ISSN 0029 - 3865

CBPF-NF-003/90

NEW SYMMETRIES FOR THE DIRAC EQUATION

by

Cesar A. LINHARES* and Juan A. MIGNACO

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*On leave of absence from Instituto de Estudos Avançados,
Centro Técnico Aeroespacial
12200 - São José dos Campos, SP - Brasil

ABSTRACT

We study the Dirac equation in four dimensions describing fermions both as 4 x 4 matrices and differential forms. We discuss in both formalisms its properties under transformations of the group SU(4). The importance of considering the minimal left ideals of the Clifford algebra of Dirac matrices and forms is stressed and their physical interpretation provided. Their relation to the Kogut-Susskind fermions on the lattice is exhibited.

Key-words: Dirac equation; Differential forms; Kogut-Susskind lattice fermions.

1 Introduction

As is well known, P.A.M. Dirac tried in 1928 [1] to reconcile the newly-born quantum mechanics where particles obeyed a Schrödinger equation for the wave functions,

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{1}$$

with the established relativistic dispersion relation for energy and momentum of a free particle:

$$E^2 = p^2c^2 + m^2c^4. \tag{2}$$

In order to achieve this, he proposed a new expression for the hamiltonian of the free particle, H, linear in the momentum:

$$H = \boldsymbol{\alpha}\cdot\boldsymbol{p}c + \beta mc^2, \tag{3}$$

with the condition that its square reproduced the relativistic dispersion relation (2) for stationary solutions:

$$E\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{4}$$

The coefficients $\boldsymbol{\alpha}$ and β were soon considered as members of a matrix algebra, and they were shown to fit in a framework known as Clifford algebras, since they had to satisfy anticommutation relations:

$$\alpha_k\alpha_\ell + \alpha_\ell\alpha_k = 2\delta_{k\ell}, \qquad l,\ell = 1,2,3 \tag{5}$$

$$\beta\alpha_k + \alpha_k\beta = 0, \qquad k = 1,2,3. \tag{6}$$

This led to dealing with the solutions ψ as being one-column matrices; they received the name of spinors, for they were considered an extension of the one-column two-component matrices required by the introduction of spin in non-relativistic quantum mechanics.

The notion of relativistic invariance was improved through the adoption of the covariant version of eq. (1), that is, a set of coupled first-order differential equations for the components of the spinor ψ, identical to the ones involved in eq. (1) with eq. (3) for the hamiltonian. The relativistic covariant equation is universally known as the Dirac equation, and reads

$$\left(i\gamma^\mu\frac{\partial}{\partial x^\mu} + \frac{mc}{\hbar}\right)\psi = 0 \tag{7}$$

with

$$\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}, \qquad \gamma^0 = \beta. \tag{8}$$

The new matrices γ^μ, $\mu = 0, \ldots, 3$, now span a Clifford algebra through their fundamental relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \tag{9}$$

where $g^{\mu\nu}$ is the metric tensor for flat spacetime.

The Dirac equation was explored and extended in the subsequent decades to spacetimes of higher dimension and varied signature, curvature was introduced, and it was further considered as a prototype formalism for relativistic particles of any spin in the Duffin–Kemmer–Pétiau approach [2].

An interesting development took place in the fifties, when the symmetry structure of the strong and specially of the weak interactions were unveiled. The relevance of time-reversal invariance and charge-conjugation invariance in several processes and the discovery of parity non-conservation for weak interactions led to a quite exhaustive investigation of the symmetry transformations of the solutions of the Dirac equation.[1]

Two decades later, when everything linked to the Dirac equation looked settled, an apparently inocuous step, namely, carrying the description of Dirac particles to a discretized spacetime, brought forth a surprise: there were apparently too many species of particles surviving after the limit back to the continuum case was taken. The origin of this feature, and how to overcome its problems, led to much work for the specialists, with many different procedures claiming to be satisfactory. There is an implicit collective feeling that one is dealing with a spurious problem, somewhat foreign to the Dirac equation, up to that time a nice, softly, well-behaved formalism.

In this article we present a new kinematical framework for the particles described by the Dirac equation, which, of course, provides the answers to the same questions dating from Dirac's time, and simultaneously embraces the more recent developments. This framework results from the union of several ingredients that were incorporated during the study of the Dirac equation along successive developments by several authors.

