Neoclassical generation of toroidal zonal flow by drift wave turbulence

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Zonal-flow instabilities due to drift-wave turbulence in the presence of toroidicity-induced parallel (neoclassical) viscosity and allowing for the toroidal flow are studied. It is shown that, as a result of the neoclassical viscosity a new type of zonal-flow instability is possible, leading to the generation of the considerable toroidal zonal flow. The toroidal instability is complementary to the previously studied instability resulting in the poloidal flow generation and occurs as a second branch of the general dispersion relation describing the evolution of the poloidal and toroidal flow. Nonlinear saturation of the new instability is studied. It is shown that saturated zonal toroidal velocity, generated in this instability, is large compared to the mean cross-field drift velocity as the ratio q/ϵ, where q is the safety factor and ϵ is the inverse aspect ratio. In addition to the broad turbulent spectrum of drift waves, a monochromatic wave packet is considered. It is revealed that for the case of sufficiently strong neoclassical viscosity such a wave packet is subjected to generation of the toroidal zonal flow due to instability of hydrodynamic type. *© 2006 American Institute of Physics*.
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I. INTRODUCTION

Large-scale convective plasma motions can be spontaneously generated by small-scale turbulence (see Ref. 1, and references therein). Convective cells can be generated as a result of decay instabilities.[2] Such large-scale coherent structures play an important role in overall dynamics of small-scale turbulence, in particular, in its saturation. Often, the convective cells are generated in the form of a band system of alternating coherent flows (zonal flows). A strong shear of plasma velocity associated with such structures has been invoked as a mechanism of the suppression of the anomalous transport in the tokamaks.

Most of the previous works on zonal flows dealt with the situation when the generated flow is in a poloidal direction. Such an ideology goes back to the approximation of slab geometry, used in studying the drift-wave turbulence. The fact is that, in the scope of this ideology, the toroidal velocity is a free parameter which can be considered for convenience as negligibly small or identical to zero. Then the poloidal velocity V_p proves to be approximately the same as the mean cross-field drift velocity $\bar{V}_E \equiv V_0$ induced by the mean radial electric field $\bar{E}_r$,

$$V_p = V_0, \tag{1}$$

where the overbar means the averaging over the small-scale oscillations. On the other hand, as known from the theory of equilibrium plasma rotation in tokamaks (see Refs. 3–5, and references therein), there is a toroidicity-induced parallel (neoclassical) viscosity that dampens the poloidal rotation. In neglect of the ion temperature gradient, the neoclassical viscosity is proportional to the poloidal velocity, with the viscosity coefficient dependent on the plasma collisionality degree. The neoclassical viscous force affects the poloidal velocity V_p so that Eq. (1) is no longer valid. In this case the approximation of negligibly small toroidal velocity is also violated. As a result, instead of Eq. (1), using the ion radial motion equation, in the case of cold ions, one has[3–5]

$$V_p = V_0 + (\epsilon/q)V_\parallel, \quad (2)$$

where ϵ is the inverse aspect ratio, q is the safety factor, and $V_\parallel$ is the mean parallel velocity, which in the case of small ϵ, $\epsilon \ll 1$, is approximately equal to the mean toroidal velocity V_t. Thus, the theory of the equilibrium plasma rotation suggests that the poloidal flow is strongly suppressed for sufficiently strong neoclassical viscosity. In this case the radial electric field results in the toroidal rather than poloidal flow. The toroidal velocity V_t is given then by

$$V_t = -(q/\epsilon)V_0. \quad (3)$$

The effects of neoclassical viscosity involving the toroidal flow have been extensively studied in the theory of linear resistive instabilities[6–10] and in the magnetic island theory.[11,12] Evidently, a complete theory of the zonal flows in a tokamak should include the effects of the neoclassical viscosity and toroidal velocity. The goal of the present paper is therefore to formulate such a theory for the case of drift-wave turbulence driven zonal flows.

The idea that the toroidal zonal-flow can be generated by turbulence goes back, probably, to Ref. 13. Then this idea was discussed in Ref. 14 and later in a series of other papers, including Ref. 15.

