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Residual Symmetries in the Presence of an EM Background

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Abstract

The symmetry algebra of a QFT in the presence of an external EM background (named "residual symmetry") is investigated within a Lie-algebraic, model independent scheme. Some results previously encountered in the literature are here extended. In particular we compute the symmetry algebra for a constant EM background in D=3 and D=4 dimensions. In D= 3 dimensions the residual symmetry algebra is isomorphic to $u(1) \oplus \mathcal{P}_c(2)$, with $\mathcal{P}_c(2)$ the centrally extended 2-dimensional Poincaré algebra. In D=4 dimension the generic residual symmetry algebra is given by a seven-dimensional solvable Lie algebra which is explicitly computed. Residual symmetry algebras are also computed for specific non-constant EM backgrounds.

Key-words: Quantum field theories; Symmetries; Lie algebras.

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1 Introduction

The issues of QFTs in an external (constant) EM background gained interest recently with the Seiberg and Witten's observation [1] that an ordinary theory in a constant EM background can be reformulated as a non-commutative gauge theory.

The problem of determining the symmetry algebra of a QFT in a constant EM background has been addressed and solved, for the very specific two-dimensional free massive complex boson model minimally coupled to an external gauge field, in [2]. It was proven in that work that its symmetry algebra coincides with the centrally extended Poincaré algebra in $1+1$ dimension, previously investigated in a series of papers [3, 4].

In the present work we extend the results of [2]. By using solely Lie-algebraic and model-independent methods we compute the symmetry algebra of different classes of QFTs coupled to a given external EM background. Throughout the text we call such symmetry algebras "residual symmetries". It is worth stressing that, due to the presence of central extensions, the residual symmetries *are not* subalgebras of the original symmetry algebra in the absence of the external EM background (such an algebra is given by the direct sum of the Poincaré algebra and a global $U(1)$ charge).

More specifically, we prove that the residual symmetry algebra of a three-dimensional Poincaré invariant QFT in a constant EM background is given by the 5-dimensional solvable Lie algebra $u(1)\oplus\mathcal{P}_c(2)$, where $\mathcal{P}_c(2)$ is the two-dimensional centrally extended Poincaré algebra whose signature, Euclidean or Minkowskian, is determined by the relative strength of the constant external electric versus magnetic field.

The results of [2] in $1+1$ dimensions are consistently recovered from our own results after performing a dimensional reduction.

Furthermore, we compute the residual symmetry algebra for a four-dimensional Poincaré invariant QFT in a generic constant EM background. The resulting symmetry is a 7-dimensional solvable Lie algebra explicitly presented in formulas (21).

The residual symmetry algebra in the presence of non-constant EM backgrounds has also been computed in various cases and the results are here presented.

The scheme of the paper is the following. In the next section we illustrate the Lie-algebraic method which allows, in a model-independent manner, to determine the residual symmetry generators and the corresponding algebra. In section **3** we present the resulting residual symmetry algebra for a $D=3$ QFT in the presence of a constant EM field. In section **4** the residual symmetry algebra is computed for a QFT in the ordinary $D=4$ Minkowski space-time in the presence of a constant EM background. In section **5** the case of a non-constant EM background is treated in some specific examples. Finally, in the Conclusions, we make some comments about our work, drawing attention to its possible applications and outlining the future investigations.

2 Residual symmetries and their generators.

Let us discuss in detail for the sake of simplicity the case of the residual symmetry for generic Poincaré-invariant field theories in $(2+1)$-dimension, coupled with an external constant EM background. The generalization of this procedure to higher-dimensional theories and non-constant EM backgrounds, such as those studied in Sections **4** and **5**, is straightforward and immediate.

