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ON THE CONVEXITY OF SETS

by

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ABSTRACT

Convexity properties of convex subsets of real vector spaces are here presented.

Key-words: Convex cone; Convex set; Cancellation rule; Vectoriality.

$\mathbb{N}^*$ is the set of all strictly positive integers, $\mathbb{R}_+^*$ is the set of all strictly positive real numbers, $\mathbb{I}$ and $\mathbb{J}$ are respectively the closed and open intervals of real numbers of extremities 0 and 1. A <u>convex set</u> X is defined by a map that to every $n \in \mathbb{N}^*$, $\lambda_i \in \mathbb{J}(i = 1,\ldots,n)$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_i \in X(i = 1,\ldots,n)$, associates a <u>convex combination</u> $\lambda_1 x_1 + \cdots + \lambda_n x_n = \sum_{1\leq i\leq n} \lambda_i x_i \in X$ so that: (1) <u>Commutativity</u>. We have $\sum_{1\leq i\leq n} \lambda_{\sigma(i)} x_{\sigma(i)} = \sum_{1\leq i\leq n} \lambda_i x_i$ for any permutation σ of the set $\{1,\ldots,n\}$. (2) <u>Associativity</u>. We have

$$\sum_{1\leq j\leq n} \mu_j \left(\sum_{1\leq i\leq m_j} \lambda_{ij} x_{ij} \right) = \sum_{\substack{1\leq i\leq m_j \\ 1\leq j\leq n}} (\lambda_{ij}\mu_j) x_{ij}$$

for $n \in \mathbb{N}^*$, $m_j \in \mathbb{N}^*(j = 1,\ldots,n)$, $\lambda_{ij} \in \mathbb{J}(i = 1,\ldots,m_j, j = 1,\ldots,n)$, $\lambda_{1j} + \cdots + \lambda_{m_j j} = 1(j = 1,\ldots,n)$, $\mu_j \in \mathbb{J}(j = 1,\ldots,n)$, $\mu_1 + \cdots + \mu = 1$, $x_{ij} \in X(j = 1,\ldots,m_j, j = 1,\ldots,n)$. <u>Distributivity</u>. We have $\lambda_1 x + \cdots + \lambda_n x_n = x$ for $n \in \mathbb{N}^*$, $\lambda_j \in \mathbb{J}(i = 1,\ldots,n)$, $\lambda_1 + \cdots + \lambda_n = 1$. A subset Y of X is a <u>convex subset</u> of X when $\lambda_1 y_1 + \cdots + \lambda_n y_n \in Y$ for $n \in \mathbb{N}^*$, $\lambda_i \in \mathbb{J}(i = 1,\ldots,n)$, $\lambda_1 + \cdots + \lambda_n = 1$, $y_i \in Y(i = 1,\ldots,n)$. Then Y is a convex set. A <u>convex set map</u> $f : X \to Y$ between convex sets is defined by $f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f_1(x_1) + \cdots + \lambda_n f_n(x_n)$ for $n \in \mathbb{N}^*$, $\lambda_i \in \mathbb{J}(i = 1,\ldots,n)$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_i \in X(i = 1,\ldots,n)$. The set $CS(X)$ of all nonvoid convex subsets of a convex set X is a convex set if we define $\lambda_1 X_1 + \cdots + \lambda_n X_n = \{\lambda_1 x_1 + \cdots + \lambda_n x_n\ ; x_i \in X(i = 1,\ldots,n)\} \in CS(X)$ for $n \in \mathbb{N}^*$, $\lambda_i \in \mathbb{J}(i = 1,\ldots,n)$, $\lambda_1 + \cdots + \lambda_n = 1$, $X_i \in CS(X)(i = 1,\ldots,n)$. We have the injective convex set map $x \in X \mapsto \{x\} \in CS(X)$.

Proposition 1. *Let X be a set where to every $\lambda \in J, x_1, x_2 \in X$, we associate a convex combination $(1-\lambda)x_1 + \lambda x_2 \in X$. There is a necessarily unique convex set structure on X defining these convex combinations if and only if: (1) Commutativity. We have $\lambda x_2 + (1-\lambda)x_1 = (1-\lambda)x_1 + \lambda x_2$ for $\lambda \in J$, $x_1, x_2 \in X$. (2) Associativity. We have $(1-\mu)[(1-\lambda)x_1 + \lambda x_2] + \mu x_3 = (1-\lambda)(1-\mu)x_1 + [\lambda(1-\mu)+\mu]\left[\frac{\lambda(1-\mu)}{\lambda(1-\mu)+\mu}x_2 + \frac{\mu}{\lambda(1-\mu)+\mu}x_3\right]$ for $\lambda, \mu \in J$, $x_1, x_2, x_3 \in X$. (3) Distributivity. We have $(1-\lambda)x + \lambda x = x$ for $\lambda \in J$, $x \in X$.*