The zonal flow theory specifically in a toroidal geometry was developed, in particular, in Refs. 15–20. The neoclassical viscosity has been included in Refs. 16, 19, and 20 in the analysis of the poloidal-flow instability. In Refs. 17 and 18 the initial value problem for poloidal flows governed by collisionless[17] and collisional[18] processes has been considered. One can show that the processes, studied in Refs. 17 and 18 by means of the kinetic theory, can be expressed in terms of the neoclassical viscosity.

The recent state of the theory of neoclassical zonal-flow instabilities can be understood turning to analogy with the theory of linear neoclassical instabilities in tokamaks.[10] Roughly speaking, these instabilities can be separated into the fast and slow ones, depending on their growth rates. In the studying the fast instabilities, such as the ideal internal kink and ballooning modes, allowing for the neoclassical viscosity, one can neglect the perturbed toroidal velocity. This approach was used, in particular, in Refs. 22 and 23, where the $m=1$ ideal internal kink mode was considered. In contrast to this, in studying the slow linear instabilities, such as the resistive modes,[6–10] allowing for this velocity proves to be necessary.

According to Refs. 22 and 23, see also Refs. 9 and 10, incorporation of the neoclassical viscosity into the theory of ideal modes does not suppress but modifies them resulting in the appearance of a family of the so-called ideal-viscous modes. At the same time, according to Refs. 6–10, one of the main neoclassical effects on the resistive modes is the renormalization of the perpendicular inertia of the type (the Callen-Shaing renormalization)[6]

$$1 + 2q^2 \rightarrow (q/\epsilon)^2. \quad (4)$$

The same inertia renormalization has been revealed in the magnetic island theory.[11,12]

Allowing for the previous text, let us discuss what are consequences of the incorporation of the neoclassical viscosity into the theory of poloidal zonal flows undertaken by Refs. 16, 19, and 20. Looking at such flows as analogy to the ideal modes, it can be seen that, in contrast to the ideal modes, the poloidal zonal flows, studied in the previous references, are suppressed by the neoclassical viscosity. Then, at a first glance, there are no neoclassical poloidal-flow instabilities, i.e., the corresponding topic is exhausted.

Meanwhile, turning to Ref. 24, one can see that the poloidal-flow instabilities, studied in Refs. 16, 19, and 20, are not complete family of such instabilities. The fact is that Refs. 16, 19, and 20 dealt with the so-called resonant zonal-flow instabilities, driven by the negative diffusion effect, and inherent for a sufficiently broad spectrum of the primary drift waves. In contrast to this, Ref. 24 studied a different type of instabilities called the instability of hydrodynamic type, inherent for a monochromatic drift-wave packet. Effect of neoclassical viscosity on the zonal-flow instability, pointed out in Ref. 24, has not yet been studied. Therefore, as a whole, the topic of neoclassical poloidal zonal-flow instabilities seems to not be exhausted. One can suggest from general considerations that a new family of neoclassical poloidal-flow instabilities, similar to the ideal-viscous linear modes, should be revealed. Following analysis confirms this suggestion.

By analogy with the linear theory, the poloidal-flow instabilities can be called the fast zonal-flow ones. In this context, the toroidal-flow instabilities can be called the slow ones. The characteristic growth rates of such instabilities are smaller than those of the standard (fast) zonal-flow instabilities.[1,19] Evidently, the slow zonal-flow instabilities are of the most interest in the conditions when the standard poloidal-flow instabilities are dampened, similarly to that the resistive modes are of interest only in conditions when the ideal modes are stable.

It is of interest for us to elucidate whether there are preceding studies on slow zonal-flow instabilities. Then we note preliminarily that there are two main approaches to studying the zonal-flow instabilities. One of them is the approach of wave kinetic equation, going back to Ref. 25. It is the approach used in Refs. 19, 20, and 24. The second one is the approach of the theory of the convective-cell generation going back to Ref. 2. This approach was used in Ref. 21 allowing for the neoclassical viscosity. Note that the term "neoclassical viscosity" has not been used in Ref. 21. However, one can see that the final results of this reference contain the inertia renormalization given by Eq. (4). This shows the presence of the neoclassical viscosity.

