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# CONVEX CONES AND CONVEX SETS

by

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**Abstract.** Convex cones (Definitions 2 and 3) and convex sets (Definition 20) are introduced without assuming they are vectorial, that is they are convex subcones and convex subsets of real vector spaces. The set of all nonvoid convex subsets of a convex set is a convex set, and the set of all nonvoid convex subsets of a convex cone is a convex cone (Definition 24). A convex cone is vectorial if and only if it satisfies the cancellation rule for convex cones (Proposition 10). A convex set is vectorial if and only if it satisfies the cancellation rule for convex sets (Proposition 34). Every convex set is a convex subset of some convex cone (Proposition 31). Convex cones with constant multiplication are simple instances (Proposition 18). Every convex cone whose power is strictly less than the power of the continuum has a constant multiplication (Proposition 40). Every convex set whose power is strictly less than the power of the continuum has a unique convex cone structure with constant multiplication defining its convex set structure (Proposition 41). There are equivalent conditions for a convex set to satisfy the cancellation rule for convex sets (Proposition 44). Linear convex sets are easily characterized (Proposition 46). For convex sets satisfying the one dimensional injection rule, there is a smallest convex set failing to satisfy the cancellation rule for convex sets (Proposition 48). Every convex cone is the image by a surjective convex cone map of a vectorial convex cone, and every convex set is the image by a surjective convex set map of a vectorial convex set (Proposition 35). Associated with any convex cone, there is a largest vectorial convex cone (Propositions 13 and 10). Associated with any convex set, there is a largest vectorial convex set (Proposition 42).

Key-words: Convex cones; Convex sets; Cancellation rules; Vectoriality.

**Notation 1.** $\mathbf{N}$ is the set of all positive integers, $\mathbf{N}^*$ is the set of all strictly positive integers, $\mathbf{R}$ is the set of all real numbers, $\mathbf{R}_+$ is the set of all positive real numbers, $\mathbf{R}_+^*$ is the set of all strictly positive real numbers, $\mathbf{I}$ is the closed interval in $\mathbf{R}$ of extremities 0,1, and $\mathbf{J}$ is the open interval in $\mathbf{R}$ of extremities $0,1$.

DEFINITION 2. *A **convex cone** is a set that either is empty, or else in which we are given maps* $(x_1,x_2) \in C \times C \mapsto x_1 + x_2 \in C$ and $(\lambda, x) \in \mathbf{R}_+^* \times C \mapsto \lambda x \in C$ *such that* $x_1+x_2 = x_2+x_1$, $(x_1+x_2)+x_3 = x_1+(x_2+x_3)$, $\lambda(x_1+x_2) = \lambda x_1+\lambda x_2$, $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$, $\lambda_1(\lambda_2 x) = (\lambda_1\lambda_2)x, 1x = x$ *for* $x_1, x_2, x_3 \in C$, $\lambda, \lambda_1, \lambda_2 \in \mathbf{R}_+^*$. *If* $n \in \mathbf{N}^*$, $\lambda_1,\ldots,\lambda_n \in \mathbf{R}_+$, $x_1,\ldots,x_n \in C$, *but some* $\lambda_i > 0$, $i = 1,\ldots,n$, *we define* $\lambda_1 x_1+\cdots+\lambda_n x_n = \lambda_{i_1}x_{i_1}+\cdots+\lambda_{i_k}x_{i_k}$, *where* $k = 1,\ldots,n$ *and* $i_1 < \cdots < i_k$ *denote the values of* $i = 1,\ldots n$ *such that* $\lambda_i > 0$. *A **convex subcone** of* $C$ *is a subset* $D$ *of* $C$ *such that* $x_1, x_2 \in D$ *imply* $x_1 + x_2 \in D$, *and* $\lambda \in \mathbf{R}_+^*$, $x \in D$ *imply* $\lambda x \in D$. *Then* $D$ *is a convex cone in a natural way. An intersection of convex subcones of a convex cone* $C$ *is a convex subcone of* $C$. *A subset* $G$ *of a convex cone* $C$ *generates a convex subcone of* $C$, *namely the intersection of all convex subcones of* $C$ *containing* $G$, *alternatively the set of all* $\lambda_1x_1 + \cdots + \lambda_n x_n$ *for* $n \in \mathbf{N}^*$, $\lambda_1,\ldots,\lambda_n \in \mathbf{R}_+^*$, $x_1,\ldots,x_n \in G$. *If* $C$ *is a convex cone,* $n \in \mathbf{N}^*$, $\lambda_1,\ldots,\lambda_n \in \mathbf{R}_+$, *but some* $\lambda_i > 0, i = 1,\ldots,n, X_1,\ldots,X_n \subset C$, *we define* $\lambda_1X_1+\cdots+\lambda_nX_n = \{\lambda_1x_1+\cdots+\lambda_nx_n;\ x_1 \in X_1,\ldots,x_n \in X_n\}$. *We say that* $f : C \to D$ *is a **convex cone map** between the convex cones* $C$ *and* $D$ *when* $f(x_1 + x_2) = f(x_1) + f(x_2)$ *and* $f(\lambda x) = \lambda f(x)$ *for* $\lambda \in \mathbf{R}_+^*$, $x_1,x_2,x \in C$. *If* $X$ *is a convex subcone of* $C$, *then* $f(X)$ *is a convex subcone of* $D$, *in particular* $f(C)$ *is a convex subcone of* $D$. *If* $Y$ *is a convex subcone of* $D$, *then* $f^{-1}(Y)$ *is a convex subcone of* $C$. *If* $f$ *is moreover injective, it is a **convex cone isomorphism**. A cartesian product of convex cones is a convex cone.*

DEFINITION 3. *A **convex cone with zero** is a convex cone* $C$ *in which there is a necessarily unique zero* $0 \in C$ *such that* $x + 0 = x$ *for all* $x \in C$. *It follows that* $\lambda 0 = 0$ *for* $\lambda \in \mathbf{R}_+^*$. *If we extend* $(\lambda, x) \in \mathbf{R}_+^* \times C \mapsto \lambda x \in C$ *to* $(\lambda, x) \in \mathbf{R}_+ \times C \mapsto \lambda x \in C$ *by defining* $0x = 0$ *for* $x \in C$, *the rules of Definition 1 remain true if* $\mathbf{R}_+^*$ *is replaced by* $\mathbf{R}_+$. *The definition of* $\lambda_1x_1 + \cdots + \lambda_nx_n$ *we have in Definition 1 coincides with the value it has now, and* $0x_1 + \cdots + 0x_n = 0$ *for* $x_1,\ldots,x_n \in C$.

*A convex subcone with zero of $C$ is a subset $D$ of $C$ such that $x_1, x_2 \in D$ imply $x_1 + x_2 \in D$, $\lambda \in \mathrm{R}_+$, $x \in D$ imply $\lambda x \in D$, and $0 \in D$. Then $D$ is a convex cone with zero in a natural way. An intersection of convex subcones with zero of a convex cone with zero $C$ is a convex subcone with zero of $C$. A subset $G$ of a convex cone with zero $C$ generates a convex subcone with zero of $C$, namely the intersection of all convex subcones with zero of $C$ containing $G$, alternatively the set of all $\lambda_1 x_1 + \cdots + \lambda_n x_n$ for $n \in \mathrm{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathrm{R}_+$, $x_1, \ldots, x_n \in G$. If $C$ is a convex cone with zero, $n \in \mathrm{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathrm{R}_+$, $X_1, \ldots, X_n \subset C$, we define $\lambda_1 X_1 + \cdots + \lambda_n X_n = \{\lambda_1 x_1 + \cdots + \lambda_n x_n;\ x_1 \in X_1, \ldots, x_n \in X_n\}$. We say that $f : C \to D$ is a convex cone with zero map between the convex cones with zero $C$ and $D$ when $f(x_1 + x_2) = f(x_1) + f(x_2)$, and $f(\lambda x) = \lambda f(x)$ for $\lambda \in \mathrm{R}_+$, $x_1, x_2, x \in C$. Then $f(0) = 0$. If $X$ is a convex subcone with zero of $C$, then $f(X)$ is a convex subcone with zero of $D$, in particular $f(C)$ is a convex subcone with zero of $D$. If $Y$ is a convex subcone with zero of $D$, then $f^{-1}(Y)$ is a convex subcone with zero of $C$, in particular $f^{-1}(0)$ is a convex subcone with zero of $C$. If $f$ is moreover injective, it is a convex cone with zero isomorphism. A cartesian product of convex cones with zero is a convex cone with zero.*

**Remark 4.** Let $C$ be a convex cone without zero. We get a convex cone with zero $C_0$ in a natural way as follows. Fix a point $0 \notin C$. Set $C_0 = C \cup \{0\}$. Extend $(x_1, x_2) \in C \times C \mapsto x_1 + x_2 \in C$, $(\lambda, x) \in \mathrm{R}_+^* \times C \mapsto \lambda x \in C$ to $(x_1, x_2) \in C_0 \times C_0 \mapsto x_1 + x_2 \in C_0$, $(\lambda, x) \in \mathrm{R}_+ \times C_0 \mapsto \lambda x \in C_0$ by defining $0 + 0 = 0$, $x + 0 = 0 + x = x$, $00 = 0, 0x = 0, \lambda 0 = 0$ for $\lambda \in \mathrm{R}_+^*$, $x \in C$. Then $C_0$ is a convex cone with zero containing $C$ as a convex subcone.

**Example 5.** A real vector space $E$ is a convex cone with zero. A convex subcone $C$ of $E$ is a convex cone. If, moreover, $0 \in C$, then $C$ is a convex cone with zero. In a real vector space $E$, the vector subspace of $E$ generated by a convex subcone $C \neq \phi$ is $C - C$. Hence $E$ is its vector subspace generated by $C$ when $E = C - C$.

