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QUANTIZATION OF SPIN-TWO FIELD IN TERMS OF FIERZ VARIABLES

THE LINEAR CASE

by

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ABSTRACT

We give a complete self-contained presentation of the description of spin-two fields using Fierz variables $A_{\alpha\beta\mu}$ instead of the conventional standard approach which deals with second order symmetric tensor $\phi_{\mu\nu}$. After a short review of the classical properties of the Fierz field we present the quantization procedure. The theory presents a striking similitude with electrodynamics which induced us to follow analogy with the Fermi-Gupta-Breuler scheme of quantization.

Key-words: Fierz variables; Quantization; Spin two field.

1 INTRODUCTION

The exact Einstein's General Theory of Relativity can be formulated in terms of a field theory in a fixed flat space-time background. That this is not a mere approximation procedure we have learned from many authors[1]. Interest on this alternative description has grown considerably in the last decade according to both the increasing attention given to unification program inspired by the success of the electro-weak unified scheme, and the recent convergence between elementary particle physics and cosmology. On the other hand, the occurrence of many unpleasant features in the standard canonical formulation of General Relativity, as it has been pointed out many times[2], has led to the search of alternative choices of the basic variables to fit into the construction of a Hamiltonian description of gravity[3]. These considerations gave us motivation to undertake the re-exam of the different approaches one may choose to describe a massless spin-two field in a Minkowski space-time. Fierz[4] claimed that there exists two distinct, although completely equivalent, ways to accomplish such task. We can either make use of a symmetric second order tensor $\varphi_{\mu\nu}$ (the so-called standard representation)[5]; or, on the other hand, we can use a third order tensor $A_{\mu\nu\lambda}$ (henceforth called the Fierz representation), which is anti-symmetric in the first pair of indices and additionally satisfies the requirement of being pseudo-trace free, that is, $A_{\mu\nu\lambda} + A_{\nu\lambda\mu} + A_{\lambda\mu\nu} = 0$. The standard description has 10 independent components, and the Fierz description has 20; thus, neither is free from unphysical variables, once we know that a massless, spin-two, field has only two degrees of freedom per each space-time point. Although one should certainly prefer to work within a conceptual scheme that dispenses with any reference to unphysical degrees of freedom, the general belief that the laws of physics can be deduced from basic symmetry principles has led physicists to deal with enlarged, generic sets of variables comprising physical and "non-physical" components. *The success of the gauge description of electro-weak forces has enforced this attitude. In the same vein, we can follow Einstein (as quoted by A. Salam), who believed that "Nature is not economical of structures: only of principles of fundamental applicability". One of these, the general covariance principle, is precisely the main responsible for the appearance of such extra non-physical variables.*

A typical example comes from the theory of Electrodynamics. Although the electromagnetic field has only two degrees of freedom, its standard description is made through the use of a four-vector A^{μ}[6]. The theory must then provide a mechanism to eliminate superfluous quantities. As it happens in many theories, in Electrodynamics there is no unique manner to perform such elimination. Consider, for instance, Fermi's program (which will provide a paradigm for our analysis in a following section, in the case of gravity). Instead of taking the conventional Maxwell's Lagrangian $L_M = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}$, Fermi deals with an extra term that destroys explicitly the previous gauge invariance of the theory to wit: $L_F = L_M - \frac{1}{2}\left(A^{\mu}{}_{,\mu}\right)^2$. The new Lagrangian L_F has the merit of yielding a wave equation $\Box A^{\mu} = 0$ and non-vanishing momenta canonically conjugated to all A'_{μ}s. However, the Hamiltonian one gets from such L_F suffers from not being positive definite. In the quantum version of this theory a direct way to solve this problem is to restrict the accessible states of the field to a physical sub-space $\mathcal{H}_{phys}$ of the complete Hilbert space $\mathcal{H}$ by imposing a subsidiary $A^{\mu(+)}{}_{,\mu}|\psi> = 0$[7]. The theory contains a typical spin-one field and a spin-zero (scalar) part. Like a miracle, the above auxiliary condition makes the energy associated to the longitudinal photon to cancel precisely the (negative) energy of its scalar part, leaving only two independent components of positive energy to survive.

Turning to the case of gravitation we shall see that a very similar behavior is found. In order to simplify our exposition we will limit our presentation here only to the linear case. We postpone the exam of non-linearities for a forthcoming paper. We can just anticipate that the procedure to deal with the non-linear case is very similar as the one done by Feynman, Gupta, Deser and others in case one uses the standard variable. However, the uses of the three-indices tensor has some new peculiarities that it does indeed merit a more extensive analysis.

Once the classical theory of spin-two field in Fierz representation has been examined in a precedent paper by two of us (Novello, M. -Pinto Neto, N.)[8], we will restrict ourselves here only to its quantum version. The paper is self-contained and it is organized as follows. In section 2 we make a review of some properties of the classical theory. From the Fierz variable $A_{\alpha\beta\mu}$ we construct the tensor $C_{\alpha\beta\mu\nu}$ which plays the role for the spin-two field as Maxwell's tensor $F_{\mu\nu}$ does for electrodynamics. The symmetries of $C_{\alpha\beta\mu\nu}$ are precisely the same as Weyl's conformal tensor of any Riemannian geometry. One should be tempted to associate $C_{\alpha\beta\mu\nu}$ to a sort of weak field limit of Einstein's theory. Although there is some support in this interpretation this similitude rests no more than an analogy. Indeed, as we will show in section 2, the tensor $C_{\alpha\beta\mu\nu}$ contains two independent spin-two fields of opposite parity. This means that we are dealing with a theory that contains more than the linearized version of Einstein's theory of gravity. We can achieve the elimination of one of these spin-2 fields by invoking Fermi's procedure to restrict the physical Hilbert space of states. We will deal with a Maxwell-like Lagrangian for the Fierz field $L = \frac{1}{8} C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$. Curiously, this theory is such that the spin-2 fields contribute to the Hamiltonian with opposite energies. We choose to associate gravity to the positive parity and positive energy part of the field. One should, however, note that only after the introduction of an interaction this choice could be properly justified. We also describe the classical part of the theory for a Lagrangian of the Fermi-type in terms of the Fierz variables.

We then use in section 3 and 4, in which we quantize the two classical theories developed in section 2, the Fermi method to eliminate additional degrees of freedom of the theory and to describe a convenient canonical quantization. When dealing with the Maxwell-like Lagrangian $L \propto C^2$ the basic two-spin-2 fields are represented by 3-tensors α_{ij} and Δ_{ij}. The momenta canonically conjugate to these variables are, respectively, given by the electric E^{ij} and the magnetic B^{ij} parts of the field $C_{\alpha\beta\mu\nu}$. This allows us to achieve a completely new set of canonical variables to represent spin-2 fields which deserve certainly to be examined further. It seems worth to point out that recent proposals of considering alternatives variables to describe canonical quantities in Einstein's theory, e.g. Ashtekar and others, deal with a combination of the electric and the magnetic potentials of Weyl conformal tensor.