In the absence of the external electric and magnetic field, the action $\mathcal{S}$ is assumed to be invariant under a 7-parameter symmetry, given by the six generators of the $(2+1)$-Poincaré symmetry which, when acting on scalar fields (the following discussion however is valid no matter which is the spin of the fields) are represented by

$$\begin{aligned} P_\mu &= -i\partial_\mu, \\ M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu), \end{aligned} \tag{1}$$

(the metric is chosen to be $+--$), plus a remaining symmetry generator corresponding to the internal global $U(1)$ charge that will be denoted as Z.

It is further assumed that in the action $\mathcal{S}$ the dependence on the classical background field is expressed in terms of the covariant gauge-derivatives

$$D_\mu = \partial_\mu - ieA_\mu,$$

with e the electric charge.

In the presence of constant external electric and magnetic fields, the $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ field-strength is constrained to satisfy

$$F^{0i} = E^i, \qquad F^{ij} = \epsilon^{ij}B, \tag{2}$$

where $\mu, \nu = 0, 1, 2$ and $i, j = 1, 2$. The fields E^i and B are constant. Without loss of generality the x^1, x^2 spatial axis can be rotated so that $E^1 \equiv E$, $E^2 = 0$. Throughout the text this convention is respected.

In order to recover (2), the gauge field A_μ must depend at most linearly on the coordinates $x^0 \equiv t$, $x^1 \equiv x$ and $x^2 \equiv y$.

The gauge-transformation

$$A_\mu \mapsto {A_\mu}' = A_\mu + \frac{1}{e}\partial_\mu\alpha(x^\nu) \tag{3}$$

allows to conveniently choose for A_μ the gauge-fixing

$$\begin{aligned} A_0 &= 0, \\ A_i &= E_i t - \frac{B}{2}\epsilon_{ij}x^j. \end{aligned} \tag{4}$$

The above choice is a good gauge-fixing since it completely fixes the gauge (no gauge-freedom is left). It will be soon evident that the residual symmetry is a truly physical symmetry, independent of the chosen gauge-fixing.

Due to (4), the action $\mathcal{S}$ explicitly depends on the x^μ coordinates entering A_μ. The simplest way to compute the symmetry property of an action such as $\mathcal{S}$ which explicitly depends on the coordinates consists in performing the following trick. At first A_μ is regarded on the same foot as the other fields entering $\mathcal{S}$ and assumed to transform as standard vector field under the global Poincaré transformations, namely

$${A_\mu}'(x^{\rho\prime}) = {\Lambda_\mu}^\nu A_\nu(x^\rho) \tag{5}$$

for $x^{\mu\prime} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu}$.

For a generic infinitesimal Poincaré transformation, however, the transformed A_μ gauge-field no longer respects the gauge-fixing condition (4). In the active transformation viewpoint only fields are entitled to transform, not the space-time coordinates themselves. A_μ plays the role of a fictitious field, inserted to take into account the dependence of the action $\mathcal{S}$ on the space-time coordinates caused by the non-trivial background. Therefore, the overall infinitesimal transformation δA_μ should be vanishing. This result can be reached if an infinitesimal gauge transformation (3) $\delta_g(A_\mu)$ can be found in order to compensate for the infinitesimal Poincaré transformation $\delta_P(A_\mu)$, i.e. if the following condition is satisfied

$$\delta(A_\mu) = \delta_P(A_\mu) + \delta_g(A_\mu) = 0. \tag{6}$$

Only those Poincaré generators which admit a compensating gauge-transformation satisfying (6) provide a symmetry of the $\mathcal{S}$ action (and therefore enter the residual symmetry algebra). This is a plain consequence of the original assumption of the Poincaré and manifest gauge invariance for the action $\mathcal{S}$ coupled to the gauge-field A_μ.