**Definition 6.** *Let $C \neq \phi$ be a convex cone, $E$ and $F$ be real vector spaces, $f : C \to E$ and $g : C \to F$ be convex cone maps, $E = f(C) - f(C)$ and $F = g(C) - g(C)$. We say that $f$ is larger than $g$ and write $f \geq g$ when there is a linear map $h : E \to F$ such that $hf = g$. Then $h$ is unique and surjective. We say that $f$ and $g$ are equivalent and write $f \sim g$ when $f \geq g$, $g \geq f$. This* *happens when $f \geq g$ and $h$ is bijective.*

PROPOSITION 7. *Let $C \neq \phi$ be a convex cone. There are a real vector space $E$ and a convex cone map $f : C \to E$ such that $E = f(C) - f(C)$, and $f$ is larger than any convex cone $g : C \to F$, where $F$ is a real vector space such that $F = g(C) - g(C)$. Moreover, $f : C \to E$ is essentially unique in the sense that, if $f_i : C \to E_i (i = 1, 2)$ are two choices for $f : C \to E$, then $f_1 \sim f_2$.*

PROOF: We may assume that $C$ has a zero (Remark 4). Consider the convex cone with zero $C \times C$, and the equivalence relation on it defined by $(t_1, x_1) \sim (t_2, x_2)$ when $t_1, x_1, t_2, x_2 \in C$ and $t_1 + x_2 + u = t_2 + x_1 + u$ for some $u \in C$. Let $E$ be the quotient set of $C \times C$, and $\pi : C \times C \to E$ be the surjective quotient map. Then $E$ is a convex cone in a unique way so that $\pi$ is a convex cone map. Actually $E$ is a real vector space because its additive nonvoid is an additive group. Indeed $\pi(x, x)$ is the zero of $E$, and $\pi(t, x)$ has the symmetric $-\pi(t, x) = \pi(x, t)$ for $t, x \in C$. Define $f : C \to E$ by $f(x) = \pi(x, 0)$ for $x \in C$. Then $f$ is a convex cone map, and $E = f(C) - f(C)$. Let $g : C \to F$ be a convex cone map, where $F$ is a real vector space such that $F = g(C) - g(C)$. Define the map $h : E \to F$ by $h[f(x) - f(t)] = g(x) - g(t)$ for $t, x \in C$. It is well defined since $f(x_1) - f(t_1) = f(x_2) - f(t_2)$ for $t_1, x_1, t_2, x_2 \in C$ implies $f(t_1 + x_2) = f(t_2 + x_1)$, hence $(t_1 + x_2, 0) \sim (t_2 + x_1, 0)$, and $t_1 + x_2 + u = t_2 + x_1 + u$ for some $u \in C$. Thus $g(t_1 + x_2 + u) = g(t_2 + x_1 + u)$ and $g(x_1) - g(t_1) = g(x_2) - g(t_2)$. Then $h$ is the unique linear map such that $hf = g$. It is surjective. Thus $f \geq g$. The essential uniqueness for $f : C \to E$ is clear. ∎

DEFINITION 8. *A convex cone is said to be vectorial when it is a convex subcone of some real vector space.*

DEFINITION 9. *A convex cone $C$ satisfies the cancellation rule for convex cones* (CRCC) when $x_1, x_2, x_3 \in C$, $x_1 + x_2 = x_1 + x_3$ imply $x_2 = x_3$.

PROPOSITION 10. *Let $C \neq \phi$ be a convex cone. If $E$ is a real vector space, and the convex cone map $f : C \to E$ is injective, then $C$ satisfies the cancellation rule for convex cones. Conversely, if $C$ satisfies this rule, then $f : C \to E$ of Proposition 7 is injective. Hence a convex cone is vectorial if and only if it satisfies the cancellation rule for convex cones.*

**Proof**: Consider the first assertion is the statement. Then $t+u=x+u$ for $t,x,u\in C$ imply $f(t)+f(u)=f(x)+f(u)$, hence $f(t)=f(x)$ and $t=x$. Thus the cancellation rule for convex cones holds in $C$. Consider the second assertion in the statement. Use notation of proof of Proposition 7. By assuming that $C$ has a zero (Remark 4), then $f(t)=f(x)$ for $t,x\in C$ imply $(t,0)\sim(x,0)$, hence $t+u=x+u$ for some $u\in C$, and $t=x$. Thus $f$ is injective. The third assertion in the statement is then clear. ∎

**Proposition 11.** *Let $C\neq\phi$ be a convex cone, $E$ and $F$ be real vector spaces, $f:C\to E$ and $g:C\to F$ be injective convex cone maps, $E=f(C)-f(C)$ and $F=g(C)-g(C)$. There is a unique linear map $h:E\to F$ such that $hf=g$. Moreover, $h$ is bijective.*

**Proof**: Define $h:E\to F$ by $h[f(x)-f(t)]=g(x)-g(t)$ for $t,x\in C$. It is well defined since $f(x_1)-f(t_1)=f(x_2)-f(t_2)$ for $t_1,x_1,t_2,x_2\in C$ imply $f(t_1+x_2)=f(t_2+x_1)$, hence $t_1+x_2=t_2+x_1$ and $g(x_1)-g(t_1)=g(x_2)-g(t_2)$. Moreover, $h$ is the unique linear map such that $hf=g$. By interchanging the roles of $f$ and $g$ we see that $h$ is bijective. ∎

Definition 6, Propositions 7 and 10 have analogous forms for convex cones satisfying the cancellation rule for convex cones in place of real vector spaces, as follows.

**Definition 12.** *Let $C\neq\phi$ be a convex cone, $E$ and $F$ be convex cones satisfying the cancellation rule for convex cones, $f:C\to E$ and $g:C\to F$ be surjective convex cone maps. We say that $f$ is larger than $g$ and write $f\geq g$ when there is a map $h:E\to F$ such that $hf=g$. Then $h$ is a convex cone map, unique and surjective. We say that $f$ and $g$ are equivalent and write $f\sim g$ when $f\geq g$, $g\geq f$. This happens when $f\geq g$ and $h$ is bijective.*

**Proposition 13.** *Let $C\neq\phi$ be a convex cone. There are a convex cone satisfying the cancellation rule $E$ and a surjective convex cone map $f:C\to E$, such that $f$ is larger than any surjective convex cone map $g:C\to F$, where $F$ is a convex cone satisfying the cancellation rule for convex cones. Moreover, $f:C\to E$ is essentially unique in the sense that, if $f_i:C\to E_i (i=1,2)$ are two choices for $f:C\to E$, then $f_1\sim f_2$.*

**Proof:** Consider the equivalence relation on $C$ defined by $t \sim x$ when $t, x \in C$ and $t + u = x + u$ for some $u \in C$. Let $E$ be the quotient set of $C$, and $f : C \to E$ be the surjective quotient map. Then $E$ is a convex cone in a unique way so that $f$ is a convex cone map. Actually $E$ satisfies the cancellation rule for convex cones. Indeed, if $f(t) + f(u) = f(x) + f(u)$ for $t, x, u \in C$, then $f(t+u) = f(x+u)$ and $t + u + v = x + u + v$ for some $v \in C$. Therefore $t \sim x$, and $f(t) = f(x)$ as wanted. Let $g : C \to F$ be a surjective convex cone map, where $F$ is a convex cone satisfying the cancellation rule for convex cones. Define $h : E \to F$ by $h[f(x)] = g(x)$ for $x \in C$. It is well defined since $f(x_1) = f(x_2)$ for $x_1, x_2 \in C$ imply $x_1 \sim x_2$, hence $x_1 + u = x_2 + u$ for some $u \in C$, hence $g(x_1) + g(u) = g(x_2) + g(u)$ and $g(x_1) = g(x_2)$. Then $h$ is the unique map satisfying $hf = g$. It is a surjective convex cone map. Thus $f \geq g$. The essential uniqueness for $h : C \to E$ is clear. ∎

**Proposition 14.** *Let $C \neq \phi$ be a convex cone. If $E$ is a convex cone satisfying the cancellation rule for convex cones, and the convex cone map $f : C \to E$ is injective, then $C$ satisfies the cancellation rule for convex cones. Conversely, if $C$ satisfies this rule, then $f : C \to E$ of Proposition 13 is bijective.*

**Proof:** Consider the first assertion in the statement. Then $t + u = x + u$ for $t, x, u \in C$ imply $f(t) + f(u) = f(x) + f(u)$, hence $f(t) = f(x)$ and $t = x$. Thus the cancellation rule holds in $C$. Consider the second assertion in the statement. Then $t \sim x$ in the proof of Proposition 13 is the same as $t = x$. ∎

**Remark 15.** If we consider $f : C \to E$ of Proposition 7, then $f : C \to f(C)$ is a convex cone map whose existence and essential uniqueness is asserted by Proposition 13. Conversely, it is easy to see that Proposition 7 is implied by Proposition 13.

**Proposition 16.** *Let $C$ be a convex cone and $x \in C$. Then either the map $\lambda \in \mathbf{R}_+^* \mapsto \lambda x \in C$ is injective, or else $\lambda x = x$ for all $\lambda \in \mathbf{R}_+^*$ (constant multiplication on $x$). The second alternative occurs if and only if $x + x = x$.*

**Proof:** Consider the multiplicative subgroup $G = \{\lambda \in \mathbf{R}_+^*; \lambda x = x\}$ of $\mathbf{R}_+^*$. If $\lambda, \mu \in G$, $\alpha, \beta \in \mathbf{R}_+^*$, $\alpha + \beta = 1$, then $(\alpha\lambda + \beta\mu)x = x$, hence $\alpha\lambda + \beta\mu \in G$.