The complete relation of our present investigation and those others can, however, be understood only after the completion of non-linear modes of vibrations of Fierz field is taken into account. This will be the subject of a future report.

2 REVIEW OF SOME CLASSICAL ASPECTS OF THE DYNAMICS OF FIERZ'S VARIABLES

Let $A_{\alpha\beta\mu}$ be a real tensor endowed with the properties (Fierz)

$$A_{\alpha\beta\mu} + A_{\beta\alpha\mu} = 0 \tag{1a}$$

$$A_{\alpha\beta\mu} + A_{\beta\mu\alpha} + A_{\mu\alpha\beta} = 0 \tag{1b}$$

or, equivalently,

$$\overset{*}{A}{}^{\alpha\beta\mu}\, \gamma_{\beta\mu} \equiv \frac{1}{2}\, \eta^{\alpha\beta\rho\sigma}\, A_{\rho\sigma}{}^{\mu}\, \gamma_{\beta\mu} = 0$$

$\gamma_{\beta\mu}$ is the metric tensor of the Minkowski space-time: and $\eta^{\alpha\beta\mu\nu}$ is the Levi-Civita completely anti-symmetric tensor. We assume henceforth that the trace $A^{\alpha\beta}{}_{\beta}$ vanishes. Thus, $A_{\alpha\beta\mu}$ has only 16 independent components. From $A_{\alpha\beta\mu}$ we construct the corresponding field $C_{\alpha\beta\mu\nu}$:

$$C_{\alpha\beta\mu\nu} = A_{\alpha\beta[\mu,\nu]} + A_{\mu\nu[\alpha,\beta]} - \text{traces}$$

$$= A_{\alpha\beta[\mu,\nu]} + A_{\mu\nu[\alpha,\beta]} + \frac{1}{2} A_{(\alpha\nu)}\gamma_{\beta\mu} + \frac{1}{2} A_{(\beta\mu)}\gamma_{\alpha\nu} +$$

$$- \frac{1}{2} A_{(\alpha\mu)}\gamma_{\beta\nu} - \frac{1}{2} A_{(\beta\nu)}\gamma_{\alpha\mu} \qquad (2)$$

in which $A_{\beta\mu} \equiv A_{\beta}{}^{\lambda}{}_{\mu,\lambda}$. Note that the derivative (represented by a comma) is taken to be covariantly defined in terms of the metric $\gamma_{\mu\nu}$; it is identified with the common derivative $\frac{\partial}{\partial x^{\mu}}$ if one chooses a cartesian coordinate system in which case $\gamma_{\mu\nu}$ reduces to diag.(+---). The tensor field $C_{\alpha\beta\mu\nu}$ has only 10 independent components, once from its definition it has the symmetries:

$$C_{\alpha\beta\mu\nu} = - C_{\alpha\beta\nu\mu} = - C_{\beta\alpha\mu\nu} = C_{\mu\nu\alpha\beta}$$

Besides, it is trace-free: $C_{\alpha\beta\mu\nu}\gamma^{\alpha\mu} = 0$ and pseudo-trace free: $\overset{*}{C}_{\alpha\beta\mu\nu}\gamma^{\alpha\mu} = 0$. This tensor $C_{\alpha\beta\mu\nu}$ plays the analogous role, for the case of the spin two field, as the tensor $F_{\mu\nu}$ does for electrodynamics. Thus, in conformity with Maxwell's theory one is led to propose for the Lagrangian of this field the expression:

$$\mathcal{L}_{(g)} = \frac{1}{8} C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} \qquad (3)$$

which implies the equation of motion

$$C^{\alpha\beta\mu\nu}{}_{,\nu} = 0 \tag{4}$$

In terms of the Fierz variables we obtain

$$\Box A_{\alpha\beta\mu} - A_{\alpha\beta}{}^{\lambda}{}_{,\lambda,\mu} + \frac{1}{2} A_{\mu\alpha}{}^{\lambda}{}_{,\lambda,\beta} - \frac{1}{2} A_{\mu\beta}{}^{\lambda}{}_{,\lambda,\alpha} + \frac{1}{2} \gamma_{\mu[\beta} A_{\alpha]}{}^{\lambda\nu}{}_{,\lambda,\nu} = 0 \tag{5}$$

The Lagrangian (3) has an internal (gauge) symmetry. Indeed, (3) is invariant under the map:

$$A_{\alpha\beta\mu} \rightarrow \tilde{A}_{\alpha\beta\mu} = A_{\alpha\beta\mu} + W_{\alpha\beta,\mu} - \frac{1}{2} W_{\mu[\alpha,\beta]} + \frac{1}{2} \gamma_{\mu[\alpha} W_{\beta]}{}^{\lambda}{}_{,\lambda} \tag{6}$$

in which $W_{\alpha\beta}$ is an arbitrary anti-symmetric tensor.

The freedom guaranteed by this invariance allows us to choose the potential $\tilde{A}_{\alpha\beta\mu}$ to satisfy a generalized Lorentz condition:

$$\tilde{A}_{\alpha\beta}{}^{\mu}{}_{,\mu} = 0$$

by just choosing $W_{\alpha\beta}$ such that:

$$\Box W_{\alpha\beta} = - A_{\alpha\beta}{}^{\mu}{}_{,\mu}$$

In this case, eq. (5) becomes simply the wave equation:

$$\Box \tilde{A}_{\alpha\beta\mu} = 0 \qquad (7)$$

with

$$\tilde{A}^{\alpha\beta\mu}{}_{,\mu} = 0 \qquad (8)$$

The situation is in complete analogy with Electrodynamics, in which one can deal with a vector potential $\tilde{A}_{\mu}$ that obeys Lorentz gauge $\tilde{A}^{\mu}{}_{,\mu} = 0$, by choosing a scalar field A(x) such that $\Box A = - A^{\mu}{}_{,\mu}$ under the transformation $\tilde{A}_{\mu} = A_{\mu} + A_{,\mu}$. In this case Maxwell's equations reduce to $\Box A_{\mu} = 0$ plus the gauge constraint $A^{\mu}{}_{,\mu} = 0$.

Due to the gauge symmetry, the theory of $A_{\alpha\beta\mu}$ must deal with 12 first class constraints in the Hamiltonian formulation (note that to fix $A_{\alpha\beta\mu}$ one must specify the six quantities $W_{\alpha\beta}$ and their corresponding time-derivatives).

