ISSN 0029 - 3865

CBPF-NF-006/87

GAUGE TRANSFORMATIONS IN DIRAC THEORY OF CONSTRAINED SYSTEMS

by

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1 INTRODUCTION

Since Dirac (Dirac 1964) developed his theory for constrained systems there has been considerable progress in the understanding of those systems. The interest on his theory is indeed justified not only for the deep insight it provides into the conceptual framework but also for the very powerful techniques it also provides, which can be applied to a very broad class of important physical systems.

One point in Dirac's theory that has been the target of critics by some authors (Gotay 1983, Schafir 1982, Sugano 1983) is his conjecture that all first class constraints are generators of gauge transformations. He also introduced the concept of an extended Hamiltonian which includes all the first class constraints, and generates the dynamical evolution of the system with full gauge freedom. In spite of the lack of a proof of his conjecture (or even a proof that it is not correct) we do not know of any physically important system to which Dirac's conjecture leads to the *wrong* result.

In order to obtain all the constraints of a theory one must use Dirac's algorithm which in some cases is very tedious. But once all the first class constraints are obtained one can construct a generator of gauge transformations as a linear combination of these constraints, the coefficients of which are, in principle, arbitrary. Application of this procedure (Sundermeyer 1982) to the case of Yang-Mills theory requires a by hand adjustement of the coefficients in order to match the result with the well known gauge transformation law for the Yang-Mills potentials.

The example of Yang-Mills theory suggested us to ask about the degree of arbitrariness of the coefficients which appear in the generator. Admiting that the evolution of a given dynamical system is generated by the total Hamiltonian H_T (we remark that this poses no restriction on the dynamics) we compared two trajectories of the system corresponding to the same initial data but to different choices of the arbitrary functions in H_T. Taking into account that the physical states of the system cannot depend on the choice of the arbitrary functions, the corresponding states along the two trajectories must be related by a gauge transformation. The result of the procedure is a differential equation relating the coefficients of the primary and secondary first class constraints. The generator so obtained has been applied to various systems yielding the correct results. (For different approaches to obtain the generator of gauge transformations see (Castellani 1982, Di Stefano 1983).)

The paper is organized as follows. In Section II we discuss some aspects of Dirac's theory which are relevant for the following sections. In Section III we present our approach to obtain the generator. Section IV is devoted to an application.

2. THE GENERATOR OF GAUGE TRANSFORMATIONS ACCORDING TO DIRAC'S THEORY

Let us consider the evolution of a mechanical system in phase space with canonical coordinates (q^n, p_n), $n = 1, \ldots, N$. We suppose that the system is singular and denote the full set of independent constraints (to be specified later on) by $C_i \approx 0$, $i = 1, \ldots, m$, which define a sub-space M in phase space, where the motion of the system actually occurs.

According to Dirac's theory the total Hamiltonian for the system is defined as

$$H_T = H_c + u^k(q,p)\phi_k \tag{2.1}$$

where H_c is the canonical Hamiltonian ϕ_k, $k = 1, \ldots, K$, are the primary constraints, and $u^k(q,p)$ are arbitrary functions. The constraints ϕ_k constitutes a sub-set of the constraints C_i. In principle the primary constraints are known once the momenta are calculated and are incorporated in the Hamiltonian by the method of Lagrange multiplers.

The consistency conditions of time preservation of the primary constraints, $\dot{\phi}_k = \{\phi_k, H_T\} \approx 0$, in general lead to the existence of new constraints, ψ_ℓ, which are called secondary constraints. (During this process some of the functions u^k can possible be determined but whether this happens or not is not important in what follows.) The set of constraints $C_i \approx 0$ is then constituted by all the primary and secondary constraints. For simplicity we will suppose that this set is first class. The important property of this set of constraints is that together with H_c it constitutes an algebra (denoted by G) under the Poisson bracket operation. Indeed, one can easily show that for arbitrary linear combinations g_i of elements of G, with arbitrary coefficients depending on (q^k, p_k), the following relations hold(*).

$$\{g_i, g_j\} = C^k{}_{ij}(q,p)\, g_k \quad ,$$
$$\{g_i, H_c\} = C^k{}_i(q,p)\, g_k \quad . \tag{2.2}$$

It follows that the set (C_i, H_c) constitutes a basis in G. As generators of infinitesimal transformations the elements of G map M on M. When the coefficients in (2.2) are constants G is a Lie algebra to which is associated the group of infinitesimal transformations on M.