Thus $G$ is a convex subset of $\mathbb{R}_+^*$. It follows that either $G = \{1\}$ or $G = \mathbb{R}_+^*$, proving the first part of the proposition. In fact, if $G \neq \{1\}$ and $\lambda \in G, \lambda \neq 1$, we may assume $\lambda > 1$, for $\lambda < 1$ implies $1/\lambda \in G$, $1/\lambda > 1$. Thus $\lambda^n \in G$ and $[1, \lambda^n] \subset G$ for $n \in \mathbb{N}$. It follows that $[1, +\infty[ \subset G$, hence $]0, 1] \subset G$. Thus $G = \mathbb{R}_+^*$. The second alternative occurs if and only if $2x = x$, by the first part of the proposition, that is $x + x = x$. ∎

**PROPOSITION 17.** *Let $C$ be a convex cone and $x_1, x_2, x_3 \in C$. If $\lambda x_1 + x_2 = \lambda x_1 + x_3$ for some $\lambda \in \mathbb{R}_+^*$, this equality holds for all $\lambda \in \mathbb{R}_+^*$.*

**PROOF:** From $\lambda x_1 + x_2 = \lambda x_1 + x_3$ we get $\lambda x_1 + 2x_2 = \lambda x_1 + x_2 + x_3$ and $\lambda x_1 + x_2 + x_3 = \lambda x_1 + 2x_3$, hence $\lambda x_1 + 2x_2 = \lambda x_1 + 2x_3$ and $(\lambda/2)x_1 + x_2 = (\lambda/2)x_1 + x_3$. By induction we have $(\lambda/2^n)x_1 + x_2 = (\lambda/2^n)x_1 + x_3$ for $n \in \mathbb{N}$. Given any $\mu \in \mathbb{R}_+^*$, choose $n \in \mathbb{N}$ so that $\lambda/2^n < \mu$. Then $(\mu - \lambda/2^n)x_1 + (\lambda/2^n)x_1 + x_2 = (\mu - \lambda/2^n)x_1 + (\lambda/2^n)x_1 + x_3$, that is $\mu x_1 + x_2 = \mu x_1 + x_3$. ∎

**PROPOSITION 18.** *(1) Let $C \neq \phi$ be an ordered set that is an inflattice. Define $x_1 + x_2 = x_1 \wedge x_2$ for $x_1, x_2 \in C$. Then $C$ is an associative, commutative, additive nonvoid, $x + x = x$, and $x_1 \leq x_2$ if and only if $x_1 + x_2 = x_1$ for $x, x_1, x_2 \in C$. Conversely, let $C \neq \phi$ be an associative, commutative, additive nonvoid such that $x + x = x$ for $x \in C$. We obtain an order on $C$ by $x_1 \leq x_2$ when $x_1 + x_2 = x_1$ for $x_1, x_2 \in C$, and $C$ is an inflattice since $x_1 + x_2 = x_1 \wedge x_2$ for $x_1, x_2 \in C$. Moreover, $C$ has a zero as an additive nonvoid if and only if $C$ has a largest element as an ordered set, this zero and largest element being identical. (2) Let $C \neq \phi$ be an associative, commutative, additive nonvoid. Define the constant multiplication $(\lambda, x) \in \mathbb{R}_+^* \times C \mapsto \lambda x = x \in C$. Then $C$ is a convex cone if and only if $x + x = x$ for $x \in C$. If $C$ has a zero $0$ as an additive nonvoid, we complete the definition of the trivial multiplication by $0x = 0$ for $x \in C$. Then $C$ is a convex cone with zero.*

**PROOF:** (1) is clear. Concerning (2), we remark only that the convex cone condition $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$ for $\lambda_1, \lambda_2 \in \mathbb{R}_+^*$, $x \in C$, is equivalent to $x = x + x$ for $x \in C$ under the circumstances. ∎

**Example 19.** Let $C$ be an associative, commutative, additive nonvoid satisfying

$x + x = x$ for $x \in C$, having at least two elements. Then $C$ is a convex cone with constant multiplication not satisfying the cancellation rule for convex cones. In fact, we can find $x_1, x_2 \in C$, $x_1 \neq x_2$, $x_1 + x_2 = x_2$, and then $x_1 + x_2 = x_2 + x_2$ does not imply $x_1 = x_2$.

DEFINITION 20. *A convex set $X$ is a set that either is empty, or else in which we are given a map that to every $n \in \mathbb{N}^*, \lambda_1, \ldots, \lambda_n \in \mathsf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \ldots, x_n \in X$, associates a point of $X$ denoted by $\lambda_1 x_1 + \cdots + \lambda_n x_n = \sum_{1 \le i \le n} \lambda_i x_i$, called a convex combination of $x_1, \ldots, x_n$ with coefficients $\lambda_1, \ldots, \lambda_n$, so that: (1) Commutativity. We have $\sum_{1 \le i \le n} \lambda_{\sigma(i)} x_{\sigma(i)} = \sum_{1 \le i \le n} \lambda_i x_i$ for any permutation $\sigma$ of $\{1, \ldots, n\}$. (2) Associativity. We have*

$$\sum_{1 \le j \le n} \mu_j \left( \sum_{1 \le i \le m_j} \lambda_{ij} x_{ij} \right) = \sum_{\substack{1 \le i \le m_j \\ 1 \le j \le n}} (\lambda_{ij} \mu_j) x_{ij}$$

*for every $n \in \mathbb{N}^*$, $m_j \in \mathbb{N}^*$ for $j = 1, \ldots, n$, $\lambda_{ij} \in \mathsf{J}$ for $i = 1, \ldots, m_j$, $j = 1, \ldots, n$ such that $\lambda_{1j} + \cdots + \lambda_{m_j j} = 1$ for $j = 1, \ldots, n$, $\mu_j \in \mathsf{J}$ for $j = 1, \ldots, n$, $\mu_1 + \cdots + \mu_n = 1$, $x_{ij} \in X$ for $i = 1, \ldots, m_j$, $j = 1, \ldots, n$, where we note that*

$$\sum_{\substack{1 \le i \le m_j \\ 1 \le j \le n}} \lambda_{ij} \mu_j = 1 \, .$$

*(3) Distributivity. We have $\lambda_1 x + \cdots + \lambda_n x = x$ for $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathsf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x \in X$. If $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathsf{I}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \ldots, x_n \in X$, we define $\lambda_1 x_1 + \cdots + \lambda_n x_n = \lambda_{i_1} x_{i_1} + \cdots + \lambda_{i_k} x_{i_k}$, where $k = 1, \ldots, n$ and $i_1 < \cdots < i_k$ denote the values of $i = 1, \ldots, n$ such that $\lambda_i > 0$. The above conditions (1), (2), (3) remain true if we replace $\mathsf{J}$ by $\mathsf{I}$. A convex subset of $X$ is a subset $Y$ of $X$ such that $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathsf{J}$, $\lambda_1 + \cdots + \lambda_n = 1, x_1, \ldots, x_n \in Y$, imply $\lambda_1 x_1 + \cdots + \lambda_n x_n \in Y$. Then $Y$ is a convex set in a natural way. An intersection of convex subsets of a convex set $X$ is a convex subset of $X$. A subset $G$ of a convex set $X$ generates a convex subset of $X$ called the convex hull of $G$ in $X$, namely the intersection of all convex subsets of $X$ containing $G$, alternatively the set of all $\lambda_1 x_1 + \cdots + \lambda_n x_n$ for $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathsf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \ldots, x_n \in G$. If $X$ is a convex set, $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathsf{I}$, $\lambda_1 + \cdots + \lambda_n = 1$, $X_1, \ldots, X_n \subset X$, we define*

$\lambda_1 X_1 + \cdots + \lambda_n X_n = \{\lambda_1 x_1 + \cdots + \lambda_n x_n;\ x_1 \in X_1, \ldots, x_n \in X_n\} \subset X$. *We say that* $f : X \to Y$ *is a **convex set map** between the convex sets* $X$ *and* $Y$ *when* $f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$ *for* $n \in \mathbb{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathsf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \ldots, x_n \in X$. *If* $V$ *is a convex subset of* $X$, *then* $f(V)$ *is a convex subset of* $Y$, *in particular* $f(X)$ *is a convex subset of* $Y$. *If* $W$ *is a convex subset of* $Y$, *then* $f^{-1}(W)$ *is a convex subset of* $X$, *in particular* $f^{-1}(y)$ *is a convex subset of* $X$ *for* $y \in Y$. *If* $f$ *is moreover injective, it is a **convex set isomorphism**. A cartesian product of convex sets is a convex set. A family of elements* $(x_i)_{i \in I}$ *of a convex set* $X$ *is **convexily independent** when* $x_j$ *does not belong to the convex hull of the set* $\{x_i; i \in I, i \neq j\}$ *for every* $j \in I$; *then* $i \in I \mapsto x_i \in X$ *is injective. Otherwise, the family is **convexily dependent**.*

**Example 21.** A convex cone is a convex set in a natural way. A convex cone map is a convex set map. A convex subset of a convex cone is a convex set. The convex subcone of a convex cone $C$ generated by a convex subset $X$ of $C$ is $\mathsf{R}_+^* X$. If $C$ has a zero, the convex subcone with zero generated by a convex subset $X \neq \phi$ of $C$ is $\mathsf{R}_+ X$.

**Notation 22.** We define

$$\frac{\lambda_1 x_1 + \cdots + \lambda_n x_n}{\lambda_1 + \cdots + \lambda_n} = \mu_1 x_1 + \cdots + \mu_n x_n$$

for $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathsf{R}_+$, $\lambda_1 + \cdots + \lambda_n > 0$, $x_1, \ldots, x_n \in X$ and $\mu_i = \lambda_i / (\lambda_1 + \cdots + \lambda_n)$ for $i = 1, \ldots, n$. In a convex cone, this notation has a meaning that agrees with the one we are giving now.