The momenta are given by:

$$\pi^{\alpha\beta\mu} = \frac{\delta L}{\delta \dot{A}_{\alpha\beta\mu}} = C^{\alpha\beta\mu o} \qquad (9)$$

Consequently one gets the six primary constraints:

$$\pi^{\alpha\beta o} = 0 \tag{10}$$

We then split $A_{\alpha\beta\mu}$ into the standard 3 + 1 decomposition of space-time obtaining for their corresponding canonical momenta:

For $\zeta_i \equiv A_{oio}$, it follows:

$$M^i \equiv \frac{\delta L}{\delta \dot{\zeta}_i} \approx 0 \tag{11a}$$

For $\beta_{ij} \equiv A_{ijo}$ it follows:

$$M^{ij} = \frac{\delta L}{\delta \dot{\beta}_{ij}} \approx 0 \tag{11b}$$

For $\alpha_{ij} \equiv A_{(i\ j)}^{\ \ o}$ it follows:

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{\alpha}_{ij}} = C^{iojo} \equiv E^{ij} \tag{12a}$$

For $\Delta_{ij} \equiv \frac{1}{2}\,\Delta_{ab(i}\varepsilon_{j)}{}^{ab}$ it follows:

$$p^{ij} = \frac{\delta L}{\delta \dot{\Delta}_{ij}} = -\frac{1}{4} C^{ab(i}{}_{o}\varepsilon_{ab}{}^{j)} \equiv B^{ij} \tag{12b}$$

in which:

$$\Delta_{(ij)} \equiv A_{ijk} + \frac{1}{2}\gamma_{k[t} A^{o}{}_{j]o} \ ,$$

$$C_{(ij)} \equiv C_{ij} + C_{ji} \ ,$$

$$C_{[ij]} \equiv C_{ij} - C_{ji}$$

and ε_{ijk} is the 3-dimensional Levi-Civita tensor.

We remark that, in contradistinction of electrodynamics in which the momenta π^{μ} is given by the electric vector uniquely, here in the case of Fierz theory, both the electric and the magnetic parts of the field appear in the expression of the momenta. This is nothing but the fact that we are dealing with two spin-two fields of opposed parity. Just for completeness, we should say that the decomposition of the tensor $C_{\alpha\beta\mu\nu}$ in the electric $(E_{\mu\nu})$ and magnetic $(B_{\mu\nu})$ parts can be made covariant. Indeed, we define, for an arbitrary observer that moves with four-velocity V^{α}:

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}V^{\alpha}V^{\beta}$$

$$B_{\mu\nu} = \overset{*}{C}_{\mu\alpha\nu\beta}V^{\alpha}V^{\beta}$$

It then follows that we can write:

$$C_{\alpha\mu\beta\nu} = (\gamma_{\alpha\mu\lambda\sigma}\gamma_{\beta\nu\tau\varepsilon} - \eta_{\alpha\mu\lambda\sigma}\eta_{\beta\nu\tau\varepsilon})V^{\lambda}V^{\tau}E^{\sigma\varepsilon} -$$

$$-(\eta_{\alpha\mu\lambda\sigma}\gamma_{\beta\nu\tau\varepsilon} + \gamma_{\alpha\mu\lambda\sigma}\eta_{\beta\nu\tau\varepsilon})V^{\lambda}V^{\tau}B^{\sigma\varepsilon} .$$

in which:

$$\gamma_{\alpha\mu\lambda\sigma} \equiv \gamma_{\alpha\lambda}\gamma_{\mu\sigma} - \gamma_{\alpha\sigma}\gamma_{\mu\lambda}$$

Constraints (10) are thus represented by eq. (11). Following Dirac's[9] procedure to deal with such quantities we obtain the corresponding secondary constraints:

$$\pi^{ij}{}_{,j} \simeq 0. \qquad (13a)$$

$$p^{ij}{}_{,j} \simeq 0. \qquad (13b)$$

The twelve constraints (11) and (13) are first class and generate the gauge transformations (6).

In order to fix the gauge we impose:

$$\zeta_i \approx 0 \tag{14a}$$

$$\beta_{ij} \approx 0 \tag{14b}$$

$$\alpha^j_{i,j} \approx 0 \tag{14c}$$

$$\Delta^j_{i,j} \approx 0 \tag{14d}$$

This choice will be called, for obvious reasons, the radiation gauge.

The constraints (11), (13) and (14) are now second class. We define Dirac brackets which will make these constraints to become strong equalities and eliminate the spurious variables.

Equation (7) becomes:

$$\Box\alpha_{ij} = 0 \tag{15a}$$

$$\Box\Delta_{ij} = 0. \tag{15b}$$

Equation (8) is automatically satisfied by (14). The reduced Hamiltonian is

$$H = \int d^3x \Big[+ \frac{1}{4}\, \dot{\alpha}_{ij}\dot{\alpha}^{ij} - \frac{1}{4}\, \alpha_{ij,m}\alpha^{ij,m} + $$

$$- \frac{1}{4}\, \dot{\Delta}_{ij}\dot{\Delta}^{ij} + \frac{1}{4}\, \Delta_{ij,m}\Delta^{ij,m} \Big] \tag{16}$$

In order to exhibit that indeed the field $A_{\alpha\beta\mu}$ is reduced only to their spin-two parts, let us consider plane wave solutions of equations (13):

$$\alpha_{ij}(x) = Q_{ij}\exp(-iK_\mu X^\mu) + h.c\ . \tag{17b}$$

$$\Delta_{ij}(x) = R_{ij}\exp(-iK_\mu X^\mu) + h.c\ . \tag{17b}$$

They satisfy (15) if $K_\mu K^\mu = 0$ and satisfy (14c) and (14d) if:

$$Q_{ij}K^j = 0 \tag{18a}$$

$$R_{ij}K^j = 0 \qquad (18b)$$

Tensors Q_{ij} and R_{ij} have the same symmetries as α_{ij} and Δ_{ij}, respectively.

Choosing a wave propagating in the X^3-direction we obtain, from (18):

$$Q_{i3} = 0 \qquad (19a)$$

$$R_{i3} = 0. \qquad (19b)$$

As a consequence of the symmetries of Q_{ij} and R_{ij} and equations (19) the unique independent components of Q_{ij} and R_{ij} are Q_{11}, Q_{21}, R_{11} and R_{21}. We then define the quantities $Q_{\pm}$ and $R_{\pm}$ such that:

$$Q_{\pm} \equiv Q_{11} \mp iQ_{21} \qquad (20a)$$

$$R_{\pm} \equiv R_{11} \mp iR_{21} \qquad (20b)$$

We then proceed to perform a coordinate transformation corresponding to a rotation of an arbitrary angle θ about the X^3-axis. Then it follows that the quantities defined in (20) behaves under the form:

$$Q'_{\pm} = e^{\pm 2i\theta} Q_{\pm} \tag{21a}$$

$$R'_{\pm} = e^{\pm 2i\theta} R_{\pm} \tag{21b}$$

which shows that the wave can be decomposed in two irreducible parts $Q_{\pm}$ and $R_{\pm}$, each one having helicity ±2. We will come back to this later on when studying its quantum version.