Ultimately, this framework allows one to perceive a basic geometrical and algebraic structure, which is, indeed, quite natural. The Dirac equation, its hamiltonian version and the symmetry transformation properties of the solutions, in the continuum as well as in the discrete spacetime, are correctly, elegantly and easily described.

To attain these results, one needs to reconsider the somewhat neglected features of the formalism, in which "Dirac matrices" γ^μ had the rôle of some auxiliary, secondary structure, needed only to frame the basic theoretical principles of the problem at hand. We shall show that they perform a more important task, at the foundation of the construction of a suitable formalism.

[1] A recent update on this matter is contained in [3].

The structure of this article, which is intended to be an introductory reference for more comprehensive work, involves a treatment of the problem based on Dirac matrices presented in the second section. The third section is a review, based on the isomorphism between Dirac matrices and differential forms [4], of the geometrical formulation of the problem. It includes an alternative derivation of the solution to the original Dirac problem, quite attractive on geometrical grounds and on its appealing relation with classical mechanics of a point particle.

To conclude, we discuss a little more the relationship between the treatment by differential forms and the original one employing matrices, and hint on future, necessary developments.

Most of the material included here has been the object of various articles, published, submitted, or under preparation [5,6,7,8,9]. We welcome the opportunity to gather all of them in our contribution to this book.

2 Dirac matrices and Dirac equation: more than sought originally

Most emphasis on the algebraic structure of Dirac matrices is customarily given to its Clifford anticommutation property. Besides, they are important to build the so-called bilinear objects:

$$\overline{\psi}\psi,\ \overline{\psi}\gamma^\mu\psi,\ \overline{\psi}\sigma_{\mu\nu}\psi,\ \overline{\psi}\gamma^\mu\gamma^5\psi,\ \overline{\psi}\gamma^5\psi,$$

which are quite relevant from the physical point of view. It is usually recalled, further, that including the identity, the $\gamma^K = \gamma^\mu, \gamma^\mu\gamma^5, \sigma_{\mu\nu}, \gamma^5$ form a ring, which we shall call the Dirac ring.

By suitably multiplying by the imaginary unit, all of them, in any metric for spacetime, become hermitian. All of them, moreover, are traceless.[2]

For a long time, the fact was ignored that the γ^K constitute a Lie algebra, due to their commutator structure. If we denote by capital latin letters indices indicating products of Dirac matrices, we have shown [7] that they are closed under commutation, i.e., for any pair of matrices of the Dirac ring, a third one corresponds to its commutator:

$$[\gamma^K, \gamma^L] = C^{KL}{}_M\gamma^M. \tag{10}$$

They are also closed under the Jacobi identity:

$$[\gamma^K, [\gamma^L, \gamma^M]] + [\gamma^L, [\gamma^M, \gamma^K]] + [\gamma^M, [\gamma^K, \gamma^L]] = 0. \tag{11}$$

[2] In particular, if one takes as the fourth coordinate $x^0 = ict$, the hermitian structure results naturally.

The fifteen hermitian matrices with null trace then form a Lie algebra. They can thus be associated with some continuous group of transformations.

To match a Lie algebra with a continuous group was accomplished by Cartan long ago, before the Dirac equation was ever written. It turns out to be essential to look at the set of mutually commuting generators, or matrices, in our case, that form what is called a Cartan subalgebra.

For the Dirac matrices, there are always three of them that are diagonal, i.e., commuting. This fixes the related continuous group of transformations to be SU(4) (or O(6), as they are isomorphic; we prefer to choose SU(4), since for other dimensions n, they are $SU(2^{n/2})$, for even n).

A glance at the table of commutators in our article [7] shows that there are several trios of commuting matrices that can span a Cartan subalgebra. The most popular ones are

$$\gamma^0, \gamma^1\gamma^2, i\gamma^0\gamma^1\gamma^2 \qquad \text{and} \qquad \gamma^0\gamma^3, i\gamma^1\gamma^2, \gamma^5.$$

People select a Cartan subalgebra when choosing to work in a given "picture" for the Dirac matrices. The choices referred to above correspond to the Dirac–Pauli and the Kramers–Weyl (chiral) pictures, respectively (see [10] for definitions).[3] There is a one-to-one relationship between the choice of picture and the choice of Cartan subalgebra. Pictures were used before anyone observed the existence of the SU(4) commutation property of Dirac matrices, but the new viewpoint expands considerably the meaning of a picture.