**Remark 23.** Let $C \neq \phi$ be a convex cone. We may ask whether the convex set structure of $C$ determines its convex cone structure. The answer is easily seen to be negative. If $C$ has a constant multiplication, the answer is affirmative since $x_1 + x_2 = (1 - \lambda) x_1 + \lambda x_2$ for $\lambda \in \mathsf{J}$, $x_1, x_2 \in C$.

DEFINITION 24. *The set* $CS(X)$ *of all nonvoid convex subsets of a convex set* $X$ *is a convex set with respect to the definition of* $\lambda_1 X_1 + \cdots + \lambda_n X_n \in CS(X)$ *we gave for* $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathsf{J}$, $\lambda_1 + \cdots + \lambda_n = 1, X_1, \ldots, X_n \in CS(X)$ *(see Definition 20). We have the injective map* $x \in X \mapsto \{x\} \in CS(X)$. *The* *set $CS(C)$ of all nonvoid convex subsets of a convex cone $C$ is a convex cone with respect to the definitions of $X_1+X_2 \in CS(C)$ and $\lambda X \in CS(C)$ we gave for $\lambda \in \mathbb{R}_+^*$, $X_1, X_2, X \in CS(C)$ (see Definition 2). The convex set structure we defined on $CS(C)$ coincides with the one we get from its convex cone structure.*

PROPOSITION 25. *(1) A subset $Y$ of a convex set $X$ is a convex subset of $X$ if and only if $(1-\lambda)x_1+\lambda x_2 \in Y$ for $\lambda \in \mathsf{J}$, $x_1, x_2 \in Y$. (2) If $X$ and $Y$ are convex sets, then $f: X \to Y$ is a convex set map if and only if $f[(1-\lambda)x_1+\lambda x_2] = (1-\lambda)f(x_1)+\lambda f(x_2)$ for $\lambda \in \mathsf{J}$, $x_1, x_2 \in X$. (3) The knowledge of all convex combinations $(1-\lambda)x_1+\lambda x_2$ for $\lambda \in \mathsf{J}$, $x_1, x_2 \in X$ determines the convex set structure of a convex set $X$.*

The proof is straightforward.

PROPOSITION 26. *Let $X$ be a set in which to every $\lambda \in \mathsf{J}$, $x_1, x_2 \in X$ we associate a point represented by $(1-\lambda)x_1+\lambda x_2 \in X$. There is a necessarily unique convex set structure on $X$ defining these convex combinations of any two points if and only if: (1) Commutativity. We have $\lambda x_2+(1-\lambda)x_1 = (1-\lambda)x_1+\lambda x_2$ for $\lambda \in \mathsf{J}$, $x_1, x_2 \in X$. Associativity. We have*

$$\begin{aligned}&(1-\lambda_2)[(1-\lambda_1)x_1+\lambda_1 x_2]+\lambda_2 x_3\\ =&(1-\lambda_1)(1-\lambda_2)x_1+[\lambda_1(1-\lambda_2)+\lambda_2]\frac{\lambda_1(1-\lambda_2)x_2+\lambda_2 x_3}{\lambda_1(1-\lambda_2)+\lambda_2}\end{aligned}$$

*for $\lambda_1, \lambda_2 \in \mathsf{J}$, $x_1, x_2, x_3 \in X$. (3) Distributivity. We have $(1-\lambda)x+\lambda x = x$ for $\lambda \in \mathsf{J}$, $x \in X$.*

PROOF: Necessity is clear. Uniqueness follows from Proposition 25, (3). Let us prove sufficiency. Set $C = \mathbb{R}_+^* \times X$. Define addition and multiplication by

$$(\lambda_1, x_1)+(\lambda_2, x_2) = \left(\lambda_1+\lambda_2,\ \frac{\lambda_1 x_1+\lambda_2 x_2}{\lambda_1+\lambda_2}\right),\ \lambda_2(\lambda_1, x_1) = (\lambda_1\lambda_2, x_1)$$

for $\lambda_1, \lambda_2 \in \mathbb{R}_+^*$, $x_1, x_2 \in X$. We claim that $C$ is a convex cone. Addition is commutative and associative because of commutativity (1) and associativity (2) in the statement. Distributivity of product on the left is true because of distributivity (3) in the statement. The remaining conditions for $C$ to be a convex cone are clear. We omit the computation. Consider $f : x \in X \mapsto (1, x) \in C$. It is injective, and $f(X)$ is a convex subset of $C$, by Proposition 25, (1). Use then the convex set structure on $f(X)$ induced by the convex cone structure of $C$ and the bijection $f : X \leftrightarrow f(X)$ to obtain a convex set structure on $X$ defining the given convex combinations of any two points of $X$. ∎

PROPOSITION 27. *(1) Let $X \neq \phi$ be a convex set, and $Y \neq \phi$ be a set in which we are given a map that to every $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathbb{J}, \lambda_1 + \cdots + \lambda_n = 1$, $y_1, \ldots, y_n \in Y$ associates a point of $Y$ represented by $\lambda_1 y_1 + \cdots + \lambda_n y_n$. Let $f : X \to Y$ be a surjective map such that $f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$ for $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathbb{J}$, $\lambda_1 + \cdots + \lambda_n = 1, x_1, \ldots, x_n \in X$. There is a unique convex set structure on $Y$ defining those given convex combinations of any $n$ points of $Y$, equivalently for which $f$ is a convex set map. (2) Let $X \neq \phi$ be a convex set, and $Y \neq \phi$ be a set in which we are given a map that to every $\lambda \in \mathbb{J}, y_1, y_2 \in Y$ associates a point of $Y$ represented by $(1-\lambda)y_1 + \lambda y_2$. Let $f : X \to Y$ be a surjective map such that $f[(1-\lambda)x_1 + \lambda x_2] = (1-\lambda)f(x_1) + \lambda f(x_2)$ for $\lambda \in \mathbb{J}$, $x_1, x_2 \in X$. There is a unique convex set structure on $Y$ defining those given convex combinations of any two points of $Y$, equivalently for which $f$ is a convex set map.*

PROOF: Part (1) is clear. It simply means that conditions (1), (2), (3) of Definition 20 that are satisfied by $X$ imply the same conditions for $Y$. As to part (2), uniqueness follows from Proposition 25, (3). To get existence, use Proposition 26. Conditions (1), (2), (3) that $Y$ needs to satisfy follow from similar conditions satisfied by $X$. Then $f$ is a convex set map by Proposition 25, (2). Conversely, if $f$ is a convex set map for a convex set structure on $Y$, this structure defines those given convex combinations of any two points of $Y$. ∎

PROPOSITION 28. *Let $X \neq \phi$ be a convex set, $Y \neq \phi$ be a set and $f : X \to Y$ be a surjective map. There is a necessarily unique convex set structure on $Y$ for which $f$ is a convex set map if and only if the following equivalent conditions hold: (1) If $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathbb{J}$, $\lambda_1 + \cdots + \lambda_n = 1, y_1, \ldots, y_n \in Y$, then $f(\lambda_1 x_1 + \cdots + \lambda_n x_n)$ depends only on $\lambda_1, \ldots, \lambda_n, y_1, \ldots, y_n$, not on the choice of $x_1, \ldots, x_n \in X$ such that $f(x_1) = y_1, \ldots, f(x_n) = y_n$. (2) If $\lambda \in \mathbb{J}$, $y_1, y_2 \in Y$,* *then $f[(1-\lambda)x_1 + \lambda x_2]$ depends only on $\lambda, y_1, y_2$, not on the choice of $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$.*

PROOF: Necessity in (1) is seen as follows. If the claimed convex set structure exists on $Y$, then (1) holds because $f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 y_1 + \cdots + \lambda_n y_n$ in the notation of (1) of the proposition. Conversely, let (1) hold. Uniqueness follows from (1), since once $\lambda_1, \ldots, \lambda_n, y_1, \ldots, y_n$ are given, then $\lambda_1 y_1 + \cdots + \lambda_n y_n = f(\lambda_1 x_1 + \cdots + \lambda_n x_n)$ whatever $x_1, \ldots, x_n \in X$ we choose so that $f(x_1) = y_1, \ldots, f(x_n) = y_n$. Existence results from (1) in the following way. Once $\lambda_1, \ldots, \lambda_n, y_1, \ldots, y_n$ are given, we define $\lambda_1 y_1 + \cdots + \lambda_n y_n = f(\lambda_1 x_1 + \cdots + \lambda_n x_n)$ whatever $x_1, \ldots, x_n$ we choose so that $f(x_1) = y_1, \ldots, f(x_n) = y_n$. Then we have $f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$. Since conditions (1), (2), (3) of Definition 20 hold for $X$, they also hold for $Y$. Thus we get a convex set structure on $Y$ for which $f$ is a convex set map. Clearly (1) implies (2). Actually (2) implies existence and (1). By (2) we define $(1-\lambda)y_1 + \lambda y_2$ for $\lambda \in \mathsf{J}$, $y_1, y_2 \in Y$ by setting $(1-\lambda)y_1 + \lambda y_2 = f[(1-\lambda)x_1 + \lambda x_2]$ whatever $x_1, x_2$ we choose so that $f(x_1) = y_1, f(x_2) = y_2$. Then $f[(1-\lambda)x_1 + \lambda x_2] = (1-\lambda)f(x_1) + \lambda f(x_2)$. Since conditions (1), (2), (3) of Proposition 26 are satisfied by $X$, they are also satisfied by $Y$. Proposition 26 implies that there is a (unique) convex set structure on $Y$ defining these convex combinations of any two points of $Y$. From Proposition 25, (2) it follows that $f$ is a convex set map. Hence (1) holds, as we saw in the necessity part. ∎

