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SELF-DUALITY CONDITION IN CHERN-SIMONS HIGGS

by

Prem P. SRIVASTAVA and K. TANAKA

Abstract

It is shown that in the Higgs and Chern-Simons-Higgs systems the 'self-duality' constraint on the scalar field (combined with the equations of motion) by itself leads to a general form for the potential. In the limits of the Chern-Simons coefficient $\kappa \to 0$ and $\kappa \to \infty$ one obtains the previous special forms of the potential. In the latter case, it is shown that the quantity $ln|a|^2$, where a is the bosonic field, satisfies the Liouville equation. The supersymmetric extensions of the theories written in terms of superfields is considered. A 'supersymmetric self-duality' constraint on the matter superfield is proposed which contains the bosonic one and it leads to the specific forms of superpotentials without invoking arguments based on an explicit N=2 supersymmetry.

Key-words: Gauge theory; Chern-Simons Higgs; Selfdual vortex solutions.

1. In (2+1) dimensional spacetime the possibility of including in abelian Higgs model the Chern-Simons (CS) term[1] has generated a great deal of interest. It was noted[2] that in the Chern-Simons-Higgs (CSH) system, the energy functional obeys a Bogomol'nyi-type[3] lower bound for a special choice of the Higgs potential. The bound is achieved if the Higg's scalar field a satisfies the following first order self-duality condition[2]

$$\mathcal{D}_1 a = -i\mathcal{D}_2 a, \quad or \qquad \mathcal{D}_i a = -i\epsilon^{ij}\mathcal{D}_j a, \tag{1}$$

where $\mathcal{D}_m = \partial_m + iev_m$, $m = 0,1,2$ are the spacetime indices while $i = 1,2$. Our metric is $\eta_{mn} = diag(-1,1,1)$ with $\epsilon^{12} \equiv \epsilon^{012} = 1$. We set

$$a = e^{iew}\,\rho^{\frac{1}{2}} \tag{2}$$

and substitute in (1). The spatial part of v_m is then found to be

$$v_i = -\partial_i\omega - \frac{1}{2e}\epsilon^{ij}\partial_j ln\rho. \tag{3}$$

The electromagnetic field is then given with the aid of v_m as

$$\begin{aligned} B = f_{12} &= \frac{1}{2e}\nabla^2\, ln\rho \\ E^i &= -\partial^i v^0 + \partial^0 v^i\,. \end{aligned} \tag{4}$$

The critical form of the potential can also be obtained by directly solving[4] the eqs. of motion with the aid of the self-duality condition. This procedure can also be extended to the scalar superfield, a supersymmetric self-duality condition[4] postulated, and the eqs. of motion solved for the supersymmetric potential (Sec. 3).

2. The Lagrangian for the *Bosonic Chern-Simons Higgs* system is

$$\mathcal{L} = -(\tilde{\mathcal{D}}^l a^*)(\mathcal{D}_l a) - V(|a|^2) - \frac{\kappa}{4}\,\epsilon^{lmn}\,v_l f_{mn} - \frac{1}{4}f_{lm}f^{lm}, \tag{5}$$

where $\tilde{\mathcal{D}}_m = \partial_m - iev_m$. The equations of motion are derived to be

$$\mathcal{D}^l\mathcal{D}_l a = V'(|a|^2)a, \tag{6}$$

and

$$-\partial_m f^{ml} + \frac{\kappa}{2}\epsilon^{lmn} f_{mn} = j^l, \tag{7}$$

Here $V'(|a|^2) = \partial V/\partial|a|^2$ and $j^l = ie(a^*\mathcal{D}^l a - a\tilde{\mathcal{D}}^l a^*)$ is the Noether current, $\partial_l j^l(v) = 0$.