Notice that the original Poincaré generators are deformed by the presence of extra-terms associated to the compensating gauge transformation. Let p denote a generator of (1) which "survives" as a symmetry in the presence of the external background. The effective generator of the residual symmetry is

$$\hat{p} \quad = \quad p + (\ldots),$$

where $(\ldots)$ denotes the extra terms arising from the compensating gauge transformation associated to p. Such $(\ldots)$ extra terms are gauge-fixing dependent. The "residual symmetry generator" $\hat{p}$ can only be expressed in a gauge-dependent manner. However, two gauge-fixing choices are related by a gauge transformation $\mathbf{g}$. The residual symmetry generator in the new gauge-fixing, denoted as $\tilde{p}$, is related to the previous one by an Adjoint transformation

$$\tilde{p} \quad = \quad \mathbf{g}\hat{p}\mathbf{g}^{-1}. \tag{7}$$

Therefore the residual symmetry algebra does not dependent on the choice of the gauge fixing and is a truly physical characterization of the action $\mathcal{S}$.

The extra-terms $(\ldots)$ are necessarily linear in the space-time coordinates when associated with a translation generator, and bilinear when associated to a surviving Lorentz generator (for a constant EM background). Their presence implies the arising of the central term in the commutator of the deformed translation generators.

3 The residual symmetry for the $(2+1)$ Poincaré case.

The residual symmetry algebra of the $(2+1)$-Poincaré theory involves, besides the global $U(1)$ generator Z, the three deformed translations and just one deformed Lorentz generator (the remaining two Lorentz generators are broken).

Within the (4) gauge-fixing choice the deformed translations are explicitly given by

$$
\begin{aligned}
P_0 &= -i\partial_t - eEx,\\
P_1 &= -i\partial_x - \frac{e}{2}By,\\
P_2 &= -i\partial_y + \frac{e}{2}Bx.
\end{aligned}
\tag{8}
$$

The deformed generator of the residual Lorentz symmetry is explicitly given, in the same gauge-fixing and for $E \neq 0$, by

$$
\begin{aligned}
M &= i(x\partial_t + t\partial_x) - i\frac{B}{E}(y\partial_x - x\partial_y) + \\
&\quad \frac{e}{2}(Et^2 + Ex^2 - Bty).
\end{aligned}
\tag{9}
$$

The residual symmetry algebra is given by

$$
\begin{aligned}
[P_0, P_1] &= iEZ,\\
[P_0, P_2] &= 0,\\
[P_1, P_2] &= iBZ,\\
[M, P_0] &= -iP_1,\\
[M, P_1] &= -iP_0 - i\frac{B}{E}P_2,\\
[M, P_2] &= i\frac{B}{E}P_1.
\end{aligned}
\tag{10}
$$

The $U(1)$ charge Z is no longer decoupled from the other symmetry generators. It appears instead in (10) as a central charge.

Please notice that the residual symmetry algebra in $(1+1)$ dimensions (computed in [2] for a specific model) is recovered from the P_0, P_1, M, Z subalgebra. It corresponds to the centrally extended $2D$ Poincaré algebra thoroughly studied in [4].

The 5-generator solvable, non-simple Lie algebra of residual symmetries admits a convenient presentation. The generator

$$
\tilde{Z} \equiv BP_0 + EP_2 \tag{11}
$$

commutes with all the other $*$ generators

$$
[\tilde{Z}, *] = 0, \tag{12}
$$

so that the residual symmetry algebra is given by a direct sum of $u(1)$ and a 4-generator algebra. The latter algebra is isomorphic to the centrally extended two-dimensional Poincaré algebra. Such an algebra is of Minkowskian or Euclidean type according to whether $E > B$ or respectively $E < B$ (the case $E = B$ is degenerate). This point can be intuitively understood due to the predominance of the electric or magnetic effect (in the absence of the electric field the theory is manifestly rotational invariant, so that the