DEFINITION 29. *An equivalence relation $t \sim x (t, x \in X)$ on a nonvoid convex set $X$ is said to be compatible with the convex set structure of $X$ when it satisfies the following condition: if $n \in \mathsf{N}^*, \lambda_1, \ldots, \lambda_n \in \mathsf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $t_1, \ldots, t_n, x_1, \ldots, x_n \in X$, $t_1 \sim x_1, \ldots, t_n \sim x_n$, then $\lambda_1 t_1 + \cdots + \lambda_n t_n \sim \lambda_1 x_1 + \cdots + \lambda_n x_n$. This is equivalent to saying that, if $n \in \mathsf{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathsf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $A_1, \ldots, A_n$ are equivalence classes, then $\lambda_1 A_1 + \cdots + \lambda_n A_n$ is contained in an equivalence class. An alternative way of defining compatibility is as follows: if $\lambda \in \mathsf{J}$, $t_1, t_2, x_1, x_2 \in X$, $t_1 \sim x_1, t_2 \sim x_2$, then $(1-\lambda)t_1 + \lambda t_2 \sim (1-\lambda)x_1 + \lambda x_2$. This is equivalent to saying that, if $\lambda \in \mathsf{J}$, $A_1, A_2$ are equivalence classes, then $(1-\lambda)A_1 + \lambda A_2$ is contained in an equivalence class. Every equivalence class $A$ for a compatible equivalence relation on $X$ is a convex subset* *of $X$. In fact, if $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathbb{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \ldots, x_n \in A$, fix any $x \in A$ to get $x_1 \sim x, \ldots, x_n \sim x$, hence $\lambda_1 x_1 + \cdots + \lambda_n x_n \sim \lambda_1 x + \cdots + \lambda_n x = x$ and $\lambda_1 x_1 + \cdots + \lambda_n x_n \in A$.*

PROPOSITION 30. *Let $X$ and $Y$ be nonvoid convex sets, and $f : X \to Y$ be a surjective convex set map. Then $f$ defines an equivalence relation on $X$, whose equivalence classes are $f^{-1}(y)$ for $y \in Y$, which is compatible with the convex set structure on $X$. Conversely, let us be given a compatible equivalence relation on a convex set $X \neq \phi$. Call $Y$ the quotient set of $X$ and $f : X \to Y$ the quotient map. There is one and only one way of making $Y$ into a convex set so that $f$ is a convex set map, namely through the map that to every $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathbb{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, and equivalence classes $A_1, \ldots, A_n$, associates the equivalence class denoted by $\lambda_n A_1 \oplus \cdots \oplus \lambda_n A_n$ containing $\lambda_1 A_1 + \cdots + \lambda_n A_n$.*

PROOF: The direct part of the proposition is clear. Let us prove its converse part. Uniqueness is seen as follows. Assume that there is a convex set structure on $Y$ so that $f$ is a convex set map. If $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathbb{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, and $A_1, \ldots, A_n \in Y$ are equivalence classes, let $\lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n$ be the corresponding convex combination in $Y$. For $x_1 \in A_1, \ldots, x_n \in A_n$ we have $f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n$ and so $\lambda_1 x_1 + \cdots \lambda_n x_n \in \lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n$ and $\lambda_1 A_1 + \cdots + \lambda_n A_n \subset \lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n$. This proves that $\lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n$ is the necessarily unique equivalence class containing $\lambda_1 A_1 + \cdots + \lambda_n A_n$. Let us prove existence. For every $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathbb{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $A_1, \ldots, A_n \in Y$, let $\lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n$ denote the equivalence class in $X$ containing $\lambda_1 A_1 + \cdots + \lambda_n A_n$. Then $f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) \oplus \cdots \oplus \lambda_n f(x_n)$ for $n \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_n \in \mathbb{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \ldots, x_n \in X$. By Proposition 27, (1), there is a unique convex set structure on $Y$ defining those given convex combinations of any $n$ points of $Y$, equivalently for which $f$ is a convex set map. ∎

PROPOSITION 31. *(1) Let $X \neq \phi$ be a convex set. Define $C = \mathbb{R}_+^* \times X$. Introduce* *addition and multiplication by*

$$(\lambda_1, x_1) + (\lambda_2, x_2) = \left(\lambda_1 + \lambda_2,\ \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}\right)$$
$$\lambda_2(\lambda_1, x_1) = (\lambda_1 \lambda_2, x_1)$$

*for* $\lambda_1, \lambda_2 \in \mathbb{R}_+^*, x_1, x_2 \in X$. *Then* $C$ *becomes a convex cone called the convex cone freely generated by* $X$. *The map* $f : x \in X \mapsto (1, x) \in C$ *is a convex set isomorphism. (2) Every convex set is a convex subset of some convex cone.*

PROOF: Concerning (1), we see as in the proof of Proposition 26 that $C$ is indeed a convex cone, and that $f$ is a convex set isomorphism. Then (2) follows from (1). ∎

DEFINITION 32. *A convex set is said to be vectorial when it is a convex subset of some real vector space.*

DEFINITION 33. *A convex set* $X$ *satisfies the cancellation rule for convex sets (CRCS) when* $\lambda \in \mathbb{J}$, $x_1, x_2, x_3 \in X$, $(1-\lambda)x_1 + \lambda x_2 = (1-\lambda)x_1 + \lambda x_3$ *imply* $x_2 = x_3$.

PROPOSITION 34. *A convex set* $X$ *is vectorial if and only if it satisfies the cancellation rule for convex sets.*

PROOF: Consider $f : X \to C$ as in Proposition 31, (1). We claim that $C$ satisfies the cancellation rule for convex cones if and only if $X$ satisfies the cancellation rule for convex sets. In fact, $C$ satisfies the cancellation rule for convex cones if and only if $(\lambda_1, x_1) + (\lambda_2, x_2) = (\lambda_1, x_1) + (\lambda_3, x_3)$ implies $(\lambda_2, x_2) = (\lambda_3, x_3)$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+^*$, $x_1, x_2, x_3 \in X$, that is if and only if

$$\left(\lambda_1 + \lambda_2,\ ,\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}\right) = \left(\lambda_1 + \lambda_3,\ \frac{\lambda_1 x_1 + \lambda_3 x_3}{\lambda_1 + \lambda_3}\right)$$

implies $\lambda_2 = \lambda_3$, $x_2 = x_3$, that is if and only if $\lambda_1 + \lambda_2 = \lambda_1 + \lambda_3$, $\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1 x_1 + \lambda_3 x_3}{\lambda_1 + \lambda_3}$ imply $\lambda_2 = \lambda_3$, $x_2 = x_3$, and finally if and only if $\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1 x_1 + \lambda_3 x_3}{\lambda_1 + \lambda_2}$ implies $x_2 = x_3$, which is the cancellation rule for convex sets by setting $\lambda = \lambda_2/(\lambda_1 + \lambda_2)$ to get that $(1-\lambda)x_1 + \lambda x_2 = (1-\lambda)x_1 + \lambda x_3$ implies

$x_2 = x_3$. This proves our claim. A real vector space satisfies the cancellation rule for convex sets. Hence a vectorial convex set satisfies the cancellation rule for convex sets. This proves necessity. To prove sufficiency, if $X$ satisfies the cancellation rule for convex sets, then $C$ satisfies the cancellation rule for convex cones, by preceding claim. Thus $C$ is vectorial, by Proposition 10. Since the convex set map $f : X \to C$ is injective, we see that $X$ is vectorial. ∎

PROPOSITION 35. *Every convex cone is the image by a surjective convex cone map of a vectorial convex cone. Every convex set is the image by a surjective convex set map of a vectorial convex set.*

PROOF: Let $C \neq \phi$ be a convex cone, and $G \neq \phi$ be a subset of $C$ that generates $C$ as a convex cone. Consider the real vector space $E$ of all real functions on $G$ that vanish outside finite subsets of $G$. Let $D$ be the convex subcone of $E$ of all $\alpha \in E$, $\alpha \neq 0$, that are positive. Define $f : D \to C$ by $f(x) = \Sigma\alpha(g)g \in C$ for $\alpha \in D$, where summation is over the support of $\alpha$. Then $f$ is a surjective convex cone map. This proves the first part. Let $X \neq \phi$ be a convex set, and $G \neq \phi$ be a subset of $X$ that generates $X$ as a convex set. Consider $E$ as before. Let $Y$ be the convex subset of $E$ of all $\alpha \in E$ that are positive and satisfy $\Sigma\alpha(g) = 1$, where summation is over the support of $\alpha$. Define $f : Y \to X$ by $f(\alpha) = \Sigma\alpha(g)g \in X$ for $\alpha \in Y$, where summation is over the support of $\alpha$. Then $f$ is a surjective convex set map. This proves the second part. ∎

LEMMA 36. *Let $X$ be a convex set and $x_1, x_2, x \in X$. Then $X_{x_1x_2}(x) = \{\lambda \in \mathbf{I}; (1-\lambda)x_1 + \lambda x_2 = x\}$ is an interval of $\mathbf{I}$. If $x_1 \neq x_2$, then $X_{x_1x_2}(x_1)$ is either $\{0\}$ or $[0,1[$, and $X_{x_1x_2}(x_2)$ is either $\{1\}$ or $]0,1]$.*

PROOF: $X_{x_1x_2}(x)$ is the inverse image of $x$ by the convex set map $\lambda \in \mathbf{I} \mapsto (1-\lambda)x_1 + \lambda x_2 \in X$, hence it is a convex subset of $\mathbf{I}$, that is an interval of $\mathbf{I}$. Since $0 \in X_{x_1x_2}(x_1)$, then $X_{x_1x_2}(x_1)$ is an interval of $\mathbf{I}$ to the right of 0. We claim that there is no $\theta \in \mathbf{J}$ such that $X_{x_1x_2}(x_1)$ is either $[0,\theta]$ or else $[0,\theta[$. Assume that $\theta$ exists. Let $0 < \epsilon \leq \theta$. Take $\lambda = \theta - \epsilon \in X_{x_1x_2}(x_1)$, $\mu = 1 \in X_{x_1x_2}(x_2)$, $\nu = \theta - \epsilon \in X_{x_1x_2}(x_1)$ in the following consequence from the associativity condition.