For static configurations eq.(6) reduces to $(i,j = 1,2)$

$$\mathcal{D}_i\mathcal{D}_i a = (V' - e^2{v_0}^2)a, \tag{8}$$

and we find from eq.(7) corresponding to $l = 0, 1$ and 2, respectively,

$$\partial_i\partial_i v_0 + \kappa f_{12} = 2e^2 v_0\,|a|^2\,, \tag{9}$$

$$\partial_2(f_{12} + \kappa v_0) = j_1, \tag{10}$$

$$\partial_1(f_{12} + \kappa v_0) = -j_2, \tag{11}$$

where the gauge $\partial_l v^l = 0$ is taken.

If we impose the self-duality condition (1),

$$\mathcal{D}_1 a = -i\mathcal{D}_2 a \qquad and \qquad \tilde{\mathcal{D}}_1 a^* = i\tilde{\mathcal{D}}_2 a^* \tag{12}$$

then eq.(8) leads to

$$e^2{v_0}^2 + ef_{12} = V'(|a|^2). \tag{13}$$

We also obtain

$$j_1 = e\partial_2|a|^2 \qquad and \qquad j_2 = -e\partial_1|a|^2 \tag{14}$$

while from (10) and (11) it follows that

$$f_{12} + \kappa v_0 = e(|a|^2 - C^2), \tag{15}$$

where C is a constant. Combining (9),(13) and (15) we derive the general result

$$V'(|a|^2) = e^2 {v_0}^2 + \frac{e}{\kappa}(2e^2|a|^2 - \partial_i^2)v_0 \tag{16}$$

where v_0 is given by

$$[\frac{1}{\kappa^2}(2e^2|a|^2 - \partial_i^2) + 1]v_0 = \frac{e}{\kappa}(|a|^2 - C^2) \tag{17}$$

Several limiting cases may be considered. When $\kappa \to 0$ (no Chern-Simons term) the eqs. (9), (13) and (15) lead to

$$\begin{aligned}(2e^2|a|^2 - \partial_i^2)v_0 &= 0,\\ V'(|a|^2) &= e^2(|a|^2 - C^2) + e^2 v_0^2\,. \end{aligned} \tag{18}$$

For the choice $v_0 = 0$ we obtain $V = (e^2/2)(|a|^2 - C^2)^2$.

In the limit $e \to \infty$, $\kappa \to \infty$ such that $(e^2/\kappa) \to$ finite the terms originating from the Maxwell term in the eqs. of motion drop out. We find from eqs. (9),(15) and (16) $ef_{12} = 2(e^2/\kappa)(ev_0)|a|^2$, $ev_0 = (e^2/\kappa)(|a|^2 - c^2)$ and

$$V'(|a|^2) = (e^2/\kappa)^2(|a|^2 - C^2)(3|a|^2 - C^2) \tag{19}$$

leading to

$$V(|a|^2) = (e^2/\kappa)^2(|a|^2 - C^2)^2|a|^2 \tag{20}$$

which is seen to saturate the lower bound of the energy functional. On making use of eq.(4) and setting $\rho = exp(\chi/2)$ we find that χ satisfies the following differential equation with the choice $C = 0$

$$\nabla^2 \chi = 8(e^2/\kappa)^2 e^{\chi} \tag{21}$$

which is the Liouville equation[6].

It may be worth remarking that in the general case if we impose in addition to the self-duality condition the ansatz $(2e^2|a|^2 - \partial_i^2)v_0 = 0$ then from eqs. (15),(16) and (17) we are led to $ev_0 = (e^2/\kappa)(|a|^2 - C^2)$, $f_{12} = 0$ and

$$V(|a|^2) = \frac{1}{3}(e^2/\kappa)^2(|a|^2 - C^2)^3 \tag{22}$$

which, however, does not saturate the lower bound. A different and more complicated potential is obtained if we impose, say, the ansatz $\partial_i^2 v_0 = 0$.