Lorentz generator is associated with the Euclidean symmetry). We have explicitly, for $B > E$, that the algebra

$$\begin{aligned} [\overline{M}, S_1] &= iS_2, \\ [\overline{M}, S_2] &= -iS_1 \end{aligned} \tag{13}$$

is reproduced by

$$\begin{aligned} \overline{M} &= \frac{E}{\sqrt{B^2 - E^2}} M, \\ S_1 &= P_0 + \frac{B}{E} P_2, \\ S_2 &= \frac{\sqrt{B^2 - E^2}}{E} P_1, \end{aligned} \tag{14}$$

while for $E > B$ the algebra

$$\begin{aligned} [\tilde{M}, T_1] &= iT_2, \\ [\tilde{M}, T_2] &= iT_1, \end{aligned} \tag{15}$$

is reproduced by

$$\begin{aligned} \tilde{M} &= \frac{E}{\sqrt{E^2 - B^2}} M, \\ T_1 &= P_0 + \frac{B}{E} P_2, \\ T_2 &= -\frac{\sqrt{E^2 - B^2}}{E} P_1. \end{aligned} \tag{16}$$

In both cases the commutator between the translation generators S_1, S_2, and respectively T_1, T_2, develops the central term proportional to Z which can be conveniently normalized.

The residual symmetry algebra of the $(2+1)$ case for generic values of E and B (the $E = B$ case is degenerate) is therefore given by the direct sum

$$u(1) \oplus \mathcal{P}_c(2). \tag{17}$$

Besides the two charges Z, $\tilde{Z}$ an extra charge is given by the order two Casimir of the centrally extended Poincaré algebra, see [4] for details.

4 The residual symmetry in 4 dimensions.

In $D = 4$ dimensions, for generic values of the constant electric and magnetic field, a convenient gauge-fixing is provided by

$$\begin{aligned} A_0 &= 0, \\ A_1 &= E_1 t, \\ A_2 &= E_2 t + Bz, \\ A_3 &= 0. \end{aligned} \tag{18}$$

We notice that, without loss of generality, we assume the constant external magnetic field $\overrightarrow{B}$ parallel to the x axis, while E_1, E_2 denote the components of the external electric field, respectively parallel and transverse to $\overrightarrow{B}$. In the following we consider the case $E_1, E_2, B \neq 0$.

The deformed translation generators are now

$$\begin{aligned} P_0 &= -i\partial_t - e(E_1 x + E_2 y), \\ P_1 &= -i\partial_x, \\ P_2 &= -i\partial_y, \\ P_3 &= -i\partial_z - eBy. \end{aligned} \tag{19}$$

For what concerns the Lorentz generators, only two of them survive as symmetry generators in the given external background. They are given by

$$\begin{aligned} M &= i\frac{B}{E_1}y\partial_t - iz\partial_x + i\frac{B}{E_1}t\partial_y + i(\frac{E_1^2+B^2}{E_1E_2})z\partial_y + ix\partial_z - i(\frac{E_1^2+B^2)}{E_1E_2})y\partial_z + \\ &\quad eB(\frac{-E_1^2+E_2^2-B^2}{2E_1E_2})y^2 + eB(\frac{E_1^2+B^2}{2E_1E_2})z^2 + e\frac{B^2}{E_1}tz + eBxy + e\frac{E_2B}{2E_1}t^2, \\ N &= ix\partial_t + i\frac{E_2}{E_1}y\partial_t + it\partial_x + i\frac{E_2}{E_1}t\partial_y + i\frac{B}{E_1}z\partial_y - i\frac{B}{E_1}y\partial_z + \\ &\quad \frac{e}{2E_1}(E_1^2+E_2^2)t^2 + \frac{e}{2}E_1x^2 + \frac{e}{2E_1}(E_2^2-B^2)y^2 + \frac{e}{2E_1}B^2z^2 + eB\frac{E_2}{E_1}tz + eE_2xy. \end{aligned} \tag{20}$$

It is convenient to normalize the generators according to

$$\begin{aligned} T_1 &= (1/\sqrt{E_2})P_0, \\ T_2 &= (1/\sqrt{E_2})P_2, \\ S_1 &= (\sqrt{E_2}/E_1)P_1, \\ S_2 &= -(\sqrt{E_2}/B)P_3. \end{aligned}$$

