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AMBIGUITIES IN THE THIRRING MODEL UNDER CHIRAL ROTATIONS

by

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Abstract

We study how different parametrizations of the Thirring model behave under finite chiral rotations with the ζ-function regularization for the fermion determinant. We show that in some cases the contribution from the chiral Jacobian may change the sign of the factor which multiplies the quadratic term in the auxiliary field leading to an exponentially divergent theory. We also propose how this problem could be remedied by exploiting the invariance of the theory under finite renormalizations.

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Since Fujikawa's discovery[1] that the chiral anomalies can be obtained by examining the change in the fermionic functional measure under chiral rotations, there has been an increasing interest in the evaluation of the Jacobians associated with these transformations in the path integral framework. Recently Gamboa-Saraví, Muschietti, Schaposnik and Solomin (G-SMSS) proposed[2],[3] a very elegant method based on the ζ-function regularization[4] and Seeley's expansion coefficients[5], which permits to extend in a natural way the evaluation of chiral Jacobians to theories which contain non-hermitian operators. They have shown[6] that their method is equivalent to Fujikawa's for the case of hermitian operators but yields different results when non-hermitian operators are present..The question as to which method is correct has not been settled yet and further investigation is necessary.

Both approaches mentioned above have one thing in common, they require an action which is quadratic in the fermionic fields. If there are quartic fermionic terms it is usual to reduce them to quadratic terms using auxiliary fields. However, it is well known that for some models, like the Thirring model[7] for example, this reduction can be performed in several ways. Some ways lead to hermitian Dirac operators while others lead to non-hermitian ones. We made use of the G-SMSS method which treats on an equal foot hermitian and non-hermitian operators to study how the different parametrizations of the Thirring model behave under chiral rotations. We are going to show that some parametrizations apparently lead to ill defined theories whose Euclidian action becomes unbounded for some choices of the chiral rotation parameter. We also show this problem can probably be cured by performing a finite renormalization.

The Thirring model is described in two dimensional Euclidian space by the Lagrangian

$$L = i\bar{\psi}\partial\!\!\!/\psi - \frac{g^2}{2}(\bar{\psi}\gamma_\mu\psi)^2 \qquad (1)$$

where $\mu = 1,2$, $\partial\!\!\!/ = \gamma_\mu\partial_\mu$ and we choose a representation in which the γ_μ matrices are hermitian, namely $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$, $\gamma_5 = \sigma_3$ and $\epsilon_{12} = 1$.

In Euclidian space

$$\{\psi_\alpha,\psi_\beta\} = \{\bar{\psi}_\alpha,\bar{\psi}_\beta\} = \{\psi_\alpha,\bar{\psi}_\beta\} = 0, \quad \alpha,\beta = 1,2\ , \qquad (2)$$

hence if M is any 2x2 matrix then

$$(\bar{\psi}M\psi)^2 = \det M(\bar{\psi}\psi)^2 \qquad (3)$$

where detM stands for the determinant of M.The identity (3) and the Pauli matrices property that det $\sigma_i = -1$ may be used to rewrite the four fermion terms in several ways, for example

$$- \frac{g^2}{2}(\bar{\psi}\gamma_\mu\psi)^2 = \frac{g'^2}{2}(\bar{\psi}\psi)^2 = - \frac{g'^2}{2}(\bar{\psi}\gamma_5\psi)^2 = \frac{g^2}{2}[(\bar{\psi}\psi)^2-(\bar{\psi}\gamma_5\psi)^2], \tag{4}$$

where $g'^2 = 2g^2$. If we substitute the four-fermion term in (1) by the second term in (4) we obtain the well known equivalence between the Thirring model and the N=1 Gross Neveu model. Each term in (4) leads to a different parametrization of the Thirring model. After reducing the quartic terms in the resulting Lagrangians using the identity

$$\exp\left[\binom{+}{-}\frac{1}{2}A^2\right] = \int \mathcal{D}\varphi \ \exp\{-\int d^2x\left[\frac{1}{2}\varphi^2+\binom{1}{i}\varphi A\right]\}. \tag{5}$$

we obtain respectively

$$L_1 = \bar{\psi}(i\not\partial+g\not A)\psi+\frac{1}{2}A_\mu^2 \equiv \bar{\psi}D_1\psi + \frac{1}{2}A_\mu^2 , \tag{6a}$$

$$L_2 = \bar{\psi}(i\not\partial+ig\sigma)\psi+\frac{1}{2}\sigma^2 \equiv \bar{\psi}D_2\psi+\frac{1}{2}\sigma^2 , \tag{6b}$$

$$L_3 = \bar{\psi}(i\not\partial+g\gamma_5\chi)\psi+\frac{1}{2}\chi^2 \equiv \bar{\psi}D_3\psi+\frac{1}{2}\chi^2 , \tag{6c}$$

$$L_4 = \bar{\psi}(i\not\partial+ig\sigma+g\gamma_5\chi)\psi+\frac{1}{2}\sigma^2+\frac{1}{2}\chi^2 \equiv \bar{\psi}D_4\psi+\frac{1}{2}\sigma^2+\frac{1}{2}\chi^2 \tag{6d}$$

where A_μ, σ, χ are auxiliary fields and we have suppressed the prime in g. Notice that only D_1 and D_3

are hermitian.