In any picture, one can build a matrix with just a single column filled with non-zero elements, those being the components of the spinor solution to the Dirac equation. For instance,

$$\Psi^{[2]} \equiv \begin{pmatrix} 0 & \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 & 0 \\ 0 & \psi_3 & 0 & 0 \\ 0 & \psi_4 & 0 & 0 \end{pmatrix}. \tag{12}$$

[3]We prefer to use the word *picture* instead of the more usual *representation*, to avoid confusion with the group-theoretical interpretation: in fact, in any picture (Dirac–Pauli, Kramers–Weyl,...) Dirac matrices are a representation of the Lie algebra of generators of SU(4).

Acting on $\Psi^{[2]}$ with the Dirac operator is analogous to acting on a Dirac spinor. This is because the Dirac ring has minimal left ideals, of which $\Psi^{[2]}$ is an example. A left ideal is defined as being invariant under *left multiplication* by another member of the ring. That is, the result of the multiplication belongs to the same ideal.

Thus, in any picture, one can associate four minimal left ideals to a given Dirac spinor, which behave the same under operation by the Dirac operator.

Moreover, there is a close link between the minimal left ideals and the picture defined by the relevant Cartan subalgebra. As a matter of fact, one can build projection operators with the unit matrix and different combinations of the three members of the Cartan subalgebra. There are four of these projection operators, and one can label a given minimal ideal by the eigenvalues of the Cartan subalgebra matrices, or, complementarily, by the signs involved in the construction. If we call $\mathcal{C}^{[1]}, \mathcal{C}^{[2]}, \mathcal{C}^{[3]}$ the three matrices, the projection operators are

$$\begin{gathered}
\tfrac{1}{4}(I + \mathcal{C}^{[1]} + \mathcal{C}^{[2]} + \mathcal{C}^{[3]}) \\
\tfrac{1}{4}(I + \mathcal{C}^{[1]} - \mathcal{C}^{[2]} - \mathcal{C}^{[3]}) \\
\tfrac{1}{4}(I - \mathcal{C}^{[1]} + \mathcal{C}^{[2]} - \mathcal{C}^{[3]}) \\
\tfrac{1}{4}(I - \mathcal{C}^{[1]} - \mathcal{C}^{[2]} + \mathcal{C}^{[3]}).
\end{gathered}$$

The relevant point is that one can show explicitly that under space inversion, P, time reversal, T, and their product, PT, the Cartan subalgebra matrices can transform among themselves (the specific transformation depending on the picture) and, consequently, the projection operators above. There is a corresponding transformation for the spinor, solution to the Dirac equation, but with some undeterminacy with regard to sign. For instance, under space inversion,

$$\psi(t, \boldsymbol{x}) = \psi(x) \to \psi'(t, -\boldsymbol{x}) = \pm\gamma^0\psi(x). \tag{13}$$

This is precisely the relative change of sign when we go from one minimal left ideal to another one by space inversion. Some keep the sign, some change. Thus, in terms of minimal left ideals, the discrete operations allow one to link them with phases that are the ones allowed by the consistency of the Dirac equation.

We have sketched the main points arising from the SU(4) algebraic structure of the Dirac matrices. But, what sort of transformations are performed by the corresponding Lie group? A change of picture, leading, for instance, from the Dirac–Pauli one to the Kramers–Weyl one, can be taken as a change of relative orientation of the reference frame in the vector space of the Lie algebra while preserving the spacetime frame. Both the Dirac equation and the Dirac–Schrödinger equation transform covariantly (form invariance):

$$(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar})\psi = 0 \quad \overset{\substack{\psi\to S\psi \\ \gamma^\mu\to S\gamma^\mu S^{-1}=\gamma'^\mu}}{\longrightarrow} \quad (i\gamma'^\mu\partial_\mu - \frac{mc}{\hbar})\psi' = 0 \tag{14}$$

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} \quad \longrightarrow \quad H'\psi' = i\hbar\frac{\partial\psi'}{\partial t}. \tag{15}$$

Usually, transformations on spacetime variables seem to decouple from transformation in the picture of Dirac matrices. The main property is the invariance of the basic anticommutation (Clifford) relation under similarity (unitary) transformations. But any unitary operator is an SU(4) element, and can be expanded in the basis of Dirac matrices and the identity.

