CBPF-NF-001/00

A Path Integral Quantization for a Damped One-Dimensional Particle

Luiz C.L. Botelho, Jorge C. de Mello and Edson P. Silva

Departamento de Física

Universidade Federal Rural do Rio de Janeiro

Itaguaí, RJ 23851, Brazil

Abstract

We propose a phenomenological set of quantum states allowing the introduction of dissipation at the quantum level. We, further, write a Feynman Path Integral for quantum propagation of such sub-set of quantum damped sates.

It is an interesting and important problem in "Turbulent Physics" to understand the effects of classical friction and damping at the quantum and classical level ([1], [2]).

In this note, we follow the previous studies of refs. [3], [4] to analyze the Botelho's dissipative anomaly factor in the modified Caldirola-Kanai action by means of a non-local discretization process.

Let us start by writing the one-dimensional Schrödinger equation in terms the polar form ([2])

$$\psi(x,t) = \rho^2(x,t)e^{\frac{i}{\hbar}}S(x,t) \tag{1}$$

namelly

$$\frac{\partial S(x,t)}{\partial t} + \frac{1}{2\mu}\left(\frac{\partial S}{\partial x}(x,t)\right)^2 + \frac{\hbar^2}{2\mu}\ \frac{\frac{\partial^2}{\partial x^2}\ \rho(x,t)}{\rho(x,t)} + V(x,t) = 0 \tag{2}$$

$$\frac{\partial^2}{\partial t}\ \left(\rho^2(x,t)\right) + \frac{\partial}{\partial x}\ \left(\rho^2(x,t)\ \frac{\partial}{\partial x}\ S(x,t)\right) = 0 \tag{3}$$

In order to relate the Schrödinger equation (1) – (3) with studies of ref. [2], we consider the space of quantum states which are subject to dissipation (ohmic-damping phenomena) composed only by W.K.B. pure phase states with $\rho(x,t) \equiv$ constant.

In order to introduce damping in our quantum system we consider the phenomenological term $-\nu S(x,t)$ in eq. (3) ([2], [3]) where ν is the classical viscosity constant. As a consequence of the above made remark we should consider the generalized Phase-Hamiltonian-Jacobi equation for the quantum phase in eq. (2).

$$\frac{\partial S(x,t)}{\partial t} + \frac{1}{2\mu}\ \left(\frac{\partial S}{\partial x}\ (x,t)\right)^2 + V(x) = -\nu S(x,t) \tag{4}$$

$$\rho(x,t) = \bar{\rho}_0 = c^{te} \tag{5}$$

Let us solve eq. (4) by considering the usual damping ansatz ([3])

$$S(x,t) = e^{-\nu t} S^{(0)}(x,t) \tag{6}$$

where $S^{(0)}(x,t)$ satisfies the mass and potential time dependent term

$$\frac{\partial S^{(0)}(x,t)}{\partial t} = \frac{1}{2\mu e^{\nu t}}\ \left(\frac{\partial S^{(0)}}{\partial x}\right)(x,t) + e^{\nu t}V(x) \tag{7}$$

which solution is given by

$$S^{(0)}(x,t) \;=\; \int_0^t d\sigma \left\{ \frac{\mu e^{\nu\sigma}}{2} \left(\frac{dx}{d\sigma}\right)^2 - e^{\nu\sigma} V\left(x(\sigma)\right) \right\} \tag{8}$$

The general dissipative quantum phase eq. (6) is, thus, given exactly by the, modified Caldirola action below and found firstly in ref. [3] in the framework of damped Hamilton-Jacobi action

$$S(x,t) = \int_0^t d\sigma e^{\nu(\sigma-t)} \left[\frac{\mu}{2} \left(\frac{dx}{d\sigma}\right)^2 - V\left(x(\sigma)\right) \right] \tag{9}$$

Let us, thus, write a Feynman path-integral representation for the W.K.B. phase quantum states propagation in the single case of $V(x) \equiv 0$. as done in refs. [2].

$$\psi^{(\nu)}_{\text{reduced}}(x,t) = e^{\frac{i}{\hbar}\ S(x,t)} \tag{10}$$

We follow, thus, Feynamn by looking the infinitesimally short time intervals $t_{k+1} - t_k = \dfrac{t-t'}{N} = \varepsilon$, where t and t' are the initial and final time propagation in his propagator quantization methods

$$\underset{reduced}{\psi^{\nu}(x_{k+1};t_{k+1})} = \int_{-\infty}^{+\infty} dx_k G^{(\nu)}\left[(x_{k+1},t_{k+1});(x_k,t_k)\right] \underset{reduced}{\psi^{\nu}(x_x t_k)} \tag{11}$$

