CBPF-NF-002/00

Some Exact Formulae for Performing Wave Field Extrapolation in Constant-Velocity and Depth-Dependent Mediums

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Abstract

We deduce well-posed formulae for wave field extrapolation in depth-dependent mediums.

1 Introduction

The basic starting point in the subject of wave field extrapolation and its application to seismic inverse procedures is the derivation of the acoustic pressure field $U(\vec{r}, z, t)$, developing in a medium (upper half-space $z > 0$) with a constant refraction index, from a known pressure field data $U(\vec{r}, z = 0, t)$ at its surface. In section II of this note, we correct mathematically some of those results of ref. [1] by considering the well-posedeness formulation of the problem. Additionally, in section III we present similar news results in the context of a depth-dependent medium for Paraxial and full wave propagation results suitable for extrapolation in water geophysical mediums and finally, we end the section by briefly sketching the constant-anisotropic medium case.

2 The Depth-Extrapolation Problem for a Medium with a Constant Refraction Index

Let us consider the acoustic wave field equation for a pressure field developing in a medium (upper half-space $z > 0$) defined by a constant refraction index and from a known pressure field data $U(\vec{r}, z = 0, t)$ but *added with depth derivative* at the surface, namelly:

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} - (k_x^2 + k_y^2)\right] U_{\vec{k}}(z; t) = 0 \tag{1}$$

$$U_{\vec{k}}(z, t)|_{z=0} = f_{\vec{k}}(t) \tag{2}$$

$$\frac{\partial U_{\vec{k}}}{\partial z}(z, t)|_{z=0} = g_{\vec{k}}(t) \tag{3}$$

where

$$U(\vec{k}, z, t) = \int_{-\infty}^{+\infty} dk_x dk_y \; exp(i\vec{k} \cdot \vec{r}) U_{\vec{k}}(z; t) \tag{4}$$

Let us analyze firstly the case of non-evanescent waves ([1]) defined by the condition $|\vec{k}|^2 < \frac{w^2}{v^2}$. Here

$$U_{\vec{k}}(z; t) = \int_{-\infty}^{+\infty} dw \; exp(iwt) U_{\vec{k}}(z; w) \tag{5}$$

The time-domain Fourier transformed field solution of eq. (1)-eq. (3) is explicitly (exactly) given by

$$U_{\vec{k}}(z;w) = F_+(w;\vec{k})exp\left(i\sqrt{\frac{w^2}{v^2}-(\vec{k})^2}\right)z + F_-(w;\vec{k})exp\left(-i\sqrt{\frac{w^2}{v^2}-(\vec{k})^2}\right)z \quad (6)$$

with

$$F_+(w;\vec{k}) = \frac{1}{2}\left(\hat{f}_k(w) - i\frac{\hat{g}_k(w)}{\sqrt{\frac{w^2}{v^2}-(\vec{k})^2}}\right) \quad (7)$$

$$F_-(w;\vec{k}) = \frac{1}{2}\left(\hat{f}_k(w) + i\frac{\hat{g}_k(w)}{\sqrt{\frac{w^2}{v^2}-(\vec{k})^2}}\right) \quad (8)$$

and

$$\hat{f}_k(w) = \int_{-\infty}^{+\infty} dt\ exp(iwt)f_k(t) \quad (9)$$

$$\hat{g}_k(w) = \int_{-\infty}^{+\infty} dt\ exp(iwt)g_k(t) \quad (10)$$

Note that only on this situation of non-evanescent case, one could claim to obtain well-posed extrapolating formulae with our new condition eq. (3), opposite to those similar results presented in ref. [1] without this condition.