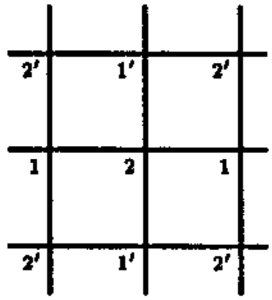
In fact, the apparent independence of change of picture for Dirac matrices and spacetime covariance is untenable. Rotations and proper Lorentz transformations are of the form

$$\exp(a_{\mu\nu}\gamma^\mu\gamma^\nu) \tag{16}$$

and to keep the equation invariant it is necessary that a simultaneous change should occur for spacetime coordinates, as well as for Dirac matrices. The nine remaining transformations of the Dirac matrices encompass the following:

$$\exp(a_\mu\gamma^\mu),\ \exp(a'_\mu\gamma^5\gamma^\mu),\ \exp(a''\gamma^5). \tag{17}$$

Needless to say, these would be expected to have many dynamical consequences when applied to spinors, and should be further investigated. The quantities a_μ, a'_μ, a'' and $a_{\mu\nu}$ above are constants. In any case, the first two classes of transformations have the form that preserves gauge invariance under vector and axial-vector transformations. The last reminds one of the chiral transformations widely used to investigate axial anomalies, or, equivalently, the non-invariance of the fermionic measure in path-integral quantization [11].



To conclude this section, we briefly refer to the proliferation problem of the fermionic spectrum on the lattice, i.e., the fact that, if spacetime is discretized, the "naive" finite-difference Dirac equation implies the appearence of further species of particles in the borders of the first Brillouin zone which remain when one takes the limit back to the continuum. This feature was first observed by Wilson himself [12], who also proposed an alteration in the lattice formalism which allowed one to keep just one species in the continuum limit. According to Wilson, 2^n species (where n is the even dimension of spacetime) should survive the continuum limit, unless some ad hoc term were added to the usual hamiltonian.

Kogut and Susskind [13] elaborated a scheme in which, instead of putting the whole spinor at each lattice site, as in the Wilson formulation, spinor components were distributed among sites. This resulted in a reduction of the number of species to $2^{n/2}$. In two dimensions, their proposal can be seen to introduce two species, such that in a square lattice the spinor components ψ_1, ψ_2, $\psi_{1'}$, $\psi_{2'}$ alternate (see figure). What we have shown [5] is that the species of the Kogut–Susskind scheme may be put in one-to-one correspondence with the relevant minimal left ideals of the algebra of 2-d γ matrices, which is su(2). Or, in other terms, discretization and the Kogut–Susskind procedure led to the need of using the full spinor content of the ideals of the algebra. Each Kogut–Susskind species represent one of the two (in two dimensions) minimal left ideals corresponding to a given Cartan subalgebra of su(2). This is also a result found by Becher and Joos [14] in the framework of differential forms that we shall consider in what follows. In four dimensions, the situation is entirely analogous and the qualitative conclusions are the same as in the bidimensional case.

What is the reason for the behaviour of found for fermions on the lattice? A simple analysis allows one to understand it. For convenience, let us consider again the two-dimensional situation. A site is determined on the lattice by a pair of integers, (M, N). Let Δ_x be the operator which translates one lattice site in the spatial direction, that is,

$$(M, N) \xrightarrow{\Delta_x} (M+1, N).$$

This operation, which is essential for lattice invariance, does not commute with the analog of parity, or space inversion, P:

$$(M, N) \xrightarrow{P} (-M, N).$$

To show this, consider the following sequences:

$$(M, N) \xrightarrow{\Delta_x} (M+1, N) \xrightarrow{P} (-(M+1), N)$$

$$(M, N) \xrightarrow{P} (-M, N) \xrightarrow{\Delta_x} (-(M-1), N).$$

The subsequent mixing of P determines that all minimal left ideals have to play a role on the lattice. By the way, looking at the figure one understands also why the Kogut–Susskind recipe works.

3 Differential forms and Dirac equations: algebra, geometry and physics meet

At the time Dirac elaborated on the problem of a relativistic equation for spin-1/2 particles, differential forms were just being introduced by Cartan. Matrices, however, became familiar with the advent of quantum mechanics, to which Dirac himself contributed almost since the beginning. It comes as no surprise, then, that as tools for his work Dirac used mainly matrices.

Though the concept of a differential form is familiar from classical mechanics and thermodynamics, where differentials whose coefficients cannot be written as gradients of some function were known since the original mathematical treatment, only the development of differential geometry allowed a systematic development.