Let us recall that a convex cone C is a set where we are given an addition $(x_1, x_2) \in C \times C \mapsto x_1 + x_2 \in C$ that is commutative and associative, as well as a multiplication $(\lambda, x) \in \mathbb{R}_+^* \times C \mapsto \lambda x \in C$ that is distributive on both sides, associative on the left side, and such that $1 \in \mathbb{R}_+^*$ behaves as the identity map of C(*). A subset D of C is a convex subcone of C when $y_1 + y_2 \in D$ for $y_1, y_2 \in D$, and $\lambda y \in D$ for $\lambda \in \mathbb{R}_+^*$, $y \in D$. Then D is a convex cone. A convex cone map $f : C \to D$ between convex cones is defined by $f(x_1 + x_2) = f(x_1) + f(x_2)$, $f(\lambda x) = \lambda f(x)$ for $\lambda \in \mathbb{R}_+^*$, $x, x_1, x_2 \in C$. A real vector space is a convex cone. A convex cone is a convex set. A convex cone C is vectorial when it is a convex subcone of some real vector space. This occurs if and only if C satisfies the cancellation rule for convex cones (CRCC): If $x_1, x_2, x_3 \in C$, $x_1 + x_2 = x_1 + x_3$, then $x_2 = x_3$. A convex set is vectorial when it is a convex subset of some real vector space. A convex cone is vectorial as a convex cone if and only if it is vectorial as a convex set. If X is a convex set, $CS(X)$ fails to be vectorial if and only if X has at least two elements. If C is a convex cone, $CS(C)$ is a convex cone if we define $X_1 + X_2 = \{x_1 + x_2; x_1 \in X_1, x_2 \in X_2\} \in CS(C)$ for $X_1, X_2 \in CS(C)$, and $\lambda X \in \{\lambda X; x \in X\} \in CS(C)$ for $\lambda \in \mathbb{R}_+^*$, $X \in CS(C)$. We have the injective convex cone map $x \in C \mapsto \{x\} \in CS(C)$. This convex cone structure on $CS(C)$ defines the previous convex set structure on it.

PROPOSITION 2. *(1) Every convex set is a convex subset of some convex cone. (2) A convex set X is vectorial if and only if it satifies the cancellation rule for convex sets ($CRCS$): If $\lambda \in \mathsf{J}$, $x_1, x_2, x_3 \in X$, $(1-\lambda)x_1 + \lambda x_2 = (1-\lambda)x_1 + \lambda x_3$, then $x_2 = x_3$. (3) Every convex set (or cone) is the image by a surjective convex set (or cone) map of some vectorial convex set (or cone).*

PROPOSITION 3. *(1) Let C be a convex cone and $x \in C$. Either the map $\lambda \in \mathbf{R}_+^* \mapsto \lambda x \in C$ is injective, or else it is a constant map, that is $\lambda x = x (\lambda \in \mathbf{R}_+^*)$, which occurs if and only if $x + x = x$. (2) Let X be a convex set and $x_1, x_2 \in X$. Either the map $\lambda \in \mathsf{I} \mapsto (1-\lambda)x_1 + \lambda x_2 \in X$ is injective, or else its restriction to J is a constant map. (3) Let C be a convex cone, and $x_1, x_2, x_3 \in C$. If $x_1 + \lambda x_2 = x_1 + \lambda x_3$ for some $\lambda \in \mathbf{R}_+^*$, this equality holds for all $\lambda \in \mathbf{R}_+^*$. (4) Let X be a convex set, and $x_1, x_2, x_3 \in X$. If $(1-\lambda)x_1 + \lambda x_2 = (1-\lambda)x_1 + \lambda x_3$ for some $\lambda \in \mathsf{J}$, this equality holds for all $\lambda \in \mathsf{J}$.*

PROPOSITION 4. *(1) Let C be an ordered set which is an inflattice. Define $x_1 + x_2 = \inf\{x_1, x_2\} \in C (x_1, x_2 \in C)$. Then C becomes an associative, commutative, additive monoid satisfying $x + x = x (x \in C)$. Moreover, $x_1 \leq x_2$ if and only if $x_1 + x_2 = x_1 (x_1, x_2 \in C)$. Conversely, let C be an associative, commutative, additive monoid satisfying $x + x = x (x \in C)$. We obtain an order on C by $x_1 \leq x_2$ when $x_1 + x_2 = x_1$, and C becomes an inflattice where $\inf\{x_1, x_2\} = x_1 + x_2 (x_1, x_2 \in C)$. Moreover, C has a zero as an additive monoid if and only if C has a largest element as an ordered set, the zero and the largest element being necessarily equal. (2) Let C be an associative, commutative, additive monoid. Introduce the constant multiplication $(\lambda, x) \in \mathbf{R}_+^* \times C \mapsto \lambda x = x \in C$. Then C becomes a convex cone if and only if $x + x = x (x \in C)$.*

Proposition 5. *(1) Let X be a convex set. There is a convex cone structure with constant multiplication on X defining the given convex set structure if and only if the map $\lambda \in \mathsf{J} \mapsto (1-\lambda)x_1 + \lambda x_2 \in X(x_1, x_2 \in X)$ is always a constant map. That convex cone structure is unique because $x_1 + x_2 = (1-\lambda)x_1 + \lambda x_2(\lambda \in \mathsf{J}, x_1, x_2 \in X)$. (2) If X is a convex set whose power is strictly less than the continuum power, there is a unique convex cone structure with constant multiplication on X defining the given convex set structure. Every convex cone whose power is strictly less than the continuum has a constant multiplication. Therefore, a convex set whose power is strictly less than the continuum power, a convex cone whose power is strictly less than the continuum power, and an inflattice whose power is strictly less than the continuum power, are three equivalent forms of the same concept.*