Now the fermionic part of the generating functional has the structure

$$F = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[-\int\bar{\psi}D\psi d^2x] = \det D \; . \qquad (7)$$

A local chiral rotation over the fermionic fields is defined as

$$\psi = e^{r\gamma_5\alpha(x)}\psi' , \; \bar{\psi} = \bar{\psi}' e^{r\gamma_5\alpha(x)} \qquad (8)$$

where r is a real parameter ($0 \leqslant r \leqslant 1$) which is used by G-SMSS to compose finite chiral rotations from infinitesimal ones by iteration(3). We define

$$D_r = e^{r\gamma_5\alpha(x)} D \, e^{r\gamma_5\alpha(x)} = i\not{\partial}+a_0 \qquad (9)$$

Following Ref. (3) the symbol of the operator D_r is
$\sigma(D_r) = -\not{\xi}+a_0$
and the Seeley's coefficients necessary for d = 2(1+1) are

$$b_{-1}(x,\xi,\lambda) = -(\not{\xi}+\lambda)^{-1}, \quad b_{-2}(x,\xi,\lambda) = -b_{-1}a_0b_{-1} \; , \qquad (10)$$

$$b_{-3}(x,\xi,\lambda) = -b_{-2}a_0b_{-1} - i \frac{\partial b_{-1}}{\partial \xi_\mu} \frac{\partial a_0}{\partial x_\mu} b_{-1} \; .$$

G-SMSS have shown that the Jacobian J associated with (8) is

$$\ln J = -\frac{2i}{(2\pi)^2}\int d^2x \int_0^1 dr \int_{|\xi|=1} d\xi \int_0^\infty du \; Tr[b_{-3}(x,\xi,iu)\gamma_5]\alpha(x). \tag{11}$$

We calculated $Tr[b_{-3}(x,\xi,\lambda)\gamma_5]$ for $a_0 = A+\gamma_5 B+\not{P}+\gamma_5\not{Q}$. where A, B, P_μ, Q_μ may depend on x_μ, r and α. The result which is general enough for our purpose is

$$Tr[b_{-3}(x,\xi,\lambda)\gamma_5] = \frac{-4AB+2\lambda\;\epsilon_{\mu\nu}\;\partial_\mu P_\nu - i2\lambda\partial_\mu Q_\mu}{(\lambda^2-\xi^2)^2} \tag{12}$$

To illustrate these ideas let us evaluate the Jacobian for L_2 (6b). In this case

$$D_{r_2} = e^{r\gamma_5\alpha(x)} D_2 e^{r\gamma_5\alpha(x)} = i\not{\partial} - ir\gamma_5\not{\partial}\alpha + ig\sigma\cosh(2r\alpha) + \tag{13}$$

$$ig\sigma\;\gamma_5\;\sinh(2r\alpha),$$

from which we read

$$A = ig\sigma\cosh 2r\alpha,\; B = ig\sigma\sinh 2r\alpha,\; P_\mu = 0,\; Q_\mu = -ir\partial_\mu\alpha \tag{14}$$

Substituting (14) into (12) and the result into (11) we obtain

$$\ln J = -\int d^2x \frac{1}{2}\sigma^2(x)\frac{g^2}{2\pi}(\cosh 4\alpha(x)-1) + \frac{1}{2\pi}\int d^2x\,\alpha(x)\partial^2\alpha(x), \tag{15}$$

and the chiral-rotated generating functional

$$Z_2 = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}\sigma \exp\{ - \int d^2x[\overline{\psi}(i\not{\partial}+ig\sigma\, e^{2\gamma_5\alpha})\psi + + \frac{1}{2}(1+\frac{g^2}{2\pi}(\cosh 4\alpha-1))\sigma^2-\frac{1}{2\pi}\alpha\partial^2\alpha]\} \qquad (16)$$

If we integrate over σ

$$Z_2 = \int\mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\{- \int d^2x[\overline{\psi}i\not{\partial}\psi + \frac{g^2(\overline{\psi}\, e^{2\gamma_5\alpha}\psi)^2}{2[1+ \frac{g^2}{2\pi}(\cosh 4\alpha-1)]} - - \frac{1}{2\pi}\alpha\partial^2\alpha]\}, \qquad (17)$$

but equation (3) implies that $(\overline{\psi}\, e^{2\gamma_5\alpha}\psi)^2=(\overline{\psi}\psi)^2$ and so the effect of the chiral rotation amounts to a finite renormalization of the charge g^2 if α is constant.

