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SEPARABLE COORDINATES AND PARTICLE CREATION III: ACCELERATING, RINDLER AND MILNE VACUA

by

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Abstract

We compare the two vacua associated with accelerating observers to Rindler vacuum and to the Milne vacua, by means of Bogolubov coefficients. This confirm previous results of the literature that say that two of these vacua are equivalent to Cartesian vacuum, and the three others behave like a thermal gas in the Cartesian vacuum.

Key-words: Quantum field theory; Non-stationary coordinates; Bogolubov coefficients.

1. Introduction

In this series of paper we study a massive scalar quantum field in coordinate systems that are non-static but also simple enough to allow the separation of the Klein-Gordon equation. We hope that the understanding of these non-trivial examples will throw some light on the concept of particles outside the frame of the Poincare group. In the first paper[1] we described the separable orthogonal coordinate systems of the two dimensional Minkowski space. In the second paper[2] we picked up one of these systems, where stationary observers are inertial in the past and become constant accelerated in the future. These observers are somehow more interesting than uniform accelerated observers as they allow to compare two particle definitions in the framework of only one coordinate system: We defined a set of positive frequency modes of the scalar field in the phase where the observers are inertial and another one in the phase of constant acceleration. We found a thermal spectrum of inertial particles in the accelerated vacuum at a temperature proportional to the asymptotic acceleration of the observers, thus confirming the well known interpretation of the Fulling effect[3] in Rindler coordinates. We also compared these vacua to the cartesian one, that is plane waves, and saw that the accelerated vacuum also has a thermal spectrum at the same temperature but that the inertial vacuum is not thermal.

Here we investigate this problem further, first trying to sample more information about the modes themselves, i. e., by calculating the Bogolubov coefficients between them and the natural modes of Rindler and Milne coordinates. These coordinate systems are in a sense more adequate than cartesian one because they are the right asymptotes of our coordinate system.

In a next paper we will finally conclude the investigation of this coordinate system by computing more physical magnitudes, like the Feynmann propagator and the Hamiltonian. We will then proceed by studying a coordinate system where the observers are inertial in both time asymptotes only suffering a boost on their velocities.

The paper is organized as follows. In the next section we define all the Fock spaces we will handle with. In section 3 we briefly compare them. The conclusions are taken in section 4.

2. Characterization of the vacuum states

We will study the quantization of a massive scalar neutral field in curvilinear coordinate systems. For that we compare six different Fock spaces which are constructed in the usual manner through the complete function set next defined.

2.1. Minkowski Cartesian modes:

$$\psi_k^M(t, x) \; := \; \frac{1}{\sqrt{4\pi\varepsilon}} \, e^{-i(\varepsilon t - kx)} \tag{1}$$

These are plane waves with mass m, wave vector k and positive frequency $\varepsilon = +\sqrt{k^2 + m^2}$. The basis is completed with their complex conjugate (which we will after this point omit). The concept of positive frequency appears in, at least, two ways: First, we note that $i\, \partial/\partial t$ is a Killing vector field and must have ψ_k as eigenfunction. Their eigenvalue is the frequency:

$$i \frac{\partial}{\partial t} \psi_k \;=\; \varepsilon\, \psi_k . \tag{2}$$

We use the additional fact that the Hamiltonian operator is defined by

$$\mathcal{H} = \frac{1}{2} \int_{-\infty}^{\infty} \left[\left(\frac{\partial\phi}{\partial t} \right)^2 + \left(\frac{\partial\phi}{\partial x} \right)^2 + m^2 \phi^2 \right] dx \qquad (3)$$

in such a way that it generate time (t) translations:

$$\left[\phi\ (t,x),\ \mathcal{H}\ (x) \right] = i\ \frac{\partial}{\partial\ t}\ \phi \quad . \qquad (4)$$

Second, we may ask that the positive frequency mode should go to zero like exp $(-\varepsilon t)$ when t goes to $-i\infty$. The plane waves obey, of course, these two criteria.