(*)
$$(1-\nu)[(1-\lambda)x_1+\lambda x_2]+\nu[(1-\mu)x_1+\mu x_2]=$$
$$[(1-\lambda)(1-\nu)+(1-\mu)\nu]x_1+[\lambda(1-\nu)+\mu\nu]x_2\,.$$

The lefthand side of (*) is then $x_1$. For the righthand side of (*) to be $x_1$, it is necessary that $\lambda(1-\nu)+\mu\nu\in X_{x_1x_2}(x_1)$, hence $\lambda(1-\nu)+\mu\nu\leq\theta$ and so $(\theta-\epsilon)(1-\theta+\epsilon)+(\theta-\epsilon)\leq\theta$. Letting $\epsilon\to 0$, we get $\theta(1-\theta)+\theta\leq\theta$, which is impossible. This proves our assertion for $X_{x_1x_2}(x_1)$. We get the assertion for $X_{x_1x_2}(x_2)$ from the one for $X_{x_1x_2}(x_1)$ by interchanging $x_1$ and $x_2$ as well as $\lambda$ and $1-\lambda$. ∎

PROPOSITION 37. *Let $X$ be a convex set and $x_1,x_2\in X$. Then either the map $f:\lambda\in\mathsf{I}\mapsto(1-\lambda)x_1+\lambda x_2\in X$ is injective, or else its restriction to $\mathsf{J}$ is a constant map.*

PROOF: Assume that $f$ is not injective. There is then $x\in X$ such that the interval $f^{-1}(x)$ of $\mathsf{I}$ is neither void nor reduced to a single point. Call $\alpha_0$ and $\alpha_1$ the smallest and largest extremities of $f^{-1}(x)$ respectively, hence $0\leq\alpha_0<\alpha_1\leq 1$. We may assume $x\neq x_1, x\neq x_2$. In fact, if $x=x_1$, then we already know by Lemma 36 that $f^{-1}(x_1)$ must be $[0,1[$ because it is not $\{0\}$, hence $f(\lambda)=x_1$ for $\lambda\in[0,1[$; and likewise for $x=x_2$. Fix $\alpha\in]\alpha_0,\alpha_1[$, hence $\alpha\in f^{-1}(x)$ and so $x=(1-\alpha)x_1+\alpha x_2$. We have $(1-\lambda)x_1+\lambda x=(1-\lambda)x_1+\lambda[(1-x)x_1+\alpha x_2]=(1-\alpha\lambda)x_1+(\alpha\lambda)x_2$ for $\lambda\in\mathsf{I}$. Note that $0<\alpha<1$, $0\leq\alpha_0/\alpha<1$. Hence $\alpha_0/\alpha<\lambda\leq 1$ imply $\alpha_0<\alpha\lambda\leq\alpha<\alpha_1$ and $\alpha\lambda\in f^{-1}(x)$, that is $x=(1-\alpha\lambda)x_1+(\alpha\lambda)x_2$ and $x=(1-\lambda)x_1+\lambda x$. It follows that $]\alpha_0/\alpha_1,1]\subset X_{x_1x}(x)$, hence $X_{x_1x}(x)=]0,1]$ because $x\neq x_1$ and by Lemma 36. Thus $x=(1-\lambda)x_1+\lambda x$ for $0<\lambda\leq 1$. Hence $x=(1-\lambda)x_1+\lambda x=(1-\alpha\lambda)x_1+(\alpha\lambda)x_2$ for $0<\lambda\leq 1$ imply $x=(1-\mu)x_1+\mu x_2$ for $0<\mu\leq\alpha$. Thus $]0,\alpha]\subset f^{-1}(x)$ and $\alpha_0=0$. Likewise $\alpha_1=1$. Thus $]0,1[\subset f^{-1}(x)$. ∎

PROPOSITION 38. *Let $X$ be a convex set and $x_1,x_2,x_3\in X$. If $(1-\lambda)x_1+\lambda x_2=(1-\lambda)x_1+\lambda x_3$ for some $\lambda\in\mathsf{J}$, this equality holds for all $\lambda\in\mathsf{J}$.*

PROOF: The set $\Lambda=\{\lambda\in\mathsf{I};(1-\lambda)x_1+\lambda x_2=(1-\lambda)x_1+\lambda x_3\}$ is a convex subset of $\mathsf{I}$ and it contains 0. Hence $[0,\lambda]\subset\Lambda$ if $\lambda\in\Lambda$. Let $\lambda\in\Lambda,\lambda>0$. Set

$\mu = 2\lambda/(1+\lambda)$. We have $0 < \lambda < \mu < 1$. We claim that $\mu \in \Lambda$. In fact, let

$$u = \frac{1-\lambda}{1+\lambda}x_1 + \frac{\lambda}{1+\lambda}x_2 + \frac{\lambda}{1+\lambda}x_3\,.$$

Then

$$u = \frac{1}{1+\lambda}[(1-\lambda)x_1 + \lambda x_2] + \frac{\lambda}{1+\lambda}x_3 =$$
$$\frac{1}{1+\lambda}[(1-\lambda)x_1 + \lambda x_3] + \frac{\lambda}{1+\lambda}x_3 = \frac{1-\lambda}{1+\lambda}x_1 \frac{2\lambda}{1+\lambda}x_3\,,$$

hence $u = (1-\mu)x_1 + \mu x_3$. Likewise, $u = (1-\mu)x_1 + \mu x_2$. Thus $\mu \in \Lambda$ as claimed. By applying this claim repeatedly, we get that $\lambda_n = 2^n\lambda/[1+(2^n-1)\lambda] \in \Lambda$ for $n \in \mathbb{N}$. Notice that $\lambda_n \to 1$ as $n \to \infty$. Hence $\lambda$ contains $[0,1[$. ∎

PROPOSITION 39. *In a convex cone $C \neq \phi$ with constant multiplication, $(1-\lambda)x_1 + \lambda x_2 = x_1 + x_2$ is independent of $\lambda \in \mathrm{J}$ for all $x_1, x_2 \in X$. If a convex set $X \neq \phi$ is a convex subset of a convex cone with constant multiplication, then $(1-\lambda)x_1 + \lambda x_2$ is independent of $\lambda \in \mathrm{J}$ for all $x_1, x_2 \in X$. Conversely, if a convex set $X \neq \phi$ satisfies this condition, then $X$ is a convex cone with constant multiplication in a unique way.*

PROOF: The first part is clear. It implies the second part. Let us check the third part. Define $x_1 + x_2 = (1-\lambda)x_1 + \lambda x_2$ for $x_1, x_2 \in X$, where $\lambda \in \mathrm{J}$. By assumption, $x_1 + x_2$ is independent of $\lambda$. It is easy to check that this addition is commutative and associative, and verifies $x + x = x$ for $x \in X$. If we introduce constant multiplication on $X$, it becomes a convex cone, by Proposition 18, (2). Uniqueness follows from Remark 23. ∎

PROPOSITION 40. *In a set $C$ whose power is strictly less than the power of the continuum, every convex cone structure has a constant multiplication, and Proposition 18, (2) applies.*

PROOF: If $x \in C$, the map $\lambda \in \mathbb{R}_+^* \mapsto \lambda x \in C$ cannot be injective. Hence it is a constant map, by Proposition 16. ∎

**Proposition** 41. *In a set $X$ whose power is strictly less than the power of the continuum, every convex set structure comes from a unique convex cone structure with constant multiplication, and Proposition 18, (2) applies.*

**Proof**: If $x_1, x_2 \in X$, the map $\lambda \in \mathbf{I} \mapsto (1-\lambda)x_1 + \lambda x_2 \in X$ cannot be injective. Hence its restriction to $\mathbf{J}$ is a constant map, by Proposition 37. There remains to apply the final part of Proposition 39. ∎

**Proposition** 42. *Let $X \neq \phi$ be a convex set, and $X^*$ denote the vector space of all convex set maps of $X$ into $\mathbf{R}$. (1) $X$ is vectorial if and only if $X^*$ is separating on $X$. (2) There is a surjective convex set map $f : X \to Y$, where $Y$ is a vectorial convex set, with the following property. For any surjective convex set map $g : X \to Z$, where $Z$ is a vectorial convex set, there is a unique map $h : Y \to Z$ such that $hf = g$, and $h$ is necessarily a surjective convex set map. Moreover, $f : X \to Y$ is essentially unique in the sense that, if $f_i : X \to Y_i (i = 1, 2)$ are two choices for $f : X \to Y$, there is a unique map $h : Y_1 \to Y_2$ such that $hf_1 = f_2$, and $h$ is necessarily a bijective convex set isomorphism.*