The resulting residual symmetry algebra is a three-graded non-simple solvable Lie algebra, given by the commutators

$$\begin{aligned} [T_i, S_j] &= i\delta_{ij}Z, \\ [T_i, T_j] &= i\epsilon_{ij}Z, \\ [S_i, S_j] &= 0, \\ [M, T_1] &= -i\frac{B}{E_1}T_2, \\ [M, S_1] &= i\frac{B}{E_1}S_2, \\ [M, T_2] &= -i(\frac{E_1^2+B^2}{E_1E_2})\frac{B}{E_2}S_2 - i\frac{B}{E_1}T_1, \end{aligned}$$

$$
\begin{aligned}
[M,S_2] &= i(\frac{E_1^2+B^2}{E_1B})T_2 - i\frac{E_1}{B}S_1,\\
[N,T_1] &= -i\frac{E_1}{E_2}S_1 - i\frac{E_2}{E_1}T_2,\\
[N,S_1] &= -i\frac{E_2}{E_1}T_1,\\
[N,T_2] &= -i\frac{B^2}{E_2E_1}S_2 - i\frac{E_2}{E_1}T_1,\\
[N,S_2] &= i\frac{E_2}{E_1}T_2,\\
[M,N] &= 0.
\end{aligned}
\qquad (21)
$$

Such an algebra admits two independent Casimir operators of order two, given by

$$
\begin{aligned}
C_1 &= T_1T_1 + 2\frac{B^2}{E_2^2}T_1S_2 - 2\frac{E_1^2}{E_2^2}S_1T_2 + (-1+\frac{B^2}{E_2^2}+\frac{E_1^2}{E_2^2})T_2T_2 + \\
&\quad (\frac{B^4}{E_2^4}+\frac{B^2E_1^2}{E_2^4})S_2S_2 - 2i\frac{BE_1}{E_2^2}E_1MZ + 2i\frac{E_1}{E_2}NZ,\\
C_2 &= 2\frac{B^2}{E_1^2}T_1S_2 + S_1S_1 - 2S_1T_2 + (1+\frac{B^2}{E_1^2})T_2T_2 + \\
&\quad (\frac{B^2}{E_1^2}+\frac{B^2}{E_2^2}+\frac{B^4}{E_1^2E_2^2})S_2S_2 - 2i\frac{B}{E_1}MZ.
\end{aligned}
\qquad (22)
$$

5 Residual symmetries in the presence of a non-constant EM background.

In the case of a non-constant EM background, the surviving symmetry generators are further constrained. Some illustrative cases are reported below.
i) Linear external EM field in $(1+1)$ dimensions.

For a field E, given by

$$
E = E_1x + E_2t, \qquad (23)
$$

a convenient gauge-fixing is

$$
\begin{aligned}
A_0 &= 0,\\
A_1 &= \frac{E_2}{2}t^2 + E_1xt.
\end{aligned}
\qquad (24)
$$

There exists only one symmetry generator left, given by

$$
P = -i\partial_t + i\frac{E_2}{E_1}\partial_x - \frac{e}{2}E_2x^2. \qquad (25)
$$

ii) Quadratic external EM field in $1+1$ dimensions.

For the external field

$$E = E_1x^2 + E_2t^2 + E_{12}xt,$$

the gauge-fixing is given by

$$\begin{aligned} A_0 &= 0, \qquad (26) \\ A_1 &= E_1x^2t + \frac{E_2}{3}t^3 + \frac{E_{12}}{2}xt^2. \end{aligned}$$

All spatial symmetries are broken unless the condition

$$E_{12} = 2\sqrt{E_1E_2}$$

is satisfied. In this particular case, there exists one symmetry generator given by

$$P = -i\partial_t + i\sqrt{\frac{E_2}{E_1}}\partial_x - \frac{e}{3}E_1x^3. \qquad (27)$$

iii) Linear external EM field in $2+1$ dimensions.