Although we could obtain by this procedure the true degrees of freedom, we lost the manifest Lorentz covariance of the theory because the radiation gauge condition (14) is not Lorentz invariant - exactly like in Electrodynamics.

The formal resemblance of both theories suggests that the Lagrangian $\mathcal{L}_g$ given by (3) must be modified by setting

$$\tilde{\mathcal{L}}_{(g)} = \frac{1}{8}\, C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} - a(A^{\alpha\beta\mu}{}_{,\mu})^2 \tag{22}$$

and we left the value of the constant a to be fixed under the condition that this Lagrangian yields the wave equation $\Box A_{\alpha\beta\mu} = 0$. It then follows that a must take the value $a = -\frac{3}{4}$. A direct manipulation with the Lagrangian $\tilde{\mathcal{L}}_{(g)}$ set it into the most convenient Fermi-like form:

$$\tilde{\mathcal{L}}_{(g)} = \frac{1}{2}\,(A_{\alpha\beta\mu,\lambda})^2 + \text{div.} \tag{23}$$

Indeed, from such Lagrangian it follows precisely the wave equation:

$$\Box A_{\alpha\beta\mu} = 0$$

The difficulty with the momenta is solved, once from (23) it follows that the new momenta canonically conjugated to $A_{\alpha\beta\mu}$ is now given by:

$$\pi_{\alpha\beta\mu} = \frac{\delta \tilde{L}'_{(g)}}{\delta A_{\alpha\beta\mu,o}} = A_{\alpha\beta\mu,o} \tag{24}$$

The Hamiltonian that follows from this theory is:

$$H = \int d^3x \left[\frac{1}{2}\,\dot{A}_{\alpha\beta\mu}\,\dot{A}^{\alpha\beta\mu} - \frac{1}{2}\,A_{\alpha\beta\mu,\kappa}\,A^{\alpha\beta\mu}{}_{,\iota}\,\delta^{\kappa\iota}\right] \tag{25}$$

Note that there are no constraints in this theory because the Lagrangian (23) is not gauge invariant.

In order to obtain the same quantum versions of the two theories developed in this section we must impose some new Fermi-like conditions on the quantum states of the last theory. This will be done in the next section.

3 QUANTIZATION OF FIERZ SPIN 2-FIELD

a) THE LAGRANGIAN C^2

The theory, in the radiation gauge, is given by:

$$\mathcal{L} = C^{i0j0}C_{i0j0} + \frac{1}{2}C^{ijk0}C_{ijk0} =$$

$$= \frac{1}{4}\left(\dot{\alpha}_{ij} - \Delta_{(i}{}^{m}{}_{j),m}\right)^2 + \frac{1}{2}\left(\dot{\Delta}_{ijk} + \frac{1}{2}\alpha_{k[i,j]}\right)^2 \qquad (26)$$

The dynamical quantities of $A_{\alpha\beta\mu}$ are represented by two second order symmetric trace-less 3-tensors α_{ij} and Δ_{ij}. The momenta canonically conjugated are then, respectively, given by:

$$\pi^{ij} = \frac{\delta L}{\delta\dot{\alpha}_{ij}} = \frac{1}{2}\dot{\alpha}^{ij} - \frac{1}{4}\Delta_{\ell}{}^{(j}{}_{,m}\varepsilon^{i)\ell m} \qquad (27a)$$

$$p^{ij} = \frac{\delta L}{\delta\dot{\Delta}_{ij}} = -\frac{1}{2}\dot{\Delta}^{ij} - \frac{1}{4}\alpha^{(i}{}_{\ell,k}\varepsilon^{j)\ell k} \qquad (27b)$$

A direct inspection on the decomposition of $C_{\alpha\beta\mu\nu}$ in terms of α_{ij} and Δ_{ij} then shows that these momenta are nothing but the electric (E^{ij}) and the magnetic (B^{ij}) parts, that is $\pi^{ij} = E^{ij}$

and $P^{ij} = B^{ij}$. Thus in the present theory we are dealing with two independent spin two fields represented by the unrelated pairs of canonical variables (α_{ij}, E^{ij}) and (Δ_{ij}, B^{ij}).

In the quantum regime the canonical variables become quantum operators satisfying the following commutation relation at equal times:

$$\left[\alpha_{ij}(x), \pi^{kl}(y)\right] = - i\left(\frac{1}{2}\, \delta_{(i}{}^{k}\delta^{l}{}_{j)} - \frac{1}{3}\, \gamma^{kl}\gamma_{ij}\right]\delta^{3}(x-y) \qquad (28a)$$

$$\left[\Delta_{ij}(x), p^{kl}(y)\right] = - i\left(\frac{1}{2}\, \delta_{(i}{}^{k}\delta^{l}{}_{j)} - \frac{1}{3}\, \gamma^{kl}\gamma_{ij}\right)\delta^{3}(x-y) \qquad (28b)$$

Using (27) we have:

$$\left[\alpha_{ij}, \dot{\alpha}_{kl}\right] = - i\left(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \frac{2}{3}\, \gamma_{kl}\gamma_{ij}\right)\delta^{3}(x-y) \qquad (29a)$$

$$\left[\Delta_{ij}, \dot{\Delta}_{kl}\right] = i\left(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \frac{2}{3}\, \gamma_{kl}\gamma_{ij}\right)\delta^{3}(x-y) \qquad (29b)$$

The Fourier expansion of the solutions of equation (15) may be written as:

$$\alpha_{ij} = \int d^{3}k\left[\tilde{e}^{(\lambda)}_{ij}\, a_{(\lambda)}(k)e^{-ikx} + \tilde{e}^{(\lambda)}_{ij}a^{+}{}_{(\lambda)}(k)e^{ikx}\right] \qquad (30a)$$

$$\Delta_{ij} = \int d^3k\left[\tilde{e}^{(\lambda)}_{ij}\; b_{(\lambda)}(k)\,e^{-ikx} + \tilde{e}^{(\lambda)}_{ij}\; b^{+}_{(\lambda)}(k)\,e^{ikx}\right] \tag{30b}$$

in which $k_\mu k^\mu = 0$ and there is a summation on λ from 1 to 5. The unit polarization tensors $\tilde{e}^{(\lambda)}_{ij}$ have the same symmetries as α_{ij} and Δ_{ij}. They constitute a basis for the 3-tensors. They satisfy:

$$\tilde{e}^{(\lambda)}_{ij}\; \tilde{e}^{(\lambda')ij} = \delta^{\lambda\lambda'} \tag{31}$$