Dealing with the standard drift-wave turbulence, one neglects the wave part of the parallel velocity $\tilde{V}_\parallel$. Meanwhile, the value $\tilde{V}_\parallel$ is important in the problem of the ion temperature gradient turbulence.[15,26,27] In accordance with Refs. 15 and 27, for $\tilde{V}_\parallel \neq 0$ there is parallel component of the Reynolds stress leading to generation of the parallel flow, and thereby, the toroidal flow. However, this generation mechanism of the parallel flow is beyond the scope of the present article.

Our starting equations are given in Sec. II. We use them initially for studying the case of sufficiently broad turbulent spectra of drift waves, Secs. III–V, and then turn to the case of the monochromatic drift-wave packet, Sec. VI. In Sec. III we derive the zonal-flow dispersion relation for the broad drift-wave spectra. Its analysis is performed in Sec. IV. This analysis reveals that, in addition to the fast zonal-flow instabilities, there are slow zonal-flow instabilities in allowing for neoclassical viscosity. In Sec. V we study saturation of instabilities considered in Sec. IV. In Sec. VI we analyze both the fast and slow zonal-flow instabilities of monochromatic wave packet. The results of the article are discussed in Sec. VII.

II. STARTING EQUATIONS

A. Description of turbulence

We assume that there is the simplest variety of the drift-wave turbulence with the wave frequencies $\omega_{\mathbf{k}}$, which are given in the rest frame by the dispersion relation $\omega_{\mathbf{k}}=\omega_{\mathbf{k}}^{(0)}$, where (see, e.g., Ref. 1, and references therein)

$$\omega_{\mathbf{k}}^{(0)} = \frac{k_y V_*}{1 + k_\perp^2 \rho_s^2}. \tag{5}$$

Here V_* is the electron diamagnetic drift velocity defined by the density gradient, $\mathbf{k}=(k_x,k_y,k_\parallel)$ is the wave vector, $k_\perp^2 = k_x^2+k_y^2$, x is the coordinate along the minor radius of tokamak, y is the angle coordinate along the small azimuth, $\parallel$ means the direction along the equilibrium magnetic field, and ρ_s is the ion sound Larmor radius, i.e., the ion Larmor radius calculated for the electron temperature.

If there is the mean cross-field drift flow characterized by the velocity V_0, the expression for the wave frequency is modified as follows (cf. Ref. 19):

$$\omega_{\mathbf{k}} = k_y V_0 + \omega_{\mathbf{k}}^{(0)}. \tag{6}$$

We describe the turbulence by means of the standard wave kinetic equation of the form[1]

$$\frac{\partial N_{\mathbf{k}}}{\partial t} + \frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{k}} \cdot \nabla N_{\mathbf{k}} - \nabla \omega_{\mathbf{k}} \cdot \frac{\partial N_{\mathbf{k}}}{\partial \mathbf{k}} = \gamma_{\mathbf{k}} N_{\mathbf{k}} - \frac{\Delta\omega_{\mathbf{k}} N_{\mathbf{k}}^2}{N_{\mathbf{k}}^{(0)}}. \tag{7}$$

Here $N_{\mathbf{k}}$ is the drift-wave action density defined by[1]

$$N_{\mathbf{k}} = (1 + k_\perp^2 \rho_s^2)^2 |\phi_{\mathbf{k}}|^2, \tag{8}$$

$\phi_{\mathbf{k}}$ is the Fourier component of the drift-wave electrostatic potential $\tilde{\phi}(\mathbf{r},t)$, $\gamma_{\mathbf{k}}/2$ is the linear growth rate of drift waves. As explained in Ref. 1, the last term on the right-hand side of Eq. (8) describes the drift-wave nonlinear damping due to self-interaction of these waves, $\Delta\omega_{\mathbf{k}}$ is the nonlinear broadening decay rate, $N_{\mathbf{k}}^{(0)}$ is the value $N_{\mathbf{k}}$ in the absence of zonal flow, i.e., for $V_0=0$ and $V_t=0$. The electrostatic potential of drift waves $\tilde{\phi}(\mathbf{r},t)$ is related to their perpendicular electric field $\tilde{\mathbf{E}}_\perp$ by

$$\tilde{\mathbf{E}}_\perp = - \nabla_\perp \tilde{\phi}, \tag{9}$$

where $\nabla_\perp$ is the perpendicular gradient (with respect to the equilibrium magnetic field). It is assumed that, in the absence of zonal flow, the right-hand side of Eq. (7) vanishes. Then

$$\Delta\omega_{\mathbf{k}} = \gamma_{\mathbf{k}}. \tag{10}$$

We allow for that the field $\tilde{\mathbf{E}}_\perp$ leads to the cross-field drift velocity of particles $\tilde{\mathbf{V}}_E$ defined by

$$\tilde{\mathbf{V}}_E = \frac{c}{B_0}[\mathbf{b} \times \nabla\tilde{\phi}]. \tag{11}$$