3. Consider the *Supersymmetric Chern-Simons-Higgs* system. The gauge vector potential in the case of 2+1 spacetime dimensions is contained in a Majorana spinor connection superfield

$$\Gamma^\alpha(x,\theta) = \chi^\alpha(x) + \bar\theta_\beta(\frac{1}{2}\epsilon^{\beta\alpha}v(x) + \gamma_l^{\beta\alpha}v^l(x)) + i\bar\theta\theta\eta^\alpha(x), \tag{23}$$

where $\eta^\alpha = \lambda^\alpha(x) - \frac{1}{2}(\gamma^l\partial_l\chi(x))^\alpha$. Here the Majorana 2-spinor field $\lambda(x)$ is the superpartner of the gauge field $v_l(x)$ while the spinor $\chi(x)$ and scalar $v(x)$ are auxiliary fields. We use a Majorana representation for gamma matrices with $(\gamma^{0\alpha}{}_\beta) = i\sigma_2$, $(\gamma^{1\alpha}{}_\beta) = \sigma_1$, $(\gamma^{2\alpha}{}_\beta) = \sigma_3$ and define $(\epsilon^{\alpha\beta}) = i\sigma_2$, $(\epsilon_{\alpha\beta}) = -i\sigma_2$ where $\alpha, \beta = 1, 2$ are spinorial indices. A Majorana spinor then has real components. The spinors with lower index carry an upperbar for convenience with $\bar\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$ and it is easily shown that $\bar\psi_\alpha\xi^\alpha \equiv \bar\psi\xi$ is Lorentz invariant.

The generator of $N = 1$ supersymmetry transformations, Q^α, is given by $iQ^\alpha = (\partial/\partial\bar\theta_\alpha) - i(\gamma^l\theta)^\alpha\partial_l$ while the covariant spinorial derivative is $D^\alpha = (\partial/\partial\bar\theta_\alpha) + i(\gamma^l\theta)^\alpha\partial_l$ and $\bar D_\alpha = \epsilon_{\alpha\beta}D^\beta$. They satisfy $\{\bar D_\alpha, D^\beta\} = -2i\gamma^{l\beta}{}_\alpha\partial_l$.

The field strength superfield is defined by

$$W^{\alpha}(x,\theta) = \frac{i}{2}\bar{D}_{\beta}D^{\alpha}\Gamma^{\beta},$$
$$= \lambda^{\alpha}(x) + \frac{1}{2}\bar{\theta}_{\beta}(\epsilon^{lmn}f_{lm})\gamma_n^{\beta\alpha} + \frac{i}{2}\bar{\theta}\theta(\gamma^l\partial_l\lambda)^{\alpha}. \tag{24}$$

where $f_{lm} = \partial_l v_m - \partial_m v_l$ and the gauge superfield action is

$$I_g = \frac{1}{8}\int d^3x d^2\theta\, \bar{W}_{\alpha}W^{\alpha} \equiv \frac{1}{8}\int d^3x\, \bar{D}D(\bar{W}_{\alpha}W^{\alpha})|_{\theta=0} \tag{25}$$

The bosonic CS term is found to be contained in $\bar{\Gamma}W = \bar{\Gamma}\gamma^l\partial_l\Gamma - \frac{i}{2}\bar{\Gamma}D\bar{D}\Gamma$ and the action for the super CS term is written as

$$I_{c.s.} = -\frac{\kappa}{8}\int d^3x d^2\theta\bar{\Gamma}W \equiv -\frac{\kappa}{8}\int d^3x\, \bar{D}D(\bar{\Gamma}W)|_{\theta=0}. \tag{26}$$

Its expression in terms of the component fields is easily obtained in the supersymmetric gauge $\bar{D}\Gamma = 0$ which corresponds to setting $v = 0$, $\partial_l v^l = 0$ and $\chi = \frac{1}{\Box}(\gamma^l\partial_l\lambda)$.