2.2. Rindler modes:

These modes are more easy defined in Rindler coordinates[4], that is

$$\begin{cases} t = X_R \ \sinh a\ T_R \\ x = X_R \ \cosh a\ T_R \end{cases} , \qquad (5)$$

for $0 < X_R < \infty$ and $-\infty < T_R < \infty$, where we write

$$\psi_\mu^R\ (T_R,\ X_R) = \frac{1}{\pi} \sqrt{\sinh \pi\mu}\quad e^{-i\mu a T_R}\ K_{i\mu}(m\ X_R) \qquad (6)$$

where $K_{i\mu}$ is a modified Bessel function[5] and $\mu>0$. These modes were first used by Fulling in his PhD thesis, and have since been well studied[6]. Rindler coordinates are adapted to observers with constant acceleration, in the sense that the coordinate lines describe world lines of constant acceleration. The field $\partial/\partial T_R$ is a Killing vector field, fact that allowed Sanchez[7] to solve even the inverse problem of finding the coordinates transformation when one is given the vacuum spectrum. ν is the frequency associated to this time in the two previous senses, as you may verify.

2.3. Sommerfield modes:

These and the next are the modes associated to geodesic observers living in Milne universe[8]. They are indeed more easy written in Milne coordinates:

$$\begin{cases} t = Y_M \cosh a X_M \\ x = Y_M \sinh a X_M \end{cases}, \tag{7}$$

for $0 < Y_M < \infty$ and $-\infty < X_M < \infty$. It is useful to introduce another variable T_M, defined by:

$$Y_M = \frac{1}{a} e^{aT_M} . \tag{8}$$

This mapping covers the future light cone. In this paper we will be also interested in the past light cone, where

$$\begin{cases} t = - Y_M \cosh a X_M \\ x = - Y_M \sinh a X_M \end{cases}, \tag{7'}$$

and

$$Y_M = \frac{1}{a} e^{-aT_M} . \tag{8'}$$

Here the two frequency definitions split: the time coordinate lines are not trajectories of a Killing vector field and the modes are not eigenfunctions of $\partial/\partial T_M$. What we can do is twice: first we may, like Sommerfield[9], choose the first frequency concept by defining a dilatation operator $\mathcal{D}$ as

$$\mathcal{D} = \frac{1}{2} \int_{-\infty}^{\infty} \left[\left(\frac{\partial \phi}{\partial T_M} \right)^2 + \left(\frac{\partial \phi}{\partial X_M} \right)^2 + e^{2aT_M} m^2 \phi^2 \right] dx \tag{9}$$

such that it generates time translations

$$\left[\phi (T_M, X_M), \mathcal{D} (T_M) \right] = i \frac{\partial \phi}{\partial T_M} . \tag{10}$$

If we expand ϕ in the basis (ψ_ν, ψ_ν^*), given by

$$\psi^{s}_{\nu}(Y_M, X_M) = \frac{-i}{2\sqrt{\sinh \pi|\nu|}} e^{i\nu a X_M} J_{-i|\nu|}(m Y_M). \tag{11}$$

and substitute it in the dilatation operator it becomes, in the light cone:

$$\mathcal{D}(T_M \to -\infty) \propto 1/2 \int_{-\infty}^{\infty} d\nu\, |\nu| \left[a^{+}(\nu)\, a(\nu) + a(\nu)\, a^{+}(\nu) \right] \tag{12}$$

You may check this using the expression

$$\lim_{T_M \to -\infty} J_{i\lambda} \propto \frac{e^{ia\lambda T_M}}{2^{i\lambda}\, \Gamma(1 + i\lambda)} . \tag{13}$$

These modes may be called positive "dilatation" frequency modes.

Second you may, with di Sessa[10], ask that the positive frequency mode should follow our second criterium. The problem is that these two criteria lead to two different modes ψ^{s}. The first is ψ^{s} and the second, ψ^{D}, we next define.