Now, given the initial data $(q^k, p_k)_{t=t_o}$ the physical state of the system is well determined at t_o. However, the time evolution of the system generated by the total Hamiltonian leads to the appearance of the arbitrary functions in the solutions of the equations of motion. This implies that there are several sets of canonical variables at $t \neq t_o$ which correspond to the same initial data. In other words, for each choice of the arbitrary functions u^k there is an extremal cur

(*) Instead of H_c we should use $H_o = H_c + \lambda_n \phi_n$, where λ_n are the multipliers which are determined during the consistency procedure. But as we said before this is not important for our purposes.

ve or trajectory of the system, starting at $(q^k, p_k)_{t=t_o}$.

From the physical point of view the choice of the arbitrary functions is irrelevant in the sense that the corresponding states of the system must be equivalent. Hence one is lead to say that the terms involving the primary first class constraints in H_T generate transformations which do not change the physical states of the system. In other words, they generate gauge transformations.

What is clear from the above discussion is that not only the primary first class constraints are generators of gauge transformations but also the secondary (first class) ones; Dirac conjectured that they should also be included in the Hamiltonian and defined the extended Hamiltonian

$$H_E = H_T + v^i(q,p)\psi_i \quad , \tag{2.3}$$

which generates the evolution of the system with full gauge freedom. In spite of their completely different physical origin it is perfectly acceptable from the physical point of view that all the first class constraints must be treated on equal foot.

According to Dirac's prescription the generator of gauge transformations for the system can be written as

$$G = F^i(q,p)\bar{\phi}_i \tag{2.4}$$

where $F^i(q,p)$ are arbitrary functions and $(\bar{\phi}_i)$ denotes all the first class constraints. A straightforward application of

(2.4) to the important case of Yang-Mills theory requires an adjustment (Sundermeyer 1982) of the "arbitrary" functions at the final step, in order to recover the correct transformation law for the gauge potentials, namely, $\delta A^{\mu}_{a} = D^{\mu}\omega_{a}(x)$. (Another procedure (Hanson et al. 1976) makes use of the equations of motion generated by H_E so as to eliminate some of the arbitrary functions and to identify the remaining ones with $A^{0}_{a}(x)$.) Guided by these facts we asked about the degree of arbitrariness of the functions $F^{i}(q,p)$ which appear in (2.4) and what we found to answer this question is showed in the next section.

3 CONSTRUCTING THE GENERATOR OF GAUGE TRANSFORMATIONS

We are going to compare two trajectories of the same physical system corresponding to the same initial data, but with two different choices of the arbitrary functions namely u^k and $u^k + \bar{u}^k$, where $\bar{u}^k$ is assumed to be a small deviation from the original functions u^k. According to the discussion of the preceeding section the corresponding physical states of the system are to be considered as equivalent and so related by a gauge transformation with the generator of the form (2.4).

Let $Q = Q[q,p,Q_o,u]$ be any dynamical variable associated with the system and Q_o its value for the initial data. (For simplicity we will assume that Q has no explicit time dependence as this will not change the results). We suppose that its evolution is generated by the total Hamiltonian so that for the choice u^k of the arbitrary functions we have

$$\dot{Q} = \{Q, H_T[u]\} = \{Q, H_c + u^k\phi_k\} \tag{3.1}$$

Denoting $\bar{Q} = Q[q,p;Q_0,u+\bar{u}]$, we also have

$$\dot{\bar{Q}} = \{Q, H_T[u+\bar{u}]\} = \{Q, H_T[u] + \bar{u}^k\phi_k\} \tag{3.2}$$

On the other hand as Q and $\bar{Q}$ must be related by a gauge transformation it follows that

$$\bar{Q} = Q[u] + \{Q,G\}_{Q[u]} = Q[u] + \{Q, F^i(q,p)\bar{\phi}_i\}_{Q[u]} \tag{3.3}$$

The time evolution of $\bar{Q}$ as given by equation (3.3) above is

$$\dot{\bar{Q}} = \{Q,H_c\} + \dot{F}^i(q,p)\bar{\phi}_i + F^i(q,p)\{\bar{\phi}_i, H_T[u]\} .$$

Thus, $\bar{Q}$ will be or solution of (3.2) if

$$\dot{F}^i(q,p)\bar{\phi}_i + F^i(q,p)\{\bar{\phi}_i , H_T[u]\} = \bar{u}^k\phi_k . \tag{3.4}$$