Proposition 6. *Let $f : X \to Y$ be a surjective convex set map between nonvoid convex sets. It defines an equivalence relation on X whose equivalence classes are $f^{-1}(y)(y \in Y)$, that is compatible with the convex set structure of X in the sense that: if $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathsf{J}$, $\lambda_1 + \ldots + \lambda_n = 1, t_1, \ldots, t_n, x_1, \ldots, x_n \in X$, $t_1 \sim x_1, \ldots, t_n \sim x_n$, then $\lambda_1 t_1 + \cdots + \lambda_n t_n \sim \lambda_1 x_1 + \cdots + \lambda_n x_n$, or equivalently, if $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathsf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $X_1, \ldots, X_n$ are equivalence classes, then $\lambda_1 X_1 + \cdots + \lambda_n X_n = \{\lambda_1 x_1 + \cdots + \lambda_n x_n;\ x \in X_1, \ldots, x_n \in X_n\}$ is contained in a necessarily unique equivalence class. Conversely, let us be given an equivalence relation on a nonvoid convex set X that is compatible with its convex set structure. Call Y the quotient set and $f : X \to Y$ the quotient map. There is one and only one convex set structure on Y so that f is a convex set map, namely $\lambda_1 Y_1 + \cdots + \lambda_n Y_n \in Y$ in the sense of Y is the unique equivalence class containing $\lambda_1 Y_1 + \cdots + \lambda_n Y_n = \{\lambda_1 y_1 + \cdots + \lambda_n y_n; y_i \in Y_i (i = 1, \ldots, n)\} \subset X$ in the sense of X, for $m \in \mathbb{N}^*$, $\lambda_i \in \mathsf{J}(i = 1, \ldots, n)$, $\lambda_1 + \cdots + \lambda_n = 1$, $Y_i \in Y(i = 1, \ldots, n)$.*

PROPOSITION 7. *Introduce on a nonvoid convex set X the binary relation $x_1 \sim x_2 (x_1, x_2 \in X)$ meaning that $(1-\lambda)x + \lambda x_1 = (1-\lambda)x + \lambda x_2$ for some $\lambda \in \mathsf{J}$, $x \in X$. It is an equivalence relation that is compatible with the convex set structure of X. Call Y the quotient set and $f : X \to Y$ the quotient map. Consider the unique convex set structure on Y so that f is a convex set map. Then Y is a convex set satisfying the cancellation rule for convex sets. If $g : X \to Z$ is any surjective convex set map, Z being a convex set satisfying the cancellation rule for convex sets, there is a map $h : Y \to Z$ satisfying $hf = g$, where h is a necessarily unique surjective convex set map.*

Introduce the one dimensional injection rule (ODIR) for a convex set X : The map $\lambda \in \mathsf{I} \mapsto (1-\lambda)x_1 + \lambda x_2 \in X (x_1, x_2 \in X, x_1 \neq x_2)$ is always injective.

PROPOSITION 8. *Fixed a closed triangle M of vertices a, b, c in R^2. Let N be the union of the closed segments $[a,d]$ and $[b,c]$ of R^2, where $d \in [b,c]$ is fixed. M is, but N is not, a convex subset of R^2. Define the surjective map $f : M \to N$ by $f(x) = x$ for $x \in [b,c]$, and as the point $f(x) \neq d$ where $[a,d]$ meets the parallel to $[b,c]$ through $x \in M$, $x \notin [b,c]$. There is one and only one convex set structure on N so that f is a convex set map for the convex set structure on M induced by R^2. Then N satisfies the one dimensional injection rule but it fails to satisfy the cancellation rule for convex sets. A convex set X fails to satisfy the cancellation rule for convex sets if and only if either X does not satisfy the one dimensional injection rule, or else X satisfies the one dimensional injection rule and it contains some convex subset isomorphic to N.*

PROPOSITION 9. *Let $x_1 \sim x_2 (x_1, x_2 \in X)$ be an equivalence relation on a nonvoid convex set X. The following conditions are equivalent: (1) There are a real vector space E containing X as a convex subset and a real vector* *subspace F of E such that the given equivalence relation on X is induced by the equivalence relation that F defines on E. (2) The given equivalence relation on X is compatible with the convex set structure on X, and both X and its quotient convex set Y are vectorial. (3) The given equivalence relation on X is compatible with its convex set structure, and if $\lambda \in$ J, $x, x_1, x_2 \in X$, then $(1-\lambda)x + \lambda x_1 = (1-\lambda)x + \lambda x_2$ implies $x_1 = x_2$, and also $(1-\lambda)x + \lambda x_1 \sim (1-\lambda)x + \lambda x_2$ implies $x_1 \sim x_2$.*

(*) See B. Fuchssteiner & W. Lusky, "Convex Cones", North-Holland, 1981.