrix} -\cos u & -\sin u \\ -\sin u & \cos u \end{pmatrix}, \tag{17}$$

where η is a constant we obtain from eq.(16)

$$u_{xy} - \sin u = 0. \tag{18}$$

which is the time independent sine-Gordon equation in two dimensions. Its solution is $u = 4\,tan^{-1}\,exp(Cx + y/C)$ where C is a constant.

When X_a are the generators of SU(N) algebra [10] the various reductions lead to two dimensional non-linear equations, Toda , affine Toda, sinh-Gordon, Bullough-Dodd, Principal chiral field, non- linear σ-model, and CP^n model.

Acknowledegements

One of the authors (P.P.S.) acknowledges the hospitality of the Department of Physics at Ohio State University and a fellowship from C.N.Pq of Brasil. The other (K.T.) thanks C. Cronstrom for the hospitality at University of Helsinki. This work was supported in part by the U.S. Department of Energy under Contract No. EY-76-C-02-1415*00.

References

[1] G.V. Dunne and C.A. Trugenberger, Phys. Rev. D43 (1991) 1323.
[2] C.Lee, K. Lee and H. Min, Phys. Letts. 252B (1990) 79.
[3] D. Bazeia, Phys. Rev. D43 (1991) 4074.
[4] J. Hong, Y. Kim and P.Y. Pac, Phys. Rev. Lett. 64 (1990) 2230.
[5] R. Jackiw and E. Weinberg, Phys. Rev. Lett. 64 (1990) 2234.
[6] P.P. Srivastava and K. Tanaka, Phys. Letts. 256B (1991) 427.
[7] E.B. Bogomol'nyi, Sov. J. Nucl. Phys. 24 (1976) 449.
[8] P.P. Srivastava and K. Tanaka, Ohio State University preprint No. DOE/ER/015145-556.
[9] C. Lee, K. Lee and E.J. Weinberg, Phys. Letts. 243B (1990) 105.
[10] G.V. Dunne, R. Jackiw, S.Y. Pi and C.A. Trugenberger, Phys. Rev. D43 (1991) 1332.
[11] K. Lee, Phys. Letts. 255B (1991) 381.
[12] P.P. Srivastava and K. Tanaka, *in preparation*.
[13] M. Crampin, F.A.E. Pirani and D.C. Robinson, Lett. Math. Phys. 2 (1977) 15.
[14] K. Tanaka, J. Math. Phys. 30 (1989) 172; Proceedings, "New Theories in Physics", Kazimierz, Poland, 1988, editors, Z. Ajduk, S. Pokorski, and A. Trautman, World Scientific, p.246.