Let us thus introduce the following variable change in the Fourier-Integral eq. (4) for the acoustic pressure field: $w' = w$; $k_x = p_x w$; $k_y = p_y\ w$ ([1]). We thus, have the following result:

$$U(t,z;\vec{r}) = \int_{-\infty}^{+\infty} d^2\vec{k}\ e^{i\vec{k}\cdot\vec{r}}\int_{-\infty}^{+\infty} dw\ e^{iwt}(U_{\vec{k}}(z;w)) =$$
$$\int_{-1/c}^{1/c} dp_x \int_{-1/c}^{1/c} dp_y \int_{-\infty}^{+\infty} dw \cdot e^{iwt}(w^2)e^{iw(\vec{p}\cdot\vec{r})}$$
$$\left\{ e^{iw\left(\sqrt{\frac{1}{c^2}-(\vec{p})^2}\right)^z} F_+(w,\vec{p}w) + e^{-iw\left(\sqrt{\frac{1}{c^2}-(\vec{p})^2}\right)^z} F_-(w,\vec{p}w)\right\} \quad (11)$$

The above expression by its turn is the sum of four Fourier integrals which are going to be analyzed. The first one is exactly given by the following expression (here $\hat{f}_{\vec{k}}(w) \equiv \hat{f}(w,\vec{k})$)

$$U^{(1)}(t,z;\vec{r}) = \int_{-1/c}^{1/c} dp_x \int_{-1/c}^{1/c} dp_y \int_{-\infty}^{+\infty} dw \cdot w^2 e^{iw\left[t+\vec{p}\cdot\vec{r}+\left(\sqrt{\frac{1}{v^2}-(\vec{p})^2}\right)\cdot z\right]}\frac{1}{2}\hat{f}(w,\vec{p}w) \quad (12)$$

As in ref. [1], we can re-write eq. (12) as a depth-extrapolator integral operator (by means of a t-convolution integral)

$$U^{(1)}(t,z;\vec{r}) = \int_{-1/c}^{1/c} dp_x \int_{-1/c}^{1/c} dp_y \; R^{(1)}(z,t,\vec{r};p)(*)_t \tilde{U}(z=0,t,p) \tag{13}$$

where the surface observed pressure field is given by

$$\tilde{U}(z=0,t,p) = \int_{-\infty}^{+\infty} dw \; e^{+iwt} \hat{f}(w,\vec{p}w) \tag{14}$$

and the depth-extrapolation Kernel is written explicitly as ([2])

$$\begin{aligned} R_1(z,t,x,p) &= \int_{-\infty}^{+\infty} dw \; e^{iwt} e^{iw\left(\vec{p}\cdot\vec{r}+\left(\sqrt{\frac{1}{v^2}-(\vec{p})^2}\right)z\right)} w^2 \\ &= -\pi\left(\frac{d^2}{d^2\xi}\left(\delta(\xi)+\frac{i}{\pi\xi}\right)\right)\Big|_{\xi=t+\vec{p}\cdot\vec{r}+\left(\sqrt{\frac{1}{v^2}-(\vec{p})^2}\right)z} \end{aligned} \tag{15}$$

The other part of the pressure field in eq. (11) corresponding to the knowledgement of the depth-derivative surface data eq. (3) and needed to turn the extrapolation problem a well-posed mathematical problem is given by a analogous Fourier integrals formulae

$$\begin{aligned} U^{(2)}(t,z,\vec{r}) &= -i\int_{-1/c}^{1/c} dp_x \int_{-1/c}^{1/c} dp_y \int_{-\infty}^{+\infty} dw \cdot w^2 \; e^{iw\left[t+\vec{p}\cdot\vec{r}\left(\sqrt{\frac{1}{v^2}-(\vec{p})^2}\right)z\right]}\left(\frac{\hat{g}(w,\vec{p}w)}{w\sqrt{\frac{1}{v^2}-(\vec{p})^2}}\right) \\ &= \int_{-1/c}^{1/c} dp_x \int_{-1/c}^{1/c} dp_y \; R^{(2)}(z,t,\vec{r};p)(*)_t \tilde{U}_t(z=0,t,\vec{p}) \end{aligned} \tag{16}$$

with

$$\tilde{U}_t(z=0,t,\vec{p}) = \int_{-\infty}^{+\infty} dw \; e^{iwt}\hat{g}(w,\vec{p}w)$$

and

$$R^{(2)}(z,t,\vec{r},\vec{p}) = \frac{1}{\sqrt{\frac{1}{v^2}-(\vec{p})^2}}\frac{\pi}{i}\left(\frac{d}{d\xi}\left(\delta(\xi)+\frac{i}{\pi\xi}\right)\right)\Big|_{\xi=t+\vec{p}\cdot\vec{r}+\left(\sqrt{\frac{1}{v^2}-(\vec{p})^2}\right)z} \tag{17}$$

The other two integrals are obtained by just changing $z \to -z$ in the above obtained formulae.