In the frame in which $K^\mu = (K,0.0.K)$ we have:

$$\tilde{e}^{(1)}_{ij} = \frac{\sqrt{2}}{2}\left(\delta^1_i\delta^1_j - \delta^2_i\delta^2_j\right)$$

$$\tilde{e}^{(2)}_{ij} = \frac{\sqrt{6}}{2}\left(\delta^3_i\delta^3_j + \frac{1}{3}\,\gamma_{ij}\right)$$

$$\tilde{e}^{(9)}_{ij} = \frac{\sqrt{2}}{2}\,\delta^1_{(i}\;\delta^2{}_{j)}$$

$$\tilde{e}^{(4)}_{ij} = \frac{\sqrt{2}}{2}\,\delta^2{}_{(i}\delta^9{}_{j)}$$

$$\tilde{e}^{(5)}_{ij} = \frac{\sqrt{2}}{2}\,\delta^3{}_{(i}\delta^1{}_{j)}$$

From eq. (29) and (30) it follows that the operators $a_{(\lambda)}(K)$ and $b_{(\lambda)}(k)$ satisfy the commutation relations:

$$\left[a_{\lambda}(k), a^{+}_{\lambda'}(k')\right] = 2i\delta_{\lambda\lambda'}(2\pi^{3})(2k_{o})\delta^{3}(k-k') \tag{32a}$$

$$\left[b_{\lambda}(k), b^{+}_{\lambda'}(k')\right] = -\ 2i\delta_{\lambda\lambda'}(2\pi^{3})(2k_{o})\delta^{3}(k-k') \tag{32b}$$

The presence of a factor 2 on the r.h.s. of eq. (32) induce us to re-define the operators by setting:

$$a'_{\lambda}(k) = \frac{\sqrt{2}}{2}\, a_{\lambda}(k) \tag{33a}$$

$$b'_{\lambda}(k) = \frac{\sqrt{2}}{2}\, b_{\lambda}(k) \tag{33b}$$

and correspondent particle number operators:

$$A_{\lambda} = a'^{+}_{\lambda} a'_{\lambda} \tag{34b}$$

$$B_{\lambda} = b'^{+}_{\lambda} b'_{\lambda} \tag{34b}$$

Inserting these definitions (33) and (34), and equations (32) and (30) into the Hamiltonian (16) we obtain:

$$H = \int \frac{d^3k}{(2\pi)^3 2k_o} k_o \left[A_1 + A_3 - B_1 - B_3\right] \tag{35}$$

It remains to prove our above assertion that we are indeed dealing with two spin-two tensors. From Noether's theorem we obtain the generic expression for the density of spin of any field theory described by Lagrangian L:

$$S^{\lambda}_{\rho\sigma} = \frac{\partial \mathcal{L}}{\delta A^{\alpha\beta\mu}{}_{,\lambda}} \quad .$$

$$\cdot \left[A_{\sigma}{}^{\beta\mu}\delta^{\alpha}_{\rho} + A^{\alpha}{}_{\sigma}{}^{\mu}\delta^{\beta}{}_{\rho} + A^{\beta\alpha}{}_{\rho}\delta^{\mu}{}_{\sigma} - A_{\rho}{}^{\beta\mu}\delta^{\alpha}_{\sigma} - A^{\alpha}{}_{\rho}{}^{\mu}\delta^{\beta}{}_{\sigma} - A^{\beta\alpha}{}_{\sigma}\delta^{\mu}{}_{\rho}\right] \tag{36}$$

Thus,

$$S^{\lambda}{}_{\rho\sigma} = 2C_{\rho\beta\mu}{}^{\lambda}\, A_{\sigma}{}^{\beta\mu} - 2C_{\sigma\beta\mu}{}^{\lambda}\, A_{\rho}{}^{\beta\mu} + C_{\mu\beta\sigma}{}^{\lambda}\, A^{\beta\mu}{}_{\rho} - C_{\mu\beta\rho}{}^{\lambda}\, A^{\beta\mu}{}_{\sigma}$$

Using the cyclic properties of $A_{\alpha\beta\mu}$ we can write:

$$S^{\lambda}{}_{\rho\sigma} = 2\left(C_{\rho\beta\mu}{}^{\lambda} + C_{\rho}{}^{\lambda}{}_{\mu\beta}\right)A_{\sigma}{}^{\beta\mu} - 2\left(C_{\sigma\beta\mu}{}^{\lambda} + C_{\sigma}{}^{\lambda}{}_{\mu\beta}\right)A_{\rho}{}^{\beta\mu} \tag{37}$$

Now, let us evaluate the spin vector in the x^3-direction. We have:

$$S_{ij} = \int d^3x\ S^{\circ}_{ij} \tag{38}$$

Using the decomposition (30) into (38) we obtain:

$$S_{12} = 2i \int d^3k \left[\left(a^{+}_{(+)} a_{(+)} - a^{+}_{(-)} a_{(-)}\right) + \left(b^{+}_{(+)} b_{(+)} - b^{+}_{(-)} b_{(-)}\right]\right.$$

where:

$$a_{(\pm)} \equiv \frac{1}{\sqrt{2}} (a_1 \mp i a_3)$$

$$b_{(\pm)} \equiv \frac{1}{\sqrt{2}} (b_1 \mp i b_3)$$

which ends our proof that out present theory deals indeed with two spin-two fields.

b) The Fermi Lagrangian

Let us now turn our attention to Lagrangian (23).

We introduce the Fourier decomposition of the field in the standard way:

$$A_{\alpha\beta\mu} = \int \sum_{n=1}^{16} \frac{d^3k}{\sqrt{(2\pi)^3 2k_o}}\, e^{(n)}{}_{\alpha\beta\mu}\left[a(n,k)\, e^{-ikx} + h.c\right) \qquad (39)$$

in which $k^{\mu}k_{\mu} = 0$ and $e^{(n)}{}_{\alpha\beta\mu}$ constitutes a set of unit polarization tensors satisfying the symmetries (1a.b). They are trace-less and normalized with an indefinite *metric* g(n,n'), that is:

$$e^{(n)}{}_{\alpha\beta\mu}\, e^{(n')\alpha\beta\mu} = g(n,n') \qquad (40)$$

$$\text{such that} \begin{cases} g(n,n') = -1 & \text{for} \quad n = n' = 1, \ldots, 8 \\ g(n,n') = +1 & \text{for} \quad n = n' = 9, \ldots, 10 \end{cases}$$

and g(n,n') vanishes otherwise.