Differentials and their wedge (exterior) product may be considered as elementary differential-geometric objects. A 1-form, or differential, represents an infinitesimal element of a line; a 2-form, the wedge product of two 1-forms, some differential area element, and so on. Choosing differentials along some coordinate reference frame, a wedge product of two 1-forms is defined as antisymmetric:

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, \tag{18}$$

with the obvious consequence that

$$dx^\mu \wedge dx^\nu = 0, \quad \text{for } \mu = \nu. \tag{19}$$

This characterizes a Graßmann algebra. In a spacetime of dimension n, the differentials allow, through wedge products, the construction of forms up to the n-th degree. Ordinary functions, which are to be associated with points in the space, are looked upon in this formalism as 0-forms, while all n-forms are proportional to the volume element, $\varepsilon = dx^0 \wedge dx^1 \wedge \ldots \wedge dx^n$.

It is easy to show that at each spacetime point p-forms span a linear vector space, of dimension $\binom{n}{p}$, which will be denoted below by Λ^p, and that the total number of independent forms for a n-dimensional spacetime is 2^n.

1-forms can be considered either globally, as objects that constitute the natural integrands for line integrals, or locally, as vectors in the cotangent space at a point of a manifold.

Operations that mix different linear p-form-vector spaces may be defined. Among them, the Hodge dual star operator:

$$* : \Lambda^p \longrightarrow \Lambda^{n-p},$$

which may be defined through its action on the basis elements of Λ^p. In four-dimensional Minkowski spacetime, it reads ($\epsilon_{0123} = 1$):

$$\begin{aligned} *1 &= \varepsilon \\ *dx^\mu &= \frac{1}{3!}\epsilon^\mu{}_{\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ *dx^\mu \wedge dx^\nu &= \frac{1}{2!}\epsilon^{\mu\nu}{}_{\rho\sigma} dx^\rho \wedge dx^\sigma \\ *dx^\mu \wedge dx^\nu \wedge dx^\rho &= \epsilon^{\mu\nu\rho}{}_\sigma dx^\sigma \\ *\varepsilon &= 1. \end{aligned}$$

Another interesting operator is the exterior differential

$$d : \Lambda^p \longrightarrow \Lambda^{p+1},$$

defined by

$$d = dx^\mu \wedge \partial_\mu. \tag{20}$$

The first allows one to introduce a metric in the vector space of differential p-forms: if ω and η are two p-forms, its scalar product is defined as

$$(\omega, \eta) = \int \omega \wedge *\eta. \tag{21}$$

(Notice that the integrand is an n-form, so that it is proportional to $d^n x$.) In terms of this product, a formal adjoint to the operator d is introduced. If now ω is a p-form and η a $(p+1)$-form, then define the operator δ as

$$(\eta, d\omega) = (\delta\eta, \omega), \tag{22}$$

which shows that δ brings a $(p+1)$-form into a p-form:

$$\delta : \Lambda^{p+1} \longrightarrow \Lambda^p.$$

In any even-dimensional spacetime,

$$\delta = - * d * . \tag{23}$$

Because of the antisymmetry of the wedge product, both differential operators are nilpotent:

$$d^2 = \delta^2 = 0. \tag{24}$$

Graf [4] saw that it was possible to establish an isomorphism between the differential forms in dimension n and the Dirac matrices in the same dimension. In order to do that, it was necessary to use the "inner calculus" formalism established in the early sixties by the mathematician Erich Kähler [15], in which a new product operator between differential forms, closely related to the wedge product, but not nilpotent, was defined. We shall call it the Clifford product, and its definition is

$$dx^\mu \vee dx^\nu = dx^\mu \wedge dx^\nu + g^{\mu\nu}, \tag{25}$$

where $g^{\mu\nu}$ is the metric tensor for flat spacetime. With this product, differential 1-forms generate a Clifford algebra:

$$dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2g^{\mu\nu} \tag{26}$$

and the isomorphism with γ matrices results. Graf then obtained a Lorentz-covariant translation of the Dirac differential operator into the language of forms. It turned out to be

$$i\gamma^\mu \partial_\mu \longrightarrow i(d+\delta). \tag{27}$$

But, on which kind of object is this operator supposed to act? As Becher and Joos [14] stressed, the analog of Dirac spinors (which we have seen in the previous section to be identified with the ideals of the matrix algebra) are now the minimal left ideals of the space of all differential forms. That is, a form belonging to a minimal left ideal under Clifford left multiplication goes into another member of the ideal. The differential equation corresponding to the Dirac equation in the present formalism is called the Dirac–Kähler equation and reads

$$[i(d+\delta) - m]\,\phi = 0, \tag{28}$$

where ϕ is now a differential-form minimal left ideal, with components identified with the Dirac spinor ones.