In the case of evanescent waves ([1]) defined by the condition $k_x^2 + k_y^2 \geq \frac{w^2}{v^2}$, the associated well-posed problem is governed by the following initial and boundary conditions imposed on eq. (1)

$$U_{\vec{k}}(z,t)\Big|_{t=0} = f_{\vec{k}}(t) \; and \; \lim_{z\to\infty} U_{\vec{k}}(z,t) = 0 \tag{18}$$

The solution takes, now, the following form

$$U^{(3)}(z,t,\vec{r},\vec{p}) = \int_{|\vec{p}|\geq \frac{1}{c}} d^2\vec{p} \int_{-\infty}^{+\infty} dw\ e^{iwt}\ w^2\ e^{iw(\vec{p}\cdot\vec{r})}\ exp\left(-w\left(\sqrt{(\vec{p})^2-\frac{1}{v^2}}\right)z\right)_{\hat{f}(w,\vec{p}w)}$$

$$= \int_{|\vec{p}|\geq \frac{1}{c}} d^2p\ R^{(3)}(z,t,\vec{r};\vec{p})(*)_t \tilde{U}_t(z=0,t,\vec{p}) \tag{19}$$

with the evanescent depth-extrapolating Kernel ([2])

$$R^{(3)}(z,t,\vec{r};\vec{p}) = -\pi\left(\frac{d^2}{d\xi^2}\left[\frac{i}{\pi\xi}\right]\right)_{\xi=t+\vec{p}\cdot\vec{r}+\left(\sqrt{(\vec{p})^2-\frac{1}{v^2}}\right)z} \tag{20}$$

3 Exact Formulae for Wave Field Extrapolation for Paraxial and Full Wave Equation in a Depth-Dependent Medium

In water geophysical mediums, the general harmonic acoustic pressure field $U(\vec{r},z,t)$ =Real $(\psi(\vec{r},z,t)exp\ i(kz-wt))$ satisfies the Paraxial wave equation

$$\left[i\frac{\partial}{\partial z}+\frac{1}{2k}\Delta_{\vec{r}}-k\mu(z)\right]\psi(\vec{r},z)=0 \tag{21}$$

where $\mu(z)=1-n^2(z)$ with $n(z)$ denoting the depth-dependent medium refraction index and $w=vk$ is the pressure field dispersation relation.

Let us consider the extrapolation problem of given the observed surface field $\psi(\vec{r},z=0)=\varphi(\vec{r})$; how one determines the full field $\psi(\vec{r},z)$ in terms of $\varphi(\vec{r})$.

In order to solve the above cited extrapolation problem, we consider the Ansatz in eq. (21)

$$\psi(\vec{r},z) = \int_{-\infty}^{+\infty} d^2\vec{\rho}\ C(\vec{\rho})\chi(\vec{\rho},z)e^{i\vec{\rho}\cdot\vec{r}} \tag{22}$$

It is straightforward to determine the coefficient $\chi(\vec{\rho},z)$, namelly:

$$\chi(\vec{\rho},z) = exp\left(-ik\int_0^z \mu(s)ds + i\frac{(\vec{\rho})^2}{2k}z\right) \tag{23}$$

and

$$C(\vec{\rho}) = \int_{-\infty}^{+\infty} d^2r\ e^{-i\vec{\rho}\cdot\vec{r}}\varphi(\vec{r}) \tag{24}$$