A convenient choice of this basis in a frame which k^{μ} = (k,o,o,k) is provided by the following set:

$$e^{(i)}_{\alpha\beta\mu} = \frac{\sqrt{3}}{2}\left[\delta^{o}{}_{[\alpha}\delta^{i}{}_{\beta]}\delta^{o}{}_{\mu} - \frac{1}{3}\,\gamma_{\mu[\alpha}\delta^{i}{}_{\beta]}\right] \qquad \text{for} \quad i = 1,2,3$$

$$e^{(4)}_{\alpha\beta\mu} = \frac{1}{2}\left(\delta^{2}{}_{[\alpha}\delta^{1}{}_{\beta]}\delta^{2}{}_{\mu} - \delta^{3}{}_{[\alpha}\delta^{1}{}_{\beta]}\delta^{3}{}_{\mu}\right)$$

$$e^{(5)}_{\alpha\beta\mu} = \frac{1}{2}\left(\delta^3{}_{[\alpha}\delta^2{}_{\beta]}\delta^3{}_{\mu} - \delta^1{}_{[\alpha}\delta^2{}_{\beta]}\delta^1{}_{\mu}\right)$$

$$e^{(6)}_{\alpha\beta\mu} = \frac{1}{2}\left(\delta^1{}_{[\alpha}\delta^3{}_{\beta]}\delta^1{}_{\mu} - \delta^2{}_{[\alpha}\delta^3{}_{\beta]}\delta^2{}_{\mu}\right)$$

$$e^{(7)}_{\alpha\beta\mu} = \frac{\sqrt{3}}{3}\left(\delta^1{}_{[\alpha}\delta^2{}_{\beta]}\delta^3{}_{\mu} - \frac{1}{2}\,\delta^1{}_{[\mu}\delta^2{}_{\alpha]}\delta^3{}_{\beta} + \frac{1}{2}\,\delta^1{}_{[\mu}\delta^2{}_{\beta]}\delta^3{}_{\alpha}\right)$$

$$e^{(8)}_{\alpha\beta\mu} = \frac{1}{2}\left(\delta^2{}_{[\alpha}\delta^3{}_{\beta]}\delta^1{}_{\mu} - \delta^3{}_{[\alpha}\delta^1{}_{\beta]}\delta^2{}_{\mu}\right)$$

$$e^{(9)}_{\alpha\beta\mu} = \frac{\sqrt{3}}{3}\left\{\delta^1{}_{[\alpha}\delta^2{}_{\beta]}\delta^0{}_{\mu} - \frac{1}{2}\,\delta^1{}_{[\mu}\delta^2{}_{\alpha]}\delta^0{}_{\beta} + \frac{1}{2}\,\delta^1{}_{[\mu}\delta^2{}_{\beta]}\delta^0{}_{\alpha}\right\}$$

$$e^{(10)}_{\alpha\beta\mu} = \frac{\sqrt{3}}{3}\left\{\delta^2{}_{[\alpha}\delta^3{}_{\beta]}\delta^0{}_{\mu} - \frac{1}{2}\,\delta^2{}_{[\mu}\delta^3{}_{\alpha]}\delta^0{}_{\beta} + \frac{1}{2}\,\delta^2{}_{[\mu}\delta^3{}_{\beta]}\delta^0{}_{\alpha}\right\}$$

$$e^{(11)}_{\alpha\beta\mu} = \frac{\sqrt{3}}{3}\left\{\delta^3{}_{[\alpha}\delta^1{}_{\beta]}\delta^0{}_{\mu} - \frac{1}{2}\,\delta^3{}_{[\mu}\delta^1{}_{\alpha]}\delta^0{}_{\beta} + \frac{1}{2}\,\delta^3{}_{[\mu}\delta^1{}_{\beta]}\delta^0{}_{\alpha}\right\}$$

$$e^{(12)}_{\alpha\beta\mu} = \frac{1}{2}\left\{\delta^0{}_{[\alpha}\delta^1{}_{\beta]}\delta^1{}_{\mu} - \delta^0{}_{[\alpha}\delta^2{}_{\beta]}\delta^2{}_{\mu}\right\}$$

$$e^{(13)}_{\alpha\beta\mu} = \frac{\sqrt{3}}{2}\left\{\delta^{0}{}_{[\alpha}\delta^{3}{}_{\beta]}\delta^{3}{}_{\mu} - \frac{1}{3}\,\gamma_{\mu[\alpha}\delta^{0}{}_{\beta]}\right\}$$

$$e^{(14)}_{\alpha\beta\mu} = \frac{1}{2}\left\{\delta^{0}{}_{[\alpha}\delta^{1}{}_{\beta]}\delta^{2}{}_{\mu} + \delta^{0}{}_{[\alpha}\delta^{2}{}_{\beta]}\delta^{1}{}_{\mu}\right\}$$

$$e^{(15)}_{\alpha\beta\mu} = \frac{1}{2}\left\{\delta^{0}{}_{[\alpha}\delta^{2}{}_{\beta]}\delta^{3}{}_{\mu} - \delta^{0}{}_{[\alpha}\delta^{3}{}_{\beta]}\delta^{2}{}_{\mu}\right)$$

$$e^{(16)}_{\alpha\beta\mu} = \frac{1}{2}\left(\delta^{0}{}_{[\alpha}\delta^{3}{}_{\beta]}\delta^{1}{}_{\mu} - \delta^{0}{}_{[\alpha}\delta^{1}{}_{\beta]}\delta^{3}{}_{\mu}\right)$$

From the canonical commutation relations of the field $A_{\alpha\beta\mu}$ and the above expansion one obtains the corresponding c.r. of the operators a(n,k), indeed, we have for equal times $x_o = y_o$.

$$\left[A_{\alpha\beta\mu}(x),\pi^{\rho\sigma\lambda}(y)\right] =$$

$$= \frac{1}{2}\left(\delta^{\rho\sigma}_{\alpha\beta}\delta^{\lambda}{}_{\mu} - \frac{1}{2}\,\delta^{\lambda\rho}_{\alpha\beta}\delta^{\sigma}{}_{\mu} - \frac{1}{2}\,\delta^{\delta\lambda}{}_{\alpha\beta}\delta^{\rho}{}_{\mu} - \delta^{[\sigma}{}_{[\beta}\gamma_{\alpha]\mu}\gamma^{\rho]\lambda}\right).\delta^{(3)}(x-y) \tag{41}$$

in which $\delta^{\rho\sigma}_{\alpha\beta} \equiv \delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta} - \delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}$.

Then, it follows:

$$\left[a(n,k),a^{+}(r,k')\right] = i\ g(n,r)(2\pi)^{3}(2k_{o})\delta^{[3]}(k-k') \qquad (42)$$

The Hamiltonian of the field then assumes the form:

$$H = \int \mathcal{H}\ d^{3}x = \int \frac{d^{3}k(k_{o})}{(2\pi)^{3}2k_{o}} \ .$$

$$\left\{-N_{1}-N_{2}-N_{3}-N_{4}-N_{5}-N_{6}-N_{7}-N_{8}+N_{9}+N_{10}+N_{11}+N_{12}+N_{13}+N_{14}+N_{15}+N_{16}\right\} \qquad (43)$$

in which $N = a^{+}a$ and we have omitted the zero point energy.