2.4. Di Sessa modes:

$$\psi^{D}_{\rho}(Y_M, X_M) = \frac{-i}{2\sqrt{2}} e^{\pi\rho/2} e^{i\rho a X_M} H^{(2)}_{i\rho}(m Y_M) \tag{14}$$

where $H^{(2)}_{i\rho}$ is a Bessel function of the third kind. You may verify that di Sessa condition is satisfied:

$$\lim_{Y_M \to -i\infty} \psi^{D}_{\rho} \propto H^{(2)}_{i\rho}(-i\, m\, \infty) = 0 \tag{15}$$

2.5. Inertial modes:

The next two modes were defined in paper II of this series and are adapted do observers that become smoothly accelerated. The coordinates are defined as

$$\begin{cases} t + x = \ \ 2/a \ \sinh a \ (T_A + X_A) \\ t - x = - 1/a \exp [-a (T_A - X_A)] \end{cases} \qquad (16)$$

We may see that the proper acceleration

$$\alpha = \sqrt{ g_{\mu\nu} \ \frac{D^2\xi^\mu}{Ds^2} \ \frac{D^2\xi^\nu}{Ds^2} } \qquad (17)$$

of a stationary observers parameterized by $\xi^\mu = (T_A, X_A)$ is given by:

$$\alpha = \frac{a \ e^{2aX_A}}{\left[e^{-2aT_A} + e^{2aX_A} \right]^{3/2}} \qquad (18)$$

To write the modes it is better to use the variables (Y_A, Z_A):

$$\begin{cases} Y_A = 1/a \exp (-a \ T_A) \\ Z_A = 1/a \exp (+a \ X_A) \end{cases} \qquad (19)$$

The modes are then ψ^I_σ and ψ^A_τ. We define ψ^I_σ as:

$$\psi^I_\sigma (Y_A, Z_A) := \frac{1}{2} \sqrt{\frac{\sigma (1 - e^{-2\pi\sigma})}{\pi}} \ H^{(1)}_{i\sigma}(m \ Y_A) \ K_{i\sigma} (m \ Z_A) \qquad (20)$$

This first mode has a quasi-classical behavior in the asymptotic region. By quasi-classical, a concept that, in this context, was exactly defined in previous paper[11], we mean:

$$\lim_{Y_A \to \infty} \psi^I_\sigma \ \propto \exp (-i \ \sigma \ m \ Y_A) \qquad (21)$$

This modes are positive frequency following di Sessa's criterium.

2.6. Accelerating modes:

These are defined as

$$\psi^A_\tau (Y_A, Z_A) := \sqrt{\frac{\tau}{\pi}} \ J_{i\tau} (m \ Y_A) \ K_{i\tau} (m \ Z_A) \qquad (22)$$

and are quasi-classical in the accelerated region:

$$\lim_{Y_A \to 0} \psi^A_\tau \propto Y_A^{\;i\tau} \tag{23}$$

They satisfy Sommerfield definition of positive frequency.

3. Comparison of the vacua

Now that we have well defined all modes we are interested on, we go on comparing them. It is well known:

$$| \langle \psi^M_k, \psi^R_\mu \rangle |^2 = \frac{a^2}{2\pi\varepsilon} \frac{1}{e^{2\pi\mu} - 1} \tag{24}$$

Here you may speak about a temperature because the equivalence principle allows you to compare the temperature measured in an inertial system with the temperature measured in the proper frame of the observer:

$$\Theta = \sqrt{g_{00}}\ \Theta_0 \tag{25}$$

so that the temperature is proportional to the proper acceleration ($\alpha_R = 1/X_R$)

$$\Theta = \frac{\alpha_R}{2\pi} . \tag{26}$$

We also know the number of Milne modes particle distributions in the Minkowski vacua:

$$| \langle \psi^M_k, \psi^S_\nu \rangle |^2 = \frac{1}{4\pi\varepsilon} \frac{1}{e^{2\pi\nu} - 1} \tag{27}$$

and

$$| \langle \psi^M_k, \psi^D_\mu \rangle |^2 = 0. \tag{28}$$

For Sommerfield quantization Milne universe behaves like a big bang, where the temperature Θ is given by

$$\Theta = \frac{1}{2\pi} e^{-aT_M}. \tag{29}$$

so that as $T_M \to -\infty$, $\Theta \to \infty$ and as $T_M \to \infty$, $\Theta \to 0$. di Sessa modes, on the contrary, lead us to zero temperature.