Or in convolution form

$$\psi(\vec{r},z)=\int_{-\infty}^{+\infty}d^2r'\cdot k(r-r';z)\varphi(r') \tag{25}$$

with

$$k(r-r',z)=\int_{-\infty}^{-\infty}d^2\vec{\rho}\;\chi(\vec{\rho},z)e^{i\vec{\rho}\cdot(\vec{r}-\vec{r}')} \tag{26}$$

Just for completeness, let us outline the generalization of the above procedure for the full wave equation in a z-dependent velocity medium (see eq. (1)-eq. (3))

$$\left[\frac{\partial^2}{\partial z^2}-\frac{1}{v^2(z)}\frac{\partial^2}{\partial t^2}-(k_x^2+k_y^2)\right]U_{\vec{k}}(z;t)=0 \tag{27}$$

$$U_{\vec{k}}(z,t)\Big|_{z=0}=f_{\vec{k}}(t) \tag{28}$$

$$\frac{\partial U_{\vec{k}}}{\partial z}(z,t)\Big|_{z=0}=g_{\vec{k}(t)} \tag{29}$$

Let us consider the depth coordinaate change

$$z'=\int_0^z\frac{ds}{v(s)} \tag{30}$$

The wave equation thus, takes the following form $\left(v'\equiv\frac{dv(z)}{dz}\right)$

$$\left\{\frac{\partial^2}{\partial z'^2}-\frac{\partial^2}{\partial t^2}+[v^2v'](z')\frac{\partial}{\partial z'}-(\vec{k})^2[v^2](z')\right\}U_{\vec{k}}(z;t)=0 \tag{31}$$

$$U_{\vec{k}}(z';t)\Big|_{z'=0}=f_{\vec{k}}(t) \tag{32}$$

$$\frac{\partial^2}{\partial z'}U_{\vec{k}}(z';t)\Big|_{z'=0}=v(0)g_{\vec{k}}(t) \tag{33}$$

We can are-write eq. (31) in the more suitable form

$$\left(e^{-w(z')}\left[\frac{\partial^2}{\partial z'^2}-\frac{\partial^2}{\partial t^2}+\Omega^2(z',(\vec{k})^2)\right]e^{+w(z')}\right)U_{\vec{k}}(z;t)=0 \tag{34}$$

with

$$2\frac{dw(z')}{dz'}=[v^2v''](z') \tag{35}$$

and

$$\Omega^2(z',(\vec{k})^2)=\left(\frac{dw}{dz'}\right)^2+\left(\frac{d^2w}{dz'^2}\right)-(\vec{k})^2(v^2)(z') \tag{36}$$

After a time-Fourier transform we are able to reduce the solution of the full wave equation to the one-dimensional depth wave equation with given initial conditions

$$\left(\frac{d^2}{dz'^2} + (w^2 + \Omega^2(z', (\vec{k})^2))\right) V_{\vec{k}}(z', w) = 0 \tag{37}$$

$$V_{\vec{k}}(z', w)\Big|_{z'=0} = \hat{f}_{\vec{k}}(w) \tag{38}$$

$$\frac{d}{dz'} V_{\vec{k}}(z', w)\Big|_{z=0} = v(0)\hat{g}_{\vec{k}}(w) + [v^2 v''](0)\hat{f}_{\vec{k}}(w) \tag{39}$$

If one is able to solve exactly (or numerically) the above written initial-value problem, the complete solution of eq. (27) will be given exactly by the extrapolation formulae below in the Fourier spatial-time domain

$$U_{\vec{k}}(z, w) = (e^{-w(z')} V_{\vec{k}}(z', w))\Big|_{z' = \int_0^z \frac{ds}{v(s)}} \tag{40}$$

Finally, let us comment the case of extrapolation problem in a homogeneous anistropic medium where the pressure vectorial field is governed by the following vectorial wave equation (with $1 \le i, j, k, \ell \le 3$)

$$\frac{\partial^2}{\partial t^2} U^i(\vec{r}, t) = C_{ijk\ell}\, \frac{\partial^2 U^k}{\partial x_j \partial x_\ell}(\vec{r}, t) \tag{41}$$

with $C_{ijk\ell}$ denoting the medium elastic constants.