**Proof**: Consider the convex set map $f : x \in X \mapsto (\varphi(x))_{\varphi \in X^*} \in \mathbf{R}^{X^*}$. (1) This map is injective if and only if $X^*$ is separating on $X$. Hence $X$ is vectorial if $X^*$ is separating on $X$. Conversely, let $X$ be vectorial, that is a convex subset of some real vector space $E$. Call $E^*$ the vector space of all linear forms on $E$, which is separating on $E$. The set of restrictions of $E^*$ to $X$ is contained in $X^*$. Hence $X^*$ is separating on $X$. (2) Set $Y = f(X)$ to obtain a convex subset of $\mathbf{R}^{X^*}$, hence a vectorial convex set. Consider the surjective convex set map $f : X \to Y$. Take any surjective convex set map $g : X \to Z$, where $Z$ is a vectorial convex set. The vector space $Z^*$ of all convex set maps of $Z$ into $\mathbf{R}$ is separating on $Z$. We claim that there is a map $h : Y \to Z$ such that $hf = g$, equivalently that $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ imply $g(x_1) = g(x_2)$. In fact, if $g(x_1) \neq g(x_2)$, there is $\psi \in Z^*$ such that $\psi[g(x_1)] \neq \psi[g(x_2)]$, and then $\varphi = \psi g \in X^*$, $\varphi(x_1) \neq \varphi(x_2)$, hence $f(x_1) \neq f(x_2)$. Clearly $h$ is unique because $f$ is surjective, and it is necessarily a surjective convex set map. The essential uniqueness for $f : X \to Y$ is clear. ∎

**Definition** 43. *Let $X$ be a convex set. We say that $X$ satisfies the one* *dimensional injection rule (ODIR) when the map* $\lambda \in \mathsf{I} \mapsto (1-\lambda)x_1 + \lambda x_2 \in X$ *is injective for any* $x_1, x_2 \in X$, $x_1 \neq x_2$. *Also that* $X$ *satisfies the two dimensional injection rule (TDIR) when the map* $(\lambda_1, \lambda_2, \lambda_3) \in \mathsf{I}_2 \mapsto \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \in X$ *is injective for any convexily independent* $x_1, x_2, x_3 \in X$, *where* $\mathsf{I}_2 = \{(\lambda_1, \lambda_2, \lambda_3) \in (\mathsf{R}_+)^3; \lambda_1 + \lambda_2 + \lambda_3 = 1\}$.

PROPOSITION 44. *Let* $X \neq \phi$ *be a convex set, and: (1)* $X$ *satisfies the cancellation rule for convex sets. (2)* $X$ *satisfies the one dimensional injection rule. (3)* $X$ *satisfies the two dimensional injection rule. Then (1) is equivalent to (2) & (3), and (1) is equivalent to (3) provided* $X$ *contains at least three convexily independent elements.*

PROOF: (1) implies that $X$ is vectorial (Proposition 34). A real vector space satisfies (2) and (3). Hence (1) implies (2) and (3). Let us now prove that (2) & (3) imply (1). Assume that $\lambda \in \mathsf{J}$, $x_1, x_2, x_3 \in X$, $(1-\lambda)x_1 + \lambda x_2 = (1-\lambda)x_1 + \lambda x_3$. We claim that $x_2 = x_3$. Let $x_1, x_2, x_3$ be convexily dependent. Then $x_1, x_2, x_3 \in [u, v]$ where $u, v = x_1, x_2, x_3$ and $[u, v]$ is the convex subset of $X$ generated by $u, v$. If $u = v$, then $x_2 = x_3$. If $u \neq v$, then $\mathsf{I}$ is isomorphic to $[u, v]$, by (2). Since (1) holds for the convex subset $\mathsf{I}$ of $\mathsf{R}$, hence for $[u, v]$, we get $x_2 = x_3$. Let now $x_1, x_2, x_3$ be convexily independent. $\mathsf{I}_2$ is isomorphic to the convex subset $[x_1, x_2, x_3]$ of $X$ generated by them, by (3). Since (1) holds for the convex subset $\mathsf{I}_2$ of $\mathsf{R}^3$, hence for $[x_1, x_2, x_3]$, we get $x_2 = x_3$. Let finally $X$ satisfy (3) and $t_1, t_2, t_3 \in X$ be convexily independent. Call $T = [t_1, t_2, t_3]$ the convex subset of $X$ generated by them. By assumption, the map $(\lambda_1, \lambda_2, \lambda_3) \in \mathsf{I}_2 \mapsto \lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3 \in T$ is injective. If $u, v \in X$, say that the convex subset $[u, v]$ of $X$ generated by them is full when the map $\lambda \in \mathsf{I} \mapsto (1-\lambda)u + \lambda v \in X$ is injective, hence $u \neq v$. We firstly claim that, if $u, v, w \in X$ are pairwise different convexily dependent, and one of the $[u, v], [u, w]$, $[v, w]$ is full, the other two are also full. Indeed, assume that $[u, v]$ is full. There are three possibilities: (i) $v$ is a convex combination of $u, w$. Then $[u, v]$ and $[v, w]$ are convex subsets of $[u, w]$. If $[u, w]$ was not full, it would consist of at most three elements (by Proposition 37), hence $[u, v]$ would consist of at most three elements and $[u, v]$ would not be full. Thus $[u, w]$ is full, hence $[v, w]$ is full too by $v \neq w$. (ii) $u$ is a convex combination of $v, w$. This case is equivalent to the preceding one by interchanging $u, v$. (iii) $w$ is a convex combination of $u, v$. Then $[u, w]$ and $[w, v]$ are convex subsets of $[u, v]$, and they are full by $u \neq v, v \neq w$. We secondly claim that, if $t \in T$, $x \in X, t \neq x$, then $[t, x]$ is full. Take indeed $u \in T, u \neq t, u \neq x$, once $T$ is infinite, by (3). Let $t, u, x$ be convexily dependent. Then $[t, u]$ is full by $t \neq u$ and the one dimensional injection rule applied to $\mathbf{I}_2$, hence to $T$ in view of (3). Thus $[t, x]$ is full by first claim. Let now $t, u, x$ be convexily independent. Then $[t, x]$ is full as a convex subset of $[t, u, x]$, by $t \neq x$ and the one dimensional injection rule applied to $\mathbf{I}_2$, hence to $[t, u, x]$. Let finally $x_1, x_2 \in X$, $x_1 \neq x_2$. Choose $t \in T, t \neq x_1, t \neq x_2$, once $T$ is infinite. If $t, x_1, x_2$ are convexily dependent, since $[t, x_1]$ and $[t, x_2]$ are fully by second claim, then $[x_1, x_2]$ is full by first claim. If $t, x_1, x_2$ are convexily independent, then $[x_1, x_2]$ is full as a convex subset of $[t, x_1, x_2]$ and by the one dimensional injection rule applied to $\mathbf{I}_2$, hence to $[t, x_1, x_2]$ in view of (3). Hence the one dimensional injection rule holds for $X$. We already know that (2) and (3) imply (1). ∎

The two dimensional injection rule is voidly satisfied by a convex set $X$ when any three points of $X$ are convexily dependent. Let us examine what happens in this case.

DEFINITION 45. *A convex set is linear when any three of its points are convexily dependent.*

PROPOSITION 46. *A convex set $X$ is isomorphic to an interval of $\mathbf{R}$ if and only if $X$ is linear, and has zero, one or at least four elements (in the last case, it has the power of the continuum).*

PROOF: Necessity is clear, because a convex subset of $\mathbf{R}$ having at least two elements has the power of the continuum. Let us prove sufficiency, and assume that $X$ is linear and has at least four elements. We first claim that, if $x_1, \ldots, x_n \in X (n \geq 3)$ are pairwise different, two of them $x_i, x_j (1 \leq i < j \leq n)$ are such that each of the remaining ones of these $n$ elements is a convex combination of $x_i, x_j$. This claim is true for $n = 3$, because $X$ is linear. Assume $n \geq 3$ and this claim true for $n$. We are going to prove that it is true for $n + 1$. Consider