Here c is the light speed and $\mathbf{b}=\mathbf{B}_0/B_0$ is the unit vector along the equilibrium magnetic field $\mathbf{B}_0$.

B. Description of mean flows

Turning to Ref. 5 and references therein (see also Chap. 19 of Ref. 10), one can arrive at the following motion equations for poloidal and toroidal velocities V_p and V_t in the presence of neoclassical viscosity:

$$(1 + 2q^2)\, \partial V_p/\partial t + \overline{(\tilde{\mathbf{V}}_E \cdot \nabla)\tilde{V}_{Ey}} = F_\theta^\pi/\rho, \tag{12}$$

$$\frac{\partial V_t}{\partial t} + \overline{(\tilde{\mathbf{V}}_E \cdot \nabla)\tilde{V}_t} - 4q\epsilon\frac{\partial V_p}{\partial t} = 0. \tag{13}$$

Here F_θ^π is the poloidal neoclassical viscosity force given by

$$F_\theta^\pi = - \kappa_\theta \rho V_p, \tag{14}$$

κ_θ is a viscosity coefficient whose specified expression depends on plasma collisionality degree, ρ is the plasma mass density, and $\tilde{V}_t$ is the wave part of the toroidal velocity. In accordance with Sec. I, the overbar means the averaging over the small-scale drift-wave oscillations. The factor $1+2q^2$ in Eq. (12) describes the perpendicular inertia renormalization due to the poloidal oscillations of the equilibrium magnetic field. The term with $q\epsilon\partial V_p/\partial t$ in Eq. (13) is a consequence of the same oscillations.

The terms with an overbar in Eqs. (12) and (13), i.e., the Reynolds stress force components, play the same role as the external forces in similar equations of Ref. 5. They have been taken in neglecting the poloidal oscillations of the equilibrium magnetic field.

Explain also that Eq. (12) is the vorticity equation (the current continuity equation) integrated over the radial coordinate. In contrast to Eq. (12), Ref. 19 started from the vorticity equation where such an integration is not performed.

In accordance with Eq. (11),

$$\tilde{V}_{Ey} = \frac{c}{B_0}\frac{\partial\tilde{\phi}}{\partial x}. \tag{15}$$

Then, similarly to Ref. 19,

$$\overline{(\tilde{\mathbf{V}}_E \cdot \nabla)\tilde{V}_{Ey}} = - \partial\Lambda/\partial x, \tag{16}$$

where the function Λ describing the Reynolds stress forces (the Reynolds stress function)[16] is given by

$$\Lambda = \frac{c^2}{B_0^2}\int \frac{k_x k_y N_{\mathbf{k}}}{(1 + k_\perp^2 \rho_s^2)^2} d\mathbf{k}. \tag{17}$$

As a result, Eq. (12) reduces to

$$\partial V_p/\partial t-(1+2q^2)^{-1}\,\partial\Lambda/\partial x=-\gamma_p V_p, \tag{18}$$

where

$$\gamma_p=\kappa_\theta/(1+2q^2). \tag{19}$$

Physically, Eq. (18) is the same as Eq. (2) of Ref. 19.