In the extrapolation problem, we make the assumption that the 3×3 matrix $C_{i3k3} = A_{ik}$ related to the depth derivative in eq. (41) is inversible. As a consequence of this assumption one can replace eq. (41) by the following depth-dependent wave propagation problem in the Fourier domain (with $(\tilde{j}, \tilde{\ell}) \in \{\{1,2\} \times \{1,2\}\}$ and

$$\frac{d^2 U^i(z, \vec{k}, w)}{d^2 z} + i(\vec{k})_\ell [A^{-1}]_{ir} C_{r3s\ell}\, \frac{dU^s(z, \vec{k}, w)}{dz} + \left([A^{-1}]_{ir} w^2\right) U^r(z, \vec{k}, w) - \left([A^{-1}]_{ir} C_{r\tilde{j}m\tilde{\ell}} (\vec{k})_{\tilde{j}} (\vec{k})_{\tilde{\ell}}\right) U^m(z, \vec{k}, w) = 0 \tag{42}$$

with the well-posedeness initial and boundary conditions

$$U^i(z, \vec{k}, w)\Big|_{z=0} = f^i(\vec{k}, w) \tag{43}$$

$$\partial_z U^i(z, \vec{k}, w)\Big|_{z=0} = g^i(\vec{k}, w) \tag{44}$$

$$\lim_{z \to +\infty} U^i(z, \vec{k}, w) \equiv 0 \tag{45}$$

or in the first-order form 6×6 system of ordinary differential equations on $C_0([0,\infty])$

$$\frac{d}{dz}\begin{bmatrix} U^i(z,\vec{k},w) \\ V^i(z,\vec{k},w)\end{bmatrix} =$$

$$\begin{bmatrix} (1)_{3\times 3} & (0)_{3\times 3} \\ \left(-i(\vec{k})_\ell[A^{-1}]C_{r3s\ell}\right)_{3\times 3} & \left(-[A^{-1}]_{im}w^2 - [A^{-1}]_{i\ell}C_{\ell\tilde{j}m\tilde{\ell}}(\vec{k})_{\tilde{j}}(\vec{k})_{\tilde{\ell}}\right)_{3\times 3}\end{bmatrix}\begin{bmatrix} \pi^i(z,\vec{k},w) \\ U^i(z,\vec{k},w)\end{bmatrix} \tag{46}$$

The formal solution of eq. (46) is straightforward given by a exponential matrix

$$\begin{bmatrix} \vec{U}(z,\vec{k},w) \\ \vec{V}(z,\vec{k},w)\end{bmatrix} =$$

$$exp\left(\begin{bmatrix} (1)_{3\times 3} & (0)_{3\times 3} \\ \left(-i(\vec{k})_\ell[A^{-1}]C_{r3s\ell}\right)_{3\times 3} & \left(-[A^{-1}]_{im}w^2 - [A^{-1}]_{i\ell}C_{\ell\tilde{j}m\tilde{\ell}}(\vec{k})_{\tilde{j}}(\vec{k})_{\tilde{\ell}}\right)_{3\times 3}\end{bmatrix}\right)$$

$$\begin{bmatrix} \vec{f}(z,\vec{k},w) \\ \vec{g}(z,\vec{k},w)\end{bmatrix} \tag{47}$$

It is worth remark that explicitly solutions for eq. (47) need the Jordan form of the matrix ([4]), a very laborious task.

At this point, we propose to make an "Anisotropic Plane-Wave" expansion similar to eq. (6)-eq. (10) to write directly Fourier integral representations for the initial value problem eq. (42)-eq. (45). Let us sketchy our procedure.