Using expressions (11) and (12) we may easily repeat the exercise for the Lagrangians (6a), (6c), (6d) and we arrive at the following generating functionals

$$Z_1 = \int\mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \exp\{-\int[\overline{\psi}(i\not{\partial}+g\not{A}-i\gamma_5\not{\partial}\alpha)\psi + \frac{1}{2}A_\mu^2+ \frac{g}{2\pi}\alpha\, \varepsilon_{\mu\nu}F_{\mu\nu}-\frac{1}{2\pi}\alpha\partial^2\alpha]d^2x\} \qquad (18)$$

$$Z_3 = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\chi \ \exp\{-\int[\bar{\Psi}(i\not{\partial}+g\chi\gamma_5 e^{2\gamma_5\alpha})\psi$$

$$+\frac{1}{2}(1-\frac{g^2}{2\pi}(\cosh 4\alpha-1))\chi^2-\frac{1}{2\pi}\alpha\partial^2\alpha]d^2x\} \qquad (19)$$

$$Z_4 = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\mathcal{D}\chi \exp\{-\int[\bar{\Psi}(i\not{\partial}+g(i\sigma+\gamma_5\chi)e^{2\gamma_5\alpha})\psi +$$

$$+\frac{1}{2}(1+\frac{g^2}{2\pi}(\cosh 4\alpha-1))\sigma^2+\frac{1}{2}(1-\frac{g^2}{2\pi}(\cosh 4\alpha-1))\chi^2$$

$$-i\sigma\chi\frac{g^2}{2\pi}\sinh 4\alpha-\frac{1}{2\pi}\alpha\partial^2\alpha]d^2x\} \qquad (20)$$

In deriving expressions (16), (18) through (20) we have not neglected any total derivative. The results were written as they were obtained from expression (11). Several comments are in order.

Examining expression (18) for Z_T, we realize that unless the term proportional to $\epsilon_{\mu\nu}\ F_{\mu\nu} \sim \epsilon_{\mu\nu}\partial_\mu A_\nu$ is neglected the theory is not invariant under global chiral rotations (α = constant). Apparently the invariance of the Thirring model under global chiral rotations holds only in the trivial topological sector where one may safely neglect total derivatives. G-SMSS made some comments along these lines although they have not mentioned the term proportional to $\epsilon_{\mu\nu}\ F_{\mu\nu}$.

Expressions (19) and (20) for Z_3 and Z_4 respectively are singular for some choices of the chiral parameters. Indeed, for any value of $g^2 > 0$ it is possible to find α_0 such that

$$1 - \frac{g^2}{2\pi}(\cosh 4\alpha_0 - 1) = 0 \qquad (21)$$

If $\alpha > \alpha_0$ the factor which multiplies χ^2 changes sign and the theory becomes exponentially divergent. However this problem may be remedied, the Thirring model requires renormalization and one must add counter terms to the Lagrangian. By choosing correctly the χ field counter term ($\sim\chi^2$) one may cancel the unwanted $(-g^2/2\pi)(\cosh 4\alpha - 1)$ rendering the theory finite. An analogous observation holds for Z_2 where one may also eliminate the term proportional to σ^2 which comes from the Jacobian (see (16)) by means of finite renormalization. If α was allowed to be imaginary $\cosh 4\alpha \to \cos 4\alpha$ and the factor which multiplies σ^2 might also change sign whenever $g^2/2\pi \geq 1$. The necessity for this renormalization is not trivial. Usually counter terms are calculated order by order in perturbation theory assuming that g is small. Our calculations, on the other hand, are non perturbative, holding for all values of g, even for those values for which the perturbative series does not converge at all.

Our calculations are easily generalized for the chiral Gross-Neveu model(7) ($L = i\psi^a \not\partial \psi^a + (g^2/2)[(\psi^a\psi^a)^2 - (\psi^a\gamma_5\psi^a)^2]$, $a = 1,\ldots,N$). After reducing the quartic terms using auxiliary fields the N components of ψ decouple and one obtains the product of N terms identical to Z_4. Hence, the Jacobian is J_4^N and the factor which multiplies χ^2 becomes $(1/2)[1-(g^2N/2\pi)(\cosh 4\alpha - 1)]$ which may change sign if we choose α conveniently.

In summary, we have studied carefully how different parametrizations of the Thirring model behave under chiral rotations using the ζ-function regularization. We showed that all parametrizations yields a non-trivial Jacobian. The standard parametrization (6a) apparently is invariant under global chiral rotations only if one neglect total deerivatives. Other parametrizations on the other hand are invariant provided that one performs a finite renormalization after the chiral rotation. We have seen that for (6c) and (6d) ill defined theories may result if this renormalization is not performed. Finally this interesting result was obtained using the Lagrangian (6c) whose Dirac operator $i\not\partial + g\gamma_5\chi$ is hermitian and in this case the G-SMSS method is equivalent to other approaches.

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