The matter superfield is a complex scalar superfield

$$\Phi(x,\theta) = a(x) + i\bar{\theta}\psi(x) + i\bar{\theta}\theta f(x). \tag{27}$$

Here $a(x)$ is a complex scalar, $\psi^{\alpha}(x)$ its complex superpartner and $f(x)$ an auxiliary complex scalar. The gauge covariant spinorial derivatives may be defined to be

$$\nabla^{\alpha}\Phi = (D^{\alpha} + e\Gamma^{\alpha})\Phi, \qquad \bar{\nabla}^{\alpha}\Phi^* = (D^{\alpha} - e\Gamma^{\alpha})\Phi^*. \tag{28}$$

The following closure relation

$$\{\bar{\nabla}_{\alpha}, \nabla^{\beta}\} = -2i\gamma^{l\beta}{}_{\alpha}\nabla_l, \tag{29}$$

where $\nabla_l = (\partial_l + e\Gamma_l)$ and $\Gamma_l = \frac{i}{2}\bar{D}\gamma_l\Gamma$, is easily established. The Bianchi identities are satisfied due to the identity $\bar{D}W = 0$.

The matter action with minimal coupling is

$$I_m = \int d^3x d^2\theta \; (\frac{1}{4}\bar{\bar{\nabla}}_\alpha \Phi^* \nabla^\alpha \Phi + iV(|\Phi|^2)), \tag{30}$$

where V is the superpotential.

From the total action we obtain the following equations of motion

$$\frac{1}{4}\bar{\nabla}_\alpha \nabla^\alpha \Phi(x,\theta) = iV'(|\Phi|^2)\Phi, \tag{31}$$

$$(\gamma^l \partial_l W)^\alpha - \kappa W^\alpha = e(\Phi^* \nabla^\alpha \Phi - \Phi \bar{\nabla}^\alpha \Phi^*). \tag{32}$$

and the conservation of Noether's current requires

$$\bar{D}_\alpha(\Phi^* \nabla^\alpha \Phi - \Phi \bar{\nabla}^\alpha \Phi^*) = 0. \tag{33}$$

We adopt the supersymmetric gauge $\bar{D}\Gamma = 0$ and consider static configurations. The self-duality constraint on the matter superfield now takes the form[4]

$$\nabla^\alpha \Phi = i(\gamma^0 \nabla)^\alpha \Phi, \qquad \bar{\nabla}^\alpha \Phi^* = -i(\gamma^0 \bar{\nabla})^\alpha \Phi^*. \tag{34}$$

The eq.(33) is easily seen to be satisfied and we derive from eq.(32)

$$\Gamma^0 = \frac{2i}{e} V'(|\Phi|^2), \tag{35}$$

where $\Gamma^l = \frac{i}{2}\bar{D}\gamma^l \Gamma$ with $l = 0,1,2$ and the supersymmetric gauge corresponds to $\partial_l \Gamma^l = 0$.

In the absence of the (super) Maxwell term we derive from eq.(32)

$$\kappa F_{12} = -\frac{e}{2}\bar{D}D|\Phi|^2, \tag{36}$$

$$\kappa \Gamma_0 = ie(|\Phi|^2 - C^2). \tag{37}$$

where $F_{12} = (\partial_1 \Gamma_2 - \partial_2 \Gamma_1)$. From eqs.(35) and (37) we derive immediately the specfic superpotential

$$V(|\Phi|^2) = -\frac{e^2}{4\kappa}(|\Phi|^2 - C^2)^2. \tag{38}$$

For the case of vanishing κ the superpotential corresponding to the self-dual solutions is found by following a similar procedure. In both cases the supersymmetric actions contain the results of the purely bosonic theory as is easily shown by integrating the superfield action over θ and eliminating the auxiliary fields by using their eqs. of motion. The same is true of the supersymmetric self-duality condition when analysed in terms of the component fields. We obtain these results without the arguments for invoking an explicit N=2 supersymmetry of the action[7].

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