Therefore, there are sixteen types of quanta, which corresponds to the 16 degrees of freedom of $A_{\alpha\beta\mu}$.

Let us then continue our strategy and submit the Fierz field $A_{\alpha\beta\mu}$ to an analogous treatment as one does for the vector field in the case of electrodynamics[5]. This means that we need to impose subsidiary conditions on $|\Psi>$ in order to obtain the physically realizable states of the field.

We decompose the states of the field $|\Psi>$ into the product $|\Psi_{T}> \times |\Psi_{R}>$ such that:

$$A^{+\alpha\beta\mu}{}_{,\mu}|\Psi_{R}> = 0. \qquad (44)$$

in which $|\Psi_R>$ represents the states that satisfies the above transverse condition (44) and $|\Psi_T>$ represents the remaining states of $\mathcal{H}$. This procedure allows us to select among the whole Hilbert space $\mathcal{H}$ those states $|\Psi>$ which obeys condition (24) and consequently reduces the freedom of the theory.

There remains to check that these six conditions on the quantum states commute with each other. Using the c.r.(41) and the definitions (24), it is straightforward to show that this is indeed the case.

Using the expansion (39), the condition (44) takes the form:

$$\left(e^{(n)}_{\ o1o} + e^{(n)}_{\ o13}\right) a_{(n)} |\Psi_R> = 0 \qquad (45a)$$

$$\left(e^{(n)}_{\ ijo} + e^{(n)}_{\ ij3}\right) a_{(n)} |\Psi_R> = 0 \qquad (45b)$$

A direct inspection on the basis $e^{(n)}_{\alpha\beta\mu}$ then yields:

$$\left(\sqrt{3}\ a_1 + 2a_{16} + a_4\right) |\Psi_R> = 0 \qquad (46a)$$

$$\left(\frac{\sqrt{3}}{2}\ a_{11} - \frac{1}{2}\ a_{16} - a_4\right)|\Psi_R> = 0 \qquad (46b)$$

$$\left(\sqrt{3}\ a_2 + 2a_{15} - a_5\right)|\Psi_R> = 0 \qquad (46c)$$

$$\left(\frac{\sqrt{3}}{2}\ a_{10} + \frac{a_{15}}{2} - a_5\right)|\Psi_R> = 0 \qquad (46d)$$

$$(a_3 - a_{13})|\psi_R> = 0 \qquad (46e)$$

$$(a_9 + a_7)|\psi_R> = 0 \qquad (46f)$$

Using conditions (46) in the calculation of the expectation value of the Hamiltonian, we obtain:

$$\frac{<\psi|H|\psi>}{<\psi|\psi>} = \frac{<\psi_T|\displaystyle\int \frac{d^3k\ k_o}{(2\pi)^3 2k_o}\left[-N_6-N_8+N_{12}+N_{14}\right]|\psi_T>}{<\psi_T|\psi_T>} \qquad (47)$$

Thus, for the theory we are considering, using Fierz variables, the generalized Lorentz-like condition (44) reduces the theory to two pure spin-2 fields of positive and negative energies. This can be set in evidence by just realizing that the only non-vanishing component that contributes to the mean value of the spin operator in the x^3-direction is given by:

$$S_{12} = 2 \int d^3k \left[a^{1+}_{(+)} a^{1}_{(+)} - a^{1+}_{(-)} a^{1}_{(-)} + a^{2+}_{(+)} a^{2}_{(+)} - a^{2+}_{(-)} a^{2}_{(-)} \right]$$

in which we define:

$$a^{1}_{(\pm)} = \frac{1}{\sqrt{2}} \left(a_{12} \mp i a_{14} \right)$$

$$a^{2}_{(\pm)} = \frac{1}{\sqrt{2}} \left(a_{8} \mp i a_{6} \right)$$

This then shows that indeed the present theory represents particles of spin 2 with two independent spin states (corresponding to helicity states parallel or anti-parallel to the direction of motion).

4 THE EXTRA SUBSIDIARY CONDITION

Let us pause for a while and see what we have achieved. From the Maxwellian Lagrangian $L = \frac{1}{8} C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu}$ we have arrived (after fixing the gauge) to the Hamiltonian (35) representing the true degrees of freedom of two spin-2 fields (see above) of opposite energy. In the precedent (Fermi) treatment the reduction to the same four degrees of freedom is achieved only after reducing the Hilbert space (through the condition $A^{+\alpha\beta\mu}{}_{,\mu}|\psi\rangle = 0$, eq. (44) above). Finally, in order to reduce the theory to a pure spin two field we must impose an extra subsidiary condition on the physically accessible state $|\psi\rangle$.

As a criterium to decide which subsidiary conditions one should impose we examine the behavior of both spin-2 fields under a space reflection. Actually, these two fields are described by the variables $A_{(i}{}^{o}{}_{j)}$ and A_{ijk}. This last one is associated to a negative parity spin 2-field. Thus we are induced to eliminate this part of the field once gravity as electrodynamics is to be described by a positive parity field (that is a true tensor, not a pseudo-tensor field).

We choose the condition:

$$\left(E_{\alpha\mu} - \frac{1}{2}\, A_{\lambda}{}^{\beta}{}_{\rho,\beta} h^{\lambda}{}_{(\alpha} h^{\rho}{}_{\mu)} - \frac{1}{2}\, A^{\lambda}{}_{\gamma}{}^{\rho}{}_{,\delta} h^{\gamma}{}_{(\alpha} h^{\delta}{}_{\mu)} V_{\rho} V_{\lambda}\right) |\psi\rangle = 0 \qquad (48)$$

where $h_{\mu\nu} \equiv \gamma_{\mu\nu} - V_{\mu}V_{\nu}$.

In a frame which $V^{\mu} = \delta^{\mu}{}_{o}$, equation (48) is nothing but:

$$\varepsilon^{ab}{}_{(j}\Delta_{1)a,b}|\psi > = 0 \tag{49}$$

It is obvious that this condition commutes with itself and with (44) in the Fermis's case.

The net consequence of (49) is to eliminate B_1 and B_3 from (35) and N_6 and N_8 from (47) which were the remnant of the pseudo-spin-2 field. The parts of the Hamiltonians that contribute to the mean value of the energy are reduced to a strictly positive definite expression for the unique spin-two field, that is.

$$H' = \int \frac{d^3k}{(2\pi)^3 2k_o} k_o(A_1 + A_3) \tag{50a}$$

and, for the Fermi's case:

$$H' = \int \frac{d^3k}{(2\pi)^3 2k_o} k_o(N_6 + N_8) \tag{50b}$$

If we had imposed instead of (48) the dual part of it, that is:

$$\left(B_{\alpha\mu} - \frac{1}{2} A_{\lambda}{}^{*}{}_{\beta\rho}{}^{,\beta} h_{(\alpha}{}^{\lambda} h^{\rho}{}_{\mu)} - \frac{1}{2} A_{\lambda}{}^{*}{}_{\gamma}{}^{\rho}{}_{,\varepsilon} h^{\gamma}{}_{(\alpha} h^{\varepsilon}{}_{\mu)} V^{\lambda} V_{\rho}\right) |\psi > = 0 \quad (51)$$

We would have eliminated the true tensor part of Hamiltonians (35) and (47) obtaining negative definites Hamiltonians.