Let us digress now and propose another solution to the original Dirac problem: to have a first-order formalism such that one could by multiplication by a suitable operator recover the relativistic dispersion relation, eq. (2). Let us try, in four dimensions, with an operator on differential forms:

$$Edx^0 = -p_k dx^k + m. \tag{29}$$

This is quite appealing from a geometrical point of view, since it involves the line element in four dimensions, and also the integrand for the classical mechanical action. Clifford left multiplying this by

$$p_k dx^k + Edx^0 + m,$$

after gathering all terms in the left-hand side, we obtain eq. (2). That is, we have another solution to the Dirac problem, or a hamiltonian

$$H = p_k dx^0 \wedge dx^k + m dx^0, \tag{30}$$

which, acting on a minimal left ideal of the space of differential forms (all) should reproduce the Dirac equation.

Now we can repeat the set of considerations made for γ matrices in the previous section. That is, denoting by dx^K, $K = 1, \ldots, 15$, any differential of the set

$$\{dx^\mu,\, dx^\mu \wedge dx^\nu\, (\mu < \nu),$$
$$dx^\mu \wedge dx^\nu \wedge dx^\rho\, (\mu < \nu < \rho),\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\},$$

we compute all possible Clifford commutators of forms, defined by [14] as

$$[dx^K, dx^L]_\vee = dx^K \vee dx^L - dx^L \vee dx^K, \tag{31}$$

The result is that one obtains a closed Lie algebra,

$$\left[dx^K, dx^L\right]_\vee = C^{KL}{}_M dx^M. \tag{32}$$

But we have no matrices here, only algebra generators, expressed as forms. However, through the Graf isomorphism, it is clear that the structure coefficients of eq. (32) are the same as those of eq. (10), thereby establishing the Lie algebra of the SU(4) group also in this context. That is, any element g of SU(4) has a representation in terms of differential forms:

$$g = \exp(a_K dx^K), \tag{33}$$

which certainly is a complete novelty. We recall that in (33) the multiplication of forms is done with the Clifford operator $\vee$.

In particular, the fact that we have again sets of three mutually commuting differential forms allows us to introduce here the concept of picture, as related to a Cartan subalgebra of differential forms. Consequently, also the minimal left ideals can be related to a picture.

An important point to be stressed is that, whereas the original Dirac operator is covariant, the Dirac–Kähler operator is invariant: it is the same whatever the picture or the reference frame chosen.

The Graf isomorphism between Dirac matrices and differential forms with Clifford product was extensively used by Becher and Joos [14], who took advantage of it to analyze also the problem of fermions on the lattice. They were able to show that the lattice Dirac-Kähler formalism was entirely equivalent to the Kogut-Susskind procedure.

We suggest that, in our framework, this isomorphism can be further exploited to deduce the general group-theoretical framework of the description of fermions in any number of dimensions. For even dimension n, the number of differential forms is 2^n. Extracting the 0-form subspace, we get $2^n - 1$, which is precisely the number of generators of $su(2^{n/2})$. It just remains to convince ourselves that the number of mutually (Clifford) commuting differential forms is $2^{n/2} - 1$.

For dimension n an odd integer, it is easy to see that the Hodge operator allows one to separate differential forms into two sets of $2 \times \frac{n-1}{2}$ components, suggesting that fermions in odd dimensions correspond to ideals of an $SU(2^{(n-1)/2}) \otimes SU(2^{(n-1)/2})$ group.