In the most general temporal-gauge case, two independent deformed translations survive as symmetry generators (all Lorentz generators are broken), if the external EM field is constrained to satisfy

$$\begin{aligned} E_1 &= \rho^2x + \rho Dy + F, \\ E_2 &= \rho Dx + Dy + G, \\ B &= B_1t + \rho B_2x + B_2y + B_3. \qquad (28) \end{aligned}$$

(ρ arbitrary).

The gauge-fixing is

$$\begin{aligned} A_0 &= 0, \\ A_1 &= \rho^2Dtx + \rho Dty + Ft, \qquad (29) \\ A_2 &= \rho\frac{B_2}{2}x^2 + \rho Dxt + Dty + B_2xy + B_3x + Gt. \qquad (30) \end{aligned}$$

The symmetry generators

$$\begin{aligned} P_0 &= -i\partial_t - \frac{e}{2}\rho^2Dx^2 - \frac{e}{2}Dy^2 - e\rho Dxy - eFx - eGy, \\ P_1 &= -i\partial_x + i\rho\partial_y - \frac{e}{2}B_2y^2 - eB_3y, \qquad (31) \end{aligned}$$

satisfy the centrally extended algebra

$$[P_0, P_1] = -ie(F - \rho G). \qquad (32)$$

Finally, let us comment that for a conformal theory in $1+1$ dimension, whose original symmetry is $Vir \oplus Vir$, in the presence of an external background all the symmetry generators (apart the Poincaré generators for the cases already considered) are broken.

6 Conclusions

In this work we have extended the results of [2] in two directions. We showed the model-independent, Lie-algebraic arising of the result of [2] (originally computed for the free massive complex boson case in $1+1$ dimension, externally coupled to a constant EM background) and later we computed the residual symmetries in the presence of constant EM backgrounds for both the $D=3$ and the ordinary Minkowskian $D=4$ theories.

For a constant EM background the residual symmetry of a $D=3$ theory corresponds to the algebra $u(1)\oplus\mathcal{P}_c(2)$, where $\mathcal{P}_c(2)$ is the centrally extended $2D$ Poincaré algebra, widely investigated, both mathematically and in physical applications, in a series of papers [3, 4]. In [3] it has been applied, e.g., to the construction of lineal-gravity theories in $1+1$ dimensions. Due to the presence of the central term, the adjoint representation of $\mathcal{P}_c(2)$ is not faithful (see [4]). On the other hand a 4-dimensional faithful representation is constructed in the light of the Kirillov's method (see [5] for details). This method is likely to be extended to compute a faithful representation for the seven-dimensional solvable Lie algebra (21) corresponding to the residual symmetry algebra in a constant EM background in $4D$.

The residual symmetry algebras as those computed above play the same role as the ordinary Poincaré algebras, in the case of QFTs living in a given constant (classical) EM background.

It is worth mentioning the connection of such residual symmetry algebras with the arising of non-commutative structures, due to the presence of the central term in the commutators of the (deformed) momenta. A corresponding dual picture can be given which manifests the non-commutativity at the level of the space-time coordinates. The connection between such two dual pictures has been fully explored (for a given specific toy model), e.g., in [6] (see also [7]). In the references [8] non-relativistic $(2+1)$-dimensional theories in an external background have been investigated.

One of the seemingly most promising line for future investigations consists in explicitly linking the role of the "deformed Poincaré generators", as those computed in section **5** in the presence of non-constant EM backgrounds, with the phenomenon of pair-production, observed for specific linear EM backgrounds such as light-cone external electric fields [9]. This would aim at a Lie-algebraical characterization of the pair-production phenomenon.

Finally, the extension of the above construction to, let's say, supersymmetric theories, will provide the deformation of the supersymmetry generators in the presence of an external EM background.

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