We now split the set of first class constraints into primary ϕ_i and secondary ψ_ℓ ones, and write the generator G as

$$G = \omega^\ell(q,p)\psi_\ell + \epsilon^n(q,p)\phi_n . \tag{3.5}$$

Equation (3.4) is then rewritten as

$$\frac{d\omega^\ell}{dt}\psi_\ell + \omega^\ell\{\psi_\ell,H_c\} + \omega^\ell u^n\{\psi_\ell,\phi_n\} + \epsilon^n\{\phi_n,H_c\} = f^n\phi_n \tag{3.6}$$

where in $f^n\phi_n$ we included all terms proportional to the primary first class constraints. Now, the quantities $\{\psi_\ell,H_k\}$ and $\{\phi_k,H_c\}$ in the above equation can be expressed as linear combinations of the secondary constraints. We write

$$\{\psi_\ell,H_c\} = \alpha_{\ell n}\psi_n \ , \ \{\phi_n,H_c\} = \beta_{n\ell}\phi_\ell \quad , \tag{3.7}$$

which when substituted in (3.6) yields

$$\frac{d\omega^\ell}{dt}\psi_\ell + \omega^n\alpha_{n\ell}\psi_\ell + \varepsilon^n\beta_{n\ell}\psi_\ell + \omega^\ell u^n\{\psi_\ell,\phi_n\} = f^n\phi_n \quad . \tag{3.8}$$

Taking into account the linear independence of the primary and secondary constraints we see that if the last term on the left hand side contains any linear combination of the secondary constraints, the coefficient ω^ℓ will depend on the arbitrary functions u^k. If this is the case the generator G will lose its meaning as it will not generate transformations between admissible trajectories. On the other hand this term is equal to zero for the interesting physical systems so that we will discard it. Then we are left with the following differential equation* relating the coefficients of the generator (3.5):

$$\frac{d\omega^\ell}{dt} + \alpha_{\ell n}\omega_n + B_{\ell i}\varepsilon_i = 0 \tag{3.9}$$

This equation shows that the coefficients ω^ℓ , ε^i are not independent of each other so that given one set the other is determined be (3.9). (In case one is looking for an explicit solution of (3.9) one must remember that the gauge transformations cannot change the initial data so that suitable initial conditions must be imposed.) We remark that we would not have obtained the equation (3.9) if we had used the extended Hamiltonian as the generator of the dynamical evolution of the system.

4 APPLICATION

In order to exhibit all features of the technique we developed to obtain the generator of gauge transformations we chose a system with somewhat higher degree of complexity than the average. We will consider a generalized Yang-Mills theory (Galvão and Pimentel 1986, Galvão 1987) described by the Lagrangian

$$L = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} - a^{2} D^{\alpha} F^{a}_{\alpha\mu} D_{\beta} F^{\beta\mu}_{a} \quad , \qquad (4.1)$$

where $\underline{a}$ is a constant and $D_{\mu}(\ )^{a} = \partial_{\mu}(\ )^{a} - C^{a}_{bc} A^{b}_{\mu}(\ )^{c}$ is the gauge covariant derivative.

This Lagrangian is clearly singular and contains second order derivatives of the gauge potentials. In order to set up a Hamiltonian formalism we used Ostrogradiskii's (Ostrogradiskii 1850) methods combined with Dirac's theory. Accordingly, we formally consider $A^{a}_{\mu}(x)$ and $\dot{A}^{a}_{\mu}(x) \equiv B^{a}_{\mu}(x)$ as independent variables. The momenta conjugated to these variables are defined as

$$p^b_\alpha = \frac{\partial L}{\partial \dot{A}^\alpha_b} - 2\partial_k \left(\frac{\partial L}{\partial(\partial_o \partial_k A^\alpha_b)}\right) - \partial_o \left(\frac{\partial L}{\partial \ddot{A}^b_\alpha}\right)$$

and

$$\Pi^b_\alpha = \frac{\partial L}{\partial \ddot{A}^b_\alpha} \quad ,$$

which for the Lagrangian (4.1) result in

$$p^b_\alpha = - F^b_{o\alpha} + 2a^2 \left[\partial_k (D_\lambda F^{k\lambda}_b)\delta^0_\alpha + \partial_k (D_\lambda F^{o\lambda}_b)\delta^k_\alpha\right]$$

$$- 2a^2 C_{abc}(2A^c_o D^\lambda F^a_{\alpha\lambda} - A^\mu_c D^\lambda F^a_{\mu\lambda}\delta_{\alpha o} - A^c_\alpha D^\lambda F^a_{o\lambda}) - \dot{\pi}^b_\alpha \ , \qquad (4.2)$$

$$\Pi^b_\alpha = - 2a^2 (D_\lambda F^{o\lambda}_b \delta^0_\alpha - D^\lambda F^b_{\alpha\lambda}) \ . \qquad (4.3)$$