The infinitesimal time (W.K.B. limit) Green function is determined by the Feynman-Dirac prescription and taking into account the complete propagation time $e^{-\nu t}$ into the discretization of the damping term in the modified Caldirola-Kanai action, and thus, leading to a non-local time discretization process

$$\begin{aligned} & G^{(\nu)}\left[(x_{k+1},t_{k+1});(x_k;t_k)\right]_{\varepsilon\to 0} \\ = \;& A(t_{k+1},t_k) \\ \times \;& \exp\left\{ \frac{i}{\hbar}\ \frac{\mu}{2}\ \exp(-\nu t)\ \exp \nu(a t_{k+1} + b t_k) \right. \\ \times \;& \left[\frac{(x_{k+1}-x_k)^2}{\varepsilon^2} \right] \varepsilon \end{aligned} \tag{12}$$

Note that in order to analize anomalous pre-factor in the searched fenomenological Feynman path integral we still followed refs. ([3],[4]) by introducing a weighted rule

$(a+b=1)$ for the discretization of the Modified-Caldirola-Kanai term $\exp\nu(t-\sigma)$ in the generalized classical action eq. (9).

We have, thus, the following short-time representation for the effective quantum propagator

$$\begin{aligned}\underset{reduced}{\psi^{\nu}(x_{k+1};t_{k+1})} &= \int_{-\infty}^{+\infty} dx_k A(t_{k+1};t_k) \\ &\exp\ \left\{\frac{i}{\hbar}\ \frac{\mu}{2\varepsilon}\ (x_{k+1}-x_k)^2 e^{-\nu t} e^{\nu(at_{k+1}+bt_k)}\right\} \\ &\psi(x_k;t_k)\end{aligned} \tag{13}$$

If we make $x_k = x_{k+1}+\eta$ and $t_{k+1}=t_k+\varepsilon$ and consider the Taylor expansions in eq. (4)

$$\underset{reduced}{\psi^{\nu}(x_{k+1};t_k+\varepsilon)} = \underset{reduced}{\psi^{\nu}(x_{k+1},t_k)} + \varepsilon\ \frac{\partial {\psi^{\nu}}_{red.}}{\partial t}\ (x_{k+1},t_k) + 0(\varepsilon^2) \tag{14}$$

$$\underset{reduced}{\psi^{\nu}(x_{k+1}+\eta;t_k)} = \underset{reduced}{\psi^{\nu}(x_{k+1},t_k)} + \eta\ \frac{\partial {\psi^{\nu}}_{red.}}{\partial x_{k+1}} + 0(\eta^2) \tag{15}$$

and by keeping the lowest-order terms, we obtain the explicit expression for the pre-factor in eq. (13).

$$(A(t_{k+1},t_k))^{-1} = \int_{-\infty}^{+\infty} d\eta \exp\left\{\frac{i}{\hbar}\ \frac{1}{2\varepsilon}\ \eta^2 e^{-\nu t} e^{\nu(at_{k+1}+bt_k)}\right\} \tag{16}$$

or explicitly

$$A(t_{k+1},t_k) = \left(\frac{\mu}{2\pi i\hbar(t_{k+1}-t_k)}\right)^{\frac{1}{2}} \exp\frac{\nu}{2}(at_{k+1}+bt_k)\exp\left(-\frac{\nu}{2}\ t\right) \tag{17}$$

As a consequence of eq. (13) and eq. (17), we can write the phenomenological Green function for our dissipative quantum system for arbitrary different time times as a Feynman Path-Integral

$$
\begin{aligned}
G^{(\nu)}\left[(x,t);(x',t')\right] &= \\
&= \lim_{N\to\infty} \int \left(\prod_{k=1}^{N-1} dx_k\right) \exp \frac{\nu}{2} \left[\sum_{k=0}^{N-1} a \left(t' + \frac{(t-t')}{N}\,(k+1)\right)\right. \\
&+ \left. b\left(t' + \frac{(t-t')k}{N}\right) - t\right] \\
&\times \prod_{k=0}^{N-1} \left(\frac{\mu}{2\pi i\hbar(t_{k+1}-t_k)}\right)^{\frac{1}{2}} \\
&\times \exp\left\{\frac{i}{\hbar}\,\frac{\mu}{2}\sum_{k=0}^{N-1} \varepsilon \exp\left[\nu(at_{k+1}+bt_k-t)\right]\right. \\
&\times \left.(x_{k+1}-x_n)^2/\varepsilon^2\right\}
\end{aligned}
\tag{18}
$$