5. CONCLUSION

We summarize our results of the theory of massless spin two field in terms of Fierz variables and their analogies with Electromagnetism in the following table:

Lagrangian	$L=-\frac{1}{4}\ F^2$	$L = \frac{1}{8}\ C^2$	$L=-\frac{1}{4}\ F^2-\frac{1}{2}(\partial A)^2$	$L=+\frac{1}{8}\ C^2+\frac{3}{4}(\partial A)^2$
Gauge Symmetry	$\delta A=\partial\Lambda$	$\delta A = dw$	—	—
Gauge Fixation	$A_0=0$ $A^i{}_{,i}=0$	$\xi_i=0.\ \beta_{ij}=0$ $\alpha_i{}^j{}_{,j}=0.\ \Delta_i{}^j{}_{,j}=0$	—	—
First Subsidiary Condition	—	—	$A^{+\mu}{}_{,\mu}\|\psi>=0$	$A^{+}_{\alpha\beta}{}^{\mu}{}_{,\mu}\|\psi>=0$
Hamiltonian	$H=\int d^3k$ (N_1+N_2)	$H=\int d^3k(A_1+A_3+$ $-B_1-B_3)$	$H=\int d^3k(N_1+$ $+N_2)$	$H=\int d^9k(-N_1-N_3+$ $N_6^2+N_8^2)$
Second Subsidiary Condition	-	1. Condition(48) 2. Condition(51)	-	1. Condition(48) 2. Condition(51)
Hamiltonian	-	1. $H'=\int d^3k(A_1+A_3)$ 2. $H'=\int d^3k(-B_1^+-B_3^+)$	—	1. $H'=\int d^3k(N_6+N_8)$ 2. $H'=\int d^3k(-N_1-N_3$
Spin	1	2	1	2

In the present paper we have then achieved an alternative description of the dynamics of spin-two field in a flat space-time that is completely analogous to Electrodynamics. The price paid for this is the appearance of not only one but two spin-two fields with opposite parities that contribute with opposite signals to the Hamiltonian. In order to eliminate one of them we have just to suppose extra subsidiary conditions on the quantum states obtaining thus a positive (or negative) definite mean value for the Hamiltonian.

Although the presence of a Hamiltonian which is not positive definite is very unconfortable, because we live in a Universe where energy is conserved. This could be a nice feature of a theory of the begining of the Universe where energy and matter seem to have appeared from nothing. One could speculate that the two spin-two fields have created each other and all matter of the Universe. The appearance of the first observers (contained in $E_{\alpha\beta}$(48) and $B_{\alpha\beta}$(51) would eliminate one of the spin-two fields thus arrising two Universes with positive (and negative) definite Hamiltonians (where the energy is indeed conserved) that would be completely disconnected from then on. To develop further this idea we must construct the non-linear extension of this theory. This will be the subject of future investigation.

APPENDIX

Dual Rotation

Maxwell's equation for electrodynamics is invariant under a constant dual rotation. This means that the dynamics described by the Lagrangian $L_M = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}$ is insensible to the mapping $F_{\mu\nu} \to \tilde{F}_{\mu\nu} = \cos\theta\, F_{\mu\nu} + \sin\theta\, F^{*}_{\mu\nu}$. The case θ becomes a space-time dependent function. The transformed Lagrangian contains a parity-violating term proportional to $F_{\mu\nu}F^{*\mu\nu}$. Thus one should suspect that a similar behavior should appear in the present theory. That this is indeed true is almost trivial. However, there is an additional property, which does not have a parallel in electrodynamics, of which it seems worth to comment.

Let us come back to the definition of the electric and magnetic tensor E_{ij} and B_{ij}

$$E_{ij} = C_{iojo} \tag{A.1a}$$

$$B_{ij} = \overset{*}{C}_{iojo} = -\frac{1}{2}\,\varepsilon^{lk}{}_{i}\, C_{lkjo} \tag{A.1b}$$

In the particular gauge we deal with (cf. section 5) we can write:

$$C_{i0j0} = \frac{1}{2}\,\dot{\alpha}_{ij} - \frac{1}{2}\,\Delta_{(i\ \ j),m}^{\ \ m} \tag{A.2a}$$

$$C_{ijk0} = \dot{\Delta}_{ijk} + \frac{1}{2}\,\alpha_{k[i,j]} \tag{A.2B}$$

Define the quantity:

$$\Delta_{ij} = \frac{1}{2}\,\varepsilon_{(i}^{\ \ lk}\Delta_{lkj)} \tag{A.3}$$

from which inverse we obtain: $\Delta_{mnj} = -\,\frac{1}{2}\,\Delta_{lj}\varepsilon^{l}_{\ mn}$.
Then, it follows that E_{ij} and B_{ij} can be written as:

$$E_{ij} = \frac{1}{2}\,\dot{\alpha}_{ij} - \frac{1}{4}\,\varepsilon^{lm}_{\ \ \ (i}\Delta_{j)l,m} \tag{A.4a}$$

$$B_{ij} = -\,\frac{1}{2}\,\dot{\Delta}_{ij} - \frac{1}{4}\,\varepsilon^{lk}_{\ \ \ (i}\alpha_{j)l,k} \tag{A.4b}$$

The quantities α_{ij} and Δ_{ij} represent the independent degrees of freedom of the field and, as we have shown in the precedent sections, describe two spin-2 fields (of opposite parity). Let us now perform a rotation in the plane (α,Δ) such that:

$$\begin{pmatrix} \alpha'_{ij} \\ \Delta'_{ij} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha_{ij} \\ \Delta_{ij} \end{pmatrix} \qquad (A.5)$$

Then, from (A.4) it implies for the electric and magnetic parts a corresponding transformation:

$$\begin{pmatrix} E'_{ij} \\ B'_{ij} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_{ij} \\ B_{ij} \end{pmatrix} \qquad (A.6)$$

that is,

$$C_{\alpha\beta\mu\nu} \to C'_{\alpha\beta\mu\nu} = \cos\theta \; C_{\alpha\beta\mu\nu} + \sin\theta \; C^{*}_{\alpha\beta\mu\nu}$$

We can, thus recognize the origin of the dual map of the field $C_{\alpha\beta\mu\nu}$: it is nothing but the rotation in the plane (α,Δ) described by (A.5).

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