$x_1, \ldots, x_{n+1} \in X$ pairwise different. One of the $x_{n-1}, x_n, x_{n+1}$, say $x_{n+1}$, is a convex combination of the other two, say $x_{n-1}, x_n$, because $X$ is linear. By the assumption that the claim holds for $n$, there are $i, j (1 \leq i < j \leq n)$ such that each of the $x_1, \ldots, x_n$ is a convex combination of $x_i, x_j$. In particular, $x_{n-1}, x_n$ are a convex combination of $x_i, x_j$, hence the same is true for $x_{n+1}$. Thus the claim is true for $n+1$. We secondly claim that $X$ satisfies the one dimensional injection rule once it has at least four elements. Take any $x_1, x_2 \in X$, $x_1 \neq x_2$. Choose $x_3, x_4 \in X$ so that $x_1, x_2, x_3, x_4$ are pairwise different. Two of these four elements are such that each of these four elements in a convex combination of those two elements. Hence there are $u, v \in X, u \neq v$, where $u, v$ are chosen among $x_1, x_2, x_3, x_4$, such that $x_1, x_2, x_3, x_4$ belong to the convex subset $[u, v]$ of $X$ generated by $u, v$. The map $\lambda \in \mathsf{I} \mapsto (1-\lambda)u + \lambda v \in X$ is injective, because the image of its restriction to $\mathsf{J}$ contains at least two elements (Proposition 37). Therefore $\lambda \in \mathsf{I} \mapsto (1-\lambda)x_1 + \lambda x_2 \in X$ is injective because $x_1, x_2 \in [u, v]$, $x_1 \neq x_2$, and the claim is true. We thirdly claim that a linear convex set satisfying the one dimensional injection rule satisfies the cancellation rule for convex sets. In fact, let $\lambda \in \mathsf{J}$, $x_1, x_2, x_3 \in X$, $(1-\lambda)x_1 + \lambda x_2 = (1-\lambda)x_1 + \lambda x_3$. There are $u, v \in X$, where $u, v$ are chosen among $x_1, x_2, x_3$, such that $x_1, x_2, x_3$ belong to the convex subset $[u, v]$ of $X$ generated by $u, v$. If $u = v$, then $x_2 = x_3$. Let $u \neq v$. By the one dimensional injection rule, $\mathsf{I}$ and $[u, v]$ are isomorphic as convex sets. Since the cancellation rule holds for $\mathsf{R}$, hence for $\mathsf{I}$, it holds for $[u, v]$ too. Hence $x_2 = x_3$, and the claim is true. It follows that $X$ is vectorial (Proposition 34). Let $E$ be a real vector space containing $X$ as a convex subset. Fix $a, b \in X$, $a \neq b$. Consider the straight line $L = \{(1-\lambda)a + \lambda b; \lambda \in \mathsf{R}\}$ in $E$ through $a, b$. We claim that $X \subset L$. In fact, let $x \in X$. Consider the possible cases: (i) $x = (1-\lambda)a + \lambda b (0 \leq \lambda \leq 1)$ in $X$, hence in $E$. Then $x \in L$. (ii) $b = (1-\lambda)a + \lambda x (0 < \lambda \leq 1)$ in $X$, hence in $E$. Then $x = (1 - 1/\lambda)a + (1/\lambda)b$ in $E$ and $x \in L$. (iii) $a = (1-\lambda)b + \lambda x (0 < \lambda \leq 1)$ in $X$, hence in $E$. Then $x = (1/\lambda)a + (1 - 1/\lambda)b$ in $E$ and $x \in L$. Thus $X \subset L$. Since $X$ is a convex subset of $E$, it is a convex subset of $L$. The map $f : \lambda \in \mathsf{R} \mapsto (1-\lambda)a + \lambda b \in L$ is a convex set isomorphism between $\mathsf{R}$ and $\mathsf{L}$. Thus $f^{-1}(X)$ is a convex subset of $\mathsf{R}$, hence an interval containing $0, 1$. Finally, $f : f^{-1}(X) \to X$ is a convex set isomorphism between $f^{-1}(X)$ and $X$. ∎

**Remark 47.** Proposition 46 excludes $X$ having two or three elements. The reason is the following. If $X$ has two or three elements, hence it is finite, its convex set structure is defined by an inflattice structure (Proposition 41). It is easy to see that (up to isomorphisms) there are three inflattices of two or three elements, and that two of them are linear, but obviously not isomorphic to a convex subset of $\mathbb{R}$, once they have at least two elements and are finite.

PROPOSITION 48. *Fix a closed triangle $M$ of vertices $a, b, c$ in $\mathbb{R}^2$. Let $N$ be the union of the closed segments $[a, d]$ and $[b, c]$ of $\mathbb{R}^2$, where $d \in [b, c]$ is fixed. $M$ is, but $N$ is not, a convex subset of $\mathbb{R}^2$. Define the surjective map $f : M \to N$ by $f(x) = x$ for $x \in [b, c]$, and as the point $f(x) \neq d$ where $[a, d]$ meets the parallel to $[b, c]$ through $x \in M, x \notin [b, c]$. There is one and only one convex set structure on $N$ so that $f$ is a convex set map for the convex set structure on $M$ induced by $\mathbb{R}^2$. Then $N$ satisfies the one dimensional injection rule, but it fails to satisfy the cancellation rule for convex sets. A convex set $X$ fails to satisfy the cancellation rule for convex sets if and only if either $X$ does not satisfy the one dimensional injection rule, or else it satisfies this rule and contains some convex subset isomorphic to $N$.*

PROOF: The surjective map $f : M \to N$ defines an equivalence relation on $M$, which is easily seen to be compatible. Hence there is one and only one convex set structure on $N$ for which $f$ is a convex set map (Proposition 30). The convex set structures of $M$ and $N$ induce on $[a, d]$ and $[b, c]$ the same convex set structures. Clearly $N$ satisfies the one-dimensional injection rule. We have $(1-\lambda)a+\lambda b = (1-\lambda)a+\lambda c$ in $N$, since both sides are equal to $(1-\lambda)a+\lambda d$ in $[a, d]$, but $c \neq c$. Therefore $N$ does not satisfy the cancellation rule for convex sets. If a convex set $X$ fails to satisfy the one dimensional injection rule, or $X$ contains some convex subset isomorphic to $N$, then $X$ fails to satisfy the cancellation rule for convex sets. Conversely, let $X$ fail to satisfy the cancellation rule for convex sets, but $X$ satisfies the one dimensional injection rule. There are $t, u, v \in X$ such that $(1-\lambda)t + \lambda u = (1-\lambda)t + \lambda v$ for some $\lambda \in \mathsf{J}$, hence for all $\lambda \in \mathsf{J}$ (Proposition 38). Let $Y$ be the union of the closed segments $[t, w]$ and $[u, v]$ in $X$, where $w \in [u, v]$ is fixed. Consider the bijective map $g : N \to Y$ defined by $g[(1-\lambda)a + \lambda d] = (1-\lambda)t + \lambda w$ and $g[(1-\lambda)b + \lambda c] = (1-\lambda)u + \lambda v$ for $\lambda \in \mathsf{I}$.

Introduce on $Y$ the convex set structure that makes $g$ a convex set isomorphism. Then $Y$ is a convex subset of $X$ isomorphic to $N$. ∎

**Proposition 49.** *Let $x_1 \sim x_2 (x_1, x_2 \in X)$ be an equivalence relation on a convex set $X \neq \phi$. The following conditions are equivalent: (1) There are a real vector space $E$ containing $X$ as a convex subset, and a real vector subspace $F$ of $E$, such that the given equivalence relation on $X$ is induced by the equivalence relation that $F$ defines on $E$. (2) The given equivalence relation on $X$ is compatible, and both $X$ and its quotient convex set $Y$ are vectorial. (3) The given equivalence relation on $X$ is compatible, and if $\lambda \in \mathsf{J}$, $x, x_1, x_2 \in X$ $(1-\lambda)x + \lambda x_1 = (1-\lambda)x + \lambda x_2$, then $x_1 = x_2$, and also if $\lambda \in \mathsf{J}$, $x, x_1, x_2 \in X$, $(1-\lambda)x + \lambda x_1 \sim (1-\lambda)x + \lambda x_2$, then $x_1 \sim x_2$.*

**Proof:** (1) implies (2). The equivalence relation defined by $F$ on $E$ is compatible. Hence the equivalence relation it induces on $X$ is compatible. It is clear that $X$ has to be vectorial. We now prove that $Y$ is vectorial. Consider the quotient real vector space $E/F$ and the quotient linear map $\pi : E \to E/F$. Then $\pi(X)$ is a convex subset of $E/F$. Hence it is vectorial. The equivalence relation defined on $X$ by the surjective convex set map $\pi : X \to \pi(X)$ is the given equivalence relation on $X$. Thus $Y$ and $\pi(X)$ are isomorphic convex sets. Hence $Y$ is vectorial. (2) implies (1). Let $X$ and $Y$ be vectorial. The convex cones $C = \mathbb{R}_+^* \times X$ and $D = \mathbb{R}_+^* \times Y$ (Proposition 31) are obviously vectorial. Consider the associated real vector spaces $E$ and $G$, and convex cone isomorphisms $f : C \to E$ and $g : D \to G$ (Proposition 10). Define the convex cone map $h : C \to D$ by $h(\lambda, x) = (\lambda, \pi(x)$ for $\lambda \in \mathbb{R}_*^*, x \in X$, where $\pi : X \to Y$ is the quotient convex cone map. We have the convex cone map $gh : C \to G$. There is a unique linear map $k : E \to G$ such that $kf = gh$ (Proposition 7). Set $F = k^{-1}(0)$, which is a real vector subspace of $E$. If $x_1, x_2 \in X$, then $x_1 \sim x_2$ if and only if $\pi(x_1) = \pi(x_2)$, or $h(1, x_1) = h(1, x_2)$, or $gh(1, x_1) = gh(1, x_2)$, or $kf(1, x_1) = kf(1, x_2)$, or $f(1, x_2) - f(1, x_1) \in F$. Thus the given equivalence relation on $X$ is the inverse image by the composite convex set isomorphism $x \in X \mapsto (1, x) \in C \mapsto f(1, x) \in E$ of the equivalence relation that $F$ defines on $E$. Finally (2) is equivalent to (3). We know that $X$ is vectorial if and only if $\lambda \in \mathsf{J}$, $x, x_1, x_2 \in X$, $(1-\lambda)x + \lambda x_1 = (1-\lambda)x + \lambda x_2$ imply $x_1 = x_2$ (Proposition 34). We now prove that $Y$ is vectorial if and only if $\lambda \in \mathrm{J}$, $x, x_1, x_2 \in X$, $(1-\lambda)x + \lambda x_1 \sim (1-\lambda)x + \lambda x_2$ imply $x_1 \sim x_2$. In fact, $Y$ is vectorial if and only if $\lambda \in \mathrm{J}$, $A, A_1, A_2 \subset Y$, $(1-\lambda)A \oplus \lambda A_1 = (1-\lambda)A \oplus \lambda A_2$ imply $A_1 = A_2$ (notation of Proposition 30). The assertion that $\lambda \in \mathrm{J}$, $A, A_1, A_2 \in Y$, $(1-\lambda)A \oplus \lambda A_1 = (1-\lambda)A \oplus \lambda A_2$ imply $A_1 = A_2$ is equivalent to the assertion that $\lambda \in \mathrm{J}$, $x, x_1, x_2 \in X$, $(1-\lambda)x + \lambda x_1 \sim (1-\lambda)x + \lambda x_2$ imply $x_1 \sim x_2$. ∎

## REFERENCE

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