Finally, we comment on the relation to topology of the description of fermions through differential forms with a Clifford product. It could be crudely said that the latter correspond to a coherent mixture of $2^{n/2}$ spinors. To check this, we have analyzed the abelian axial anomaly in a theory of a Yang-Mills gauge field coupled to fermions described by the Dirac-Kähler scheme [6]. We have studied the topological index of the gauge-covariant Dirac-Kähler (signature) operator by explicit computation of the path-integral fermionic determinant through the heat-kernel method, expressed in terms of the Seeley coefficients, along the lines exposed in [11]. We been able to show that the index is $2^{n/2}$ times the index for spinors, and, correspondingly, the value for the anomaly is $2^{n/2}$ larger, in agreement with the lattice calculation, following the work of Becher and Joos [14], performed by Göckeler [16]. As the Dirac-Kähler formalism on the lattice is equivalent to the Kogut-Susskind treatment of spinors, one should also expect this to be the result for the latter.

4 Final comments and possible lines of development

We think we have already reached two important conclusions of a rather lengthy study:

- The natural kinematical framework for studying the Dirac equation follows from the SU(4) (in four dimensions) algebra spanned by Dirac matrices of differential forms;
- The description of relativistic spin-1/2 particles through differential forms should deserve greater consideration, because of its geometrical and algebraic features.

A possible development of great interest has been mentioned above, namely, to make clear the group-theoretical framework for any dimension. Also, the generalization to curved spaces of the whole scheme is highly desirable, as departures from the results of the general-relativistic treatment of the spinorial formalism have been detected [17]. Another incongruence appears in the expression of the topological index for the signature operator, as its spacetime-curvature term has a different factor with respect to the corresponding one for the Dirac operator (see [18]). In connection with these considerations, if one permits the SU(4) symmetry to be local, we see from the generators in eq. (16) that the theory should contain gravitation.

It is also interesting to look at the problem backwards from the framework of differential forms. A recent work by Veltman on gamma matrices [19] introduces in fact several properties that are the translation of similar features for differential forms.

What looks to be of considerable interest is to define the picture in the treatment of the equation as a possibly local property. This may lead to a gauge-like treatment of spin-1/2 particles that in some sense would apply to them the same considerations regarding locality in the choice of frames which are usually applied to Yang-Mills gauge fields.

Another path of possibly interesting developments corresponds to considering carefully what would be the nature of the associated fiber bundle for describing fermions, taking into account properly the natural kinematical group framework referred to in this article.

References

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A117**, 610 (1928).

[2] See E. M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave Equations*, Blackie and Son, Glasgow, 1953.

[3] R. N. Decker, Fortsch. der Phys. **37**, 657 (1989).

[4] W. Graf, Ann. Inst. H. Poincaré **A29**, 85 (1978).

[5] C. A. Linhares and J. A. Mignaco, Phys. Lett. **153B**, 82 (1985).

[6] C. A. Linhares, J. A. Mignaco and M. A. Rego Monteiro, Lett. Math. Phys. **10**, 79 (1985).

[7] C. A. Linhares and J. A. Mignaco, "SU(4) properties of the Dirac equation", IEAv–CBPF preprint, submitted for publication.

[8] C. A. Linhares and J. A. Mignaco, "SU(4) properties of the Dirac–Kähler equation", in preparation.

[9] C. A. Linhares and J. A. Mignaco, "The 'true' Dirac equation", in preparation.

[10] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1981.

[11] R. E. Gamboa-Saraví, M. A. Muschietti, F. A. Schaposnik and J. E. Solomin, Ann. Phys. (NY) **157**, 360 (1984).

[12] K. G. Wilson, in *New Phenomena in Subnuclear Physics*, A. Zichichi, ed. (Erice 1975), Plenum Press, New York, 1979.

[13] J. B. Kogut and L. Susskind, Phys. Rev. **D11**, 295; 3594 (1975); J. B. Kogut, Rev. Mod. Phys. **55**, 775 (1983); L. Susskind, Phys. Rev. **D16**, 3031 (1977).

[14] P. Becher and H. Joos, Z. Phys. C **15**, 343 (1982).

[15] E. Kähler, Abh. Dt. Akad. Wiss. Berlin, Kl. für Math., Phys. u. Tech., Jahrg. 1960, Nr. 4; 1961, Nr. 1; Rendiconti di Matematica (Roma), Ser. V, **21**, 425 (1962).

[16] M. Göckeler, Nucl. Phys. B**244**, 508 (1983).

[17] T. Banks, Y. Dothan and D. Horn, Phys. Lett. **117B**, 413 (1982).

[18] T. Eguchi, P. B. Gilkey and A. J. Hanson, Phys. Rep. **66**, 213 (1980).

[19] M. Veltman, Nucl. Phys. B**319**, 253 (1989).