In a previous paper we calculated the Bogolubov coefficients between Minkowski and ψ^A_τ-mode and between Minkowski and ψ^I_σ-modes:

$$|\langle\psi^M_k, \psi^A_\tau\rangle|^2 = \frac{1}{2\pi\varepsilon a} \frac{1}{e^{2\pi\tau} - 1} \tag{30}$$

(note that the associated proper temperature is proportional to α_∞) and

$$|\langle\psi^M_k, \psi^I_\sigma\rangle|^2 = \sinh^{-2}\pi\sigma \frac{\sigma}{\varepsilon a} \mathrm{Re}^2 \left[\frac{((\varepsilon-k)/2a)^{-i\sigma}}{(-i\sigma)!} \right]. \tag{31}$$

This resonates in $(\varepsilon-k)/2a$ and σ for $\varepsilon \gg k$ and goes to zero for $\varepsilon \sim k$.

We also know the relationships

$$|\langle\psi^S_\nu, \psi^D_\rho\rangle|^2 = \frac{1}{e^{2\pi\nu} - 1} \delta^2(\nu-\rho) \tag{32}$$

and

$$|\langle\psi^A_\tau, \psi^I_\sigma\rangle|^2 = \frac{1}{e^{2\pi\sigma} - 1} \delta^2(\tau-\sigma). \tag{33}$$

The new results we are presenting in this paper come from the comparison of the accelerating vacua with Rindler and Milne ones. They are:

$$|\langle\psi^R_\mu, \psi^A_\tau\rangle|^2 = 0. \tag{34}$$

This is because the proper acceleration of gaussian observers coincide in the asymptotic region where both modes are quasi-classical. Second we have:

$$|\langle\psi^{R}_{\mu},\ \psi^{I}_{\sigma}\rangle|^{2} = \frac{1}{e^{2\pi\sigma}-1}\,\delta^{2}(\tau-\sigma). \tag{35}$$

Finally we have:

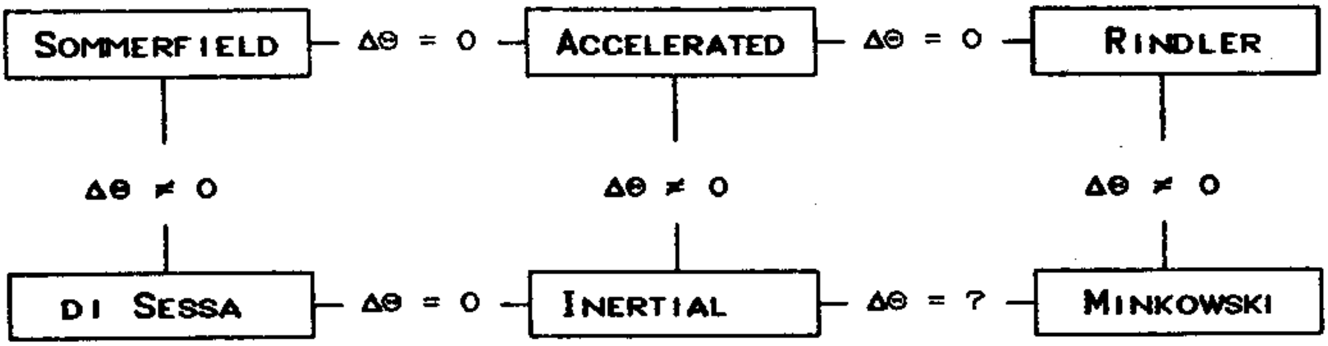
$$|\langle\psi^{S}_{\nu},\ \psi^{A}_{\tau}\rangle|^{2} = 0 \tag{36}$$

and

$$|\langle\psi^{D}_{\rho},\ \psi^{I}_{\sigma}\rangle|^{2} = 0 \tag{37}$$

4. Conclusion

If you put this in a diagram, you see that this concept of temperature may be seen as an equivalence relation.



The only exception is the Bogolubov coefficient between the inertial and Minkowski modes. Castagnino[12] suggested that this result is due to the fact that the associated observers fluid is not rigid. Notice that we didn't compute the Bogolubov coefficients between Sommerfield and inertial modes and between Di Sessa and accelerated modes because the positive frequency definitions are incompatible so that it will be non sense.

In a next paper we will check this results by calculating the Feynmann propagator, Wightmann functions and the Hamiltonian which are needed to construct a detector.

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