From the above expressions we obtain two primary constraints

$$\phi^b_{(1)} = \pi^b_o \approx 0 \quad , \qquad (4.4)$$

$$\phi^b_{(2)} = p^b_o - D^k \pi^b_k \approx 0 \ . \qquad (4.5)$$

By eliminating $\dot{B}^k_a(x)$ from the canonical Hamiltonian

$$H_c = \int d^3x \left[p^a_\alpha \dot{A}^\alpha_a + \pi^a_\alpha \dot{B}^\alpha_a - L\right]$$

we obtained

$$H = \int d^3x \Big[(p^a_k - C^a_{bc} A^0_b \pi^c_k) B^k_a - \frac{1}{4a^2} \Pi^a_k \Pi^k_a + \Pi^k_a D^i F^a_{ki}$$

$$+ a^2 D_i F^{oi}_a D_j F^{oj}_a + \frac{1}{4} F^a_{ij} F^{ij}_a + \frac{1}{2} F^a_{oj} F^{oj}_a$$

$$- C_{bcd} \Pi^k_b A^0_d F^c_{ko} + \overset{(1)}{C}_a (x) + (\overset{(2)}{C}_a (x) + B^0_a)(p^a_o - D^k \Pi^a_k) \Big] \qquad (4.6)$$

Introducing the fundamental Poisson brackets

$$\{p^a_\alpha(x), A^\beta{}_b(x') = -\delta^a_b \delta^\alpha_\beta \overset{(3)}{\delta} (\vec{x} - \vec{x}'), \{\pi^a_\alpha(x), B^\beta_b(x')\} = - \delta^a_b \delta^\alpha_\beta \overset{(3)}{\delta} (\vec{x} - \vec{x}')$$

the consistency conditions lead to the secondary constraint

$$\psi^a_{(1)} = D^k (p^a_k + C_{abc} \pi^b_k A^c_o) - C_{abc} \pi^k_b F^c_{ko} = 0 \qquad (4.7)$$

It is easy to check that the constraints (4.4,5,7) are first class.

The generator (3.3) and the equations (3.6,7) are now written as

$$G = - \int dz (\omega^b \psi_b + \dot{\varepsilon}^b_{(i)} \phi^i_b) \qquad (i = 1,2) \qquad (4.8)$$

$$\frac{d\omega^b}{dz^0} + \omega^a a_{ab} + \varepsilon^a_{(i)} B_{(i)ab} = 0 \qquad (4.9)$$

with

$$\{\psi_a, H\} = a_{ab} \psi_b \ , \ \{\phi^a_{(i)}, H\} = b_{(i)ab} \psi_b \quad .$$

We obtained

$$a_{ab} = C_{abc}A^0_c \ , \ b_{(1)ab} = \delta_{ab} \ , \ b_{(2)ab} = 0$$

It follows that $\varepsilon^b_{(2)} \equiv \varepsilon^b$ remains arbitrary, while (4.9) leads to

$$\varepsilon_{(1)b} = - D_o\omega_b$$

so that

$$G = - \int dz(\omega^b\psi^b_{(1)} - D_o\omega^b\phi_{(1)} + \varepsilon^b\phi^b_{(2)})$$

Using expressions (4.4,5,7) we obtained after some partial integrations and discording some unimportant surface terms:

$$G = \int dz(p^\mu_a - C_{abc}A^c_o\pi^\mu_b)D^\mu\omega^a + \int dz\pi^\mu_a D_\mu\varepsilon^b$$
$$+ \int dz(C_{abc}\pi^\mu_b F^c_{\mu o})\omega^a \ . \qquad (4.10)$$

It is easy to check that the above generator leads to the correct gauge transformations for the dynamical variables:

$$\delta A^b_k = \{A^b_k(x), G\} = D_k\omega^b \ , \qquad (4.11)$$

$$\delta B^b_k = - (C_{abc}A^c_o D_n\omega^\ell + D_n\varepsilon^a + C_{abc}F^c_{no}\omega^a = \partial_o(D_k\omega^b) \ . \qquad (4.12)$$

One should not be disapointed by the non gauge-covariance of (4.12) because this is due to the definition of the variables B_k^a.

We observe that for the choice $\omega^b \to \delta_d^b$ both A_k^a and B_k^a transform under G_d as the adjoint representation of the group. Finally, we mention that the generators $G(\omega)$ form or closed algebra.

CONCLUSIONS

We made an analysis of some aspects of gauge transformations in the context of Dirac's theory of constrained systems. Accepting Dirac's conjecture that all first class constraints associated with a given physical system generate gauge transformations but using the total Hamiltonian to describe its dynamical evolution we have been able to construct a generator for gauge transformations by comparing phase space trajectories with the same initial data but different choices of the arbitrary functions. The coefficients of the primary and secondary constraints in our generator are not completely free but related by a differential equation . We observe that this does not means any distinction between those constraints from the dynamical point of view. The generator we obtained obeys a closed algebra and leads to correct results when applied to the most familiar examples.

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