Now we can define formally the limits in eq. (18) as a well defined product Feynman measure over paths multiplied by a general damping anomaly factor as firstly found in refs. ([3],[4])

$$
G^{(\nu)}\left[(x,t);(x',t')\right] = \exp\left[\frac{\nu}{4}\,(a-b)(t-t')\right] \int \lim_{N\to\infty} \left(\prod_{k=1}^{N-1} dx_k\right) \tag{19}
$$

$$
\times \left(\frac{\mu}{2\pi i\hbar\bar{\varepsilon}^{(\nu)}}\right)^{\frac{N}{2}} \exp\left\{\frac{i}{\hbar}\,\frac{\mu}{2}\sum_{k=0}^{N-1} \varepsilon \exp\left[\nu(at_{k+1}+bt_k-t)\right]\frac{(x_{k+1}-x_k)^2}{\varepsilon^2}\right\} \tag{20}
$$

Here the infinitesimal step in the dissipative anomaly factor in eq. (20) is given by the expression below and is independent of our original weighted time-interval partition rule used for the dissipative term $\exp\nu(\sigma-t)$ in the Generalized action eq. (9)

$$
\bar{\varepsilon}(\nu) = \frac{(t-t')}{N}\exp\left(\frac{\nu}{2}(t-t')\right) \tag{21}
$$

We finally obtain the main result of this note similar in its structural form to that obtained in refs. ([3],[4]) from the usual Caldirola-Kanai action but differing from the result of ref. [3] by its dissipative anomaly factor similar to that of ref. [4]

$$
G^{(\nu)}\left[(x,t);(x',t)\right] = e^{\frac{\nu}{4}(a-b)(t-t')} \int_{\substack{x(t)=x \\ x(t')=x'}} U^F[x(\sigma)]e^{\frac{i}{\hbar}\int_{t'} dve^{\nu(\sigma-t)}} \times \left[\frac{\mu}{2}\,(\dot{x}(\sigma))^2 - V(x(\sigma))\right] \tag{22}
$$

where we have reintroduced the potential in our analysis since its presence does not alter the above cited dissipative anomaly factor.

Work is in progress to analyze the electronic conductivity by introducing a electric field $V(x) = -eE \cdot x$ and evaluating the resulting electronic current $f(x,t) = \psi(x,t)\left(-\frac{i}{\hbar}\frac{\partial}{\partial x}\right)\psi(x,t)$ by means of the effective-phenomenological Green function eq. (22) (at a W.K.B. limit $\hbar \to 0$) and with a plane wave (free electron) initial condiction, namelly:

$$\psi(x,t) = \int dy G^{(\nu)}_{(\hbar\to 0)}\left[(x,t),(y,0)\right]\left(Ae^{ipy}\right) \tag{23}$$

Acknowledgements: The authors are thankful to Professor J.M. Bassalo from UFPA for some discussions and to Professor Helayël-Neto from CBPF for scientific support.

References

[1] Luiz C.L. Botelho, Brazilian Journal of Physics, vol. 21, n. 2, 290 (1991).

Luiz C.L. Botelho, Modern Physics Letters B, vol. 5, 391 (1991).

Luiz C.L. Botelho and R. Vilhena, Phys. Rev. E52, 6183 (1995).

Luiz C.L. Botelho, E.P. da Silva, M.L. Lyra and S.B. Cavalcanti, Modern Physics Letters B, vol. 12, n. 8, 301-308 (1998).

Luiz C.L. Botelho and E.P. da Silva, International Journal of Modern Physics B, vol. 12, n. 27 e 28, 2857 (1998).

Luiz C.L. Botelho and E.P. da Silva, J. Phys. A23, (1990).

Luiz C.L. Botelho, Modern Physics Letters B, vol. 12, 1153 (1998).

Luiz C.L. Botelho, International Journal of Modern Physics B, vol. 13, n. 13, 1663 (1999).

Luiz C.L. Botelho, Modern Physics Letters B, vol. 13, n. 11, 363 (1999).

Luiz C.L. Botelho, Modern Physics Letters B, vol. 13, n. 9810, 317 (1999).

Luiz C.L. Botelho, Modern Physics Letters B, vol. 13, n. 6 e 7, 203 (1999).

Luiz C.L. Botelho, Modern Physics Letters B, vol. 6, n. 4, 203 (1992).

[2] Luiz C.L. Botelho and E.P. da Silva, Modern Physics Letters B, vol. 12, n. 14 e 15, 569 (1998).

[3] Luiz C.L. Botelho and E.P. da Silva, Modern Physics Letters B, vol. 12, n. 28, 1191 (1998).

[4] Luiz C.L. Botelho and E.P. da Silva, Phys. Rev. 58E, 1141 (1998).