As first step, let us remark that any system of ordinary differential equation of the form

$$\frac{d^2U^i(z)}{d^2z} + \alpha_{ij}\,\frac{dU^j(z)}{dz} + \beta_{ik}U^k(z) = 0 \tag{48}$$

$$U^i(0) = a^i \tag{49}$$

$$\frac{dU^i}{dz}(0) = b^i \tag{50}$$

can be put in the following somewhat canonical form without the first order derivative term d/dz

$$\left(\frac{d^2}{dz^2}\delta_{ij}\left([\beta] - \frac{[\alpha]^2}{4}\right)\right)_{ij}\left(e^{-\frac{[\alpha]}{2}z}\vec{U}(z)\right)_j = 0 \tag{51}$$

$$e^{-\frac{[\alpha]}{2}z}\vec{U}(z)\Big|)_{z=0} = a^i \tag{52}$$

$$\frac{d}{d^2}\left(e^{-\frac{[\alpha]}{2}z}\vec{U}(z)\right)\Big|_{z=0} = -\frac{[\alpha]}{2}ija_j + b_i \tag{53}$$

In order to solve eq. (51) in a straightforward way, let us make the Plane-Wave onsatz for $\vec{S}(z) = e^{-\frac{[\alpha]}{2}z}\vec{U}(z)$, namelly:

$$\vec{S}(z) = \vec{A}e^{iwz} \tag{54}$$

As a consequence, we have that $w^2 = \{\lambda^+, \lambda^-, \lambda^0\}$ are the eigenvalues of the matrix $[\beta] - [\alpha]^2/4$ and $\vec{A} = \{\vec{A}_+, \vec{A}_-, \vec{A}_0\}$ are its associated linear independent eigenvectors.

The general solution of eq. (51), thus, takes the simple complex form with six unknow constants $\{c_+, \tilde{c}_+, c_-, \tilde{c}_-, c_0, \tilde{c}_-\}$

$$\vec{S}(z) = c_+\vec{A}_+e^{i(\sqrt{\lambda^+})z} + \tilde{c}_+\vec{A}_+e^{i(\sqrt{\lambda^+})z} + c_-\vec{A}_-e^{i(\sqrt{\lambda^-})z} + \tilde{c}_-\vec{A}_-e^{i(\sqrt{\lambda^-})z}$$
$$+c_0\vec{A}_0e^{i(\sqrt{\lambda^0})z} + \tilde{c}_0\vec{A}_0e^{i(\sqrt{\lambda^0})z} \tag{55}$$

The six unknow constants $\{c_+, \tilde{c}_+, c_-, \tilde{c}_-, c_0, \tilde{c}_0\}$ are easily evaluated by adjusting eq. (55) to the initial conditions eq. (52)-eq. (53) (a 46×6 linear system)

$$\begin{cases} c_+\vec{A}_+ + \tilde{c}_+\vec{A}_+ + c_-\vec{A}_- + \tilde{c}_-\vec{A}_- + c_0\vec{A}_0 + \tilde{c}_0\vec{A}_0 = a^i \\ (\sqrt{\lambda^+})c_+\vec{A}_+ - (\sqrt{\lambda^+})\tilde{c}_+\vec{A}_+ + \sqrt{\lambda^-}c_-\vec{A}_- - \sqrt{\lambda^-}\tilde{c}_-\vec{A}_- \\ +\sqrt{\lambda^0}c_0\vec{A}_0 - \sqrt{\lambda^0}\tilde{c}_0\vec{A}_0 = -\frac{[\alpha]}{2}ija_j + b_i \end{cases} \tag{56}$$

In our case eq. (42)-eq. (45), the Fourier-integral solution based on the ansatz eq. (55) will involves the explicitly expressions for the medium tensor c_{rsen} and a $(\vec{k}, w)$ – variables dependence as a consequence a detailed analysis of the algebraic singularities on the associated integration formulae similar to those analyzed on ref. (5) and reference therein will be needed. This work will be presented elsewhere.

Acknowledgements: Luiz C.L. Botelho is supported by CNPq, the Brazilian Science Agency. I am very thankful to Professor Helayël-Neto from CBPF for scientific support.

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