In order to calculate the left-hand side of Eq. (13), let us recall that in deriving the drift-wave dispersion relation given by Eq. (5), the perturbed ion parallel plasma velocity has been assumed to be vanishing (cf. the discussion in Sec. I), i.e.,

$$\tilde{V}_\parallel=\tilde{V}_t+(\epsilon/q)\tilde{V}_{Ey}=0. \tag{20}$$

Therefore

$$\tilde{V}_t=-(\epsilon/q)\tilde{V}_{Ey}. \tag{21}$$

As a result, in terms of $V_\parallel$ and V_0, Eq. (13) reduces to

$$\frac{\partial V_\parallel}{\partial t}-\frac{\epsilon}{q}(1+4q^2)\frac{\partial V_p}{\partial t}+\frac{\epsilon}{q}\frac{\partial\Lambda}{\partial x}=0. \tag{22}$$

Allowing for Eq. (18), Eq. (22) can be represented as follows:

$$\partial V_\parallel/\partial t-2\epsilon q\,\partial V_p/\partial t=-\gamma_\parallel V_p, \tag{23}$$

where

$$\gamma_\parallel=\epsilon\kappa_\theta/q. \tag{24}$$

The term with ϵq in Eq. (23) proves to be unimportant for our problem. Therefore we neglect it and use the following simpler version of Eq. (23):

$$\partial V_\parallel/\partial t=-\gamma_\parallel V_p. \tag{25}$$

Equations (18) and (25) together with Eqs. (17) and (2) are starting ones in description of poloidal and toroidal flows. Physically, Eq. (25) means the parallel motion equation. It is remarkable that this equation does not depend on the Reynolds stress forces and has the same form as in the theory of equilibrium rotation in the absence of the parallel external force.

III. DERIVING DISPERSION RELATION OF ZONAL-FLOW RESONANT INSTABILITIES

Similarly to Ref. 19, we take

$$(\hat{N}_{\mathbf{k}},V_0,V_p,V_\parallel)\sim\exp(-i\Omega t+iq_x x), \tag{26}$$

where $\hat{N}_{\mathbf{k}}$ is a small perturbation of $N_{\mathbf{k}}$, V_0, V_p, and $V_\parallel$ are also considered to be small perturbations, Ω and q_x are their frequency and radial wave vector. We linearize Eq. (7) allowing for Eqs. (6) and (10). Then, similarly to Ref. 19, we find that, for $\Omega\ll q_xV_g$, the resonant and nonresonant parts of $\hat{N}_{\mathbf{k}}$, $N_{\mathbf{k}}^{(r)}$, and $N_{\mathbf{k}}^{(1)}$, respectively, are given by

$$N_{\mathbf{k}}^{(r)}=iq_xk_yV_0R(\mathbf{k})\,\partial N_{\mathbf{k}}^{(0)}/\partial k_x, \tag{27}$$

$$N_{\mathbf{k}}^{(1)}=k_yV_0\frac{1}{V_g}\frac{\partial N_{\mathbf{k}}^{(0)}}{\partial k_x}. \tag{28}$$

Here $V_g\equiv\partial\omega_{\mathbf{k}}^{(0)}/\partial k_x$ is the drift-wave group velocity, $R(\mathbf{k})$ is the response function, which in our particular case $\Omega\ll q_xV_g$ is defined by (see in detail Ref. 1)

$$R(\mathbf{k})=1/\Delta\omega_{\mathbf{k}}. \tag{29}$$

On the other hand, linearizing Eq. (17) and using Eqs. (27) and (28), we have

$$\hat{\Lambda}=(u-iq_xD_{xx})V_0. \tag{30}$$

Here $\hat{\Lambda}$ is the linear part of Λ. Thus, according to Eq. (30), the Reynolds stress force is governed by the mean cross-field drift velocity, whereas, according to Eqs. (18) and (25)—by the mean poloidal velocity V_p.

By analogy with Refs. 1 and 19, the value D_{xx} can be called the coefficient of radial diffusion. It is given by

$$D_{xx}=-\frac{c^2}{B_0^2}\int\frac{k_xk_y^2R(\mathbf{k})}{(1+k_\perp^2\rho_s^2)^2}\frac{\partial N_{\mathbf{k}}^{(0)}}{\partial k_x}d\mathbf{k}. \tag{31}$$

Similarly to Ref. 19, the value u can be called the parameter of radial propagation of perturbations. It is defined by

$$u=\frac{c^2}{B_0^2}\int\frac{k_xk_y^2}{(1+k_\perp^2\rho_s^2)^2}\frac{1}{V_g}\frac{\partial N_{\mathbf{k}}^{(0)}}{\partial k_x}d\mathbf{k}. \tag{32}$$

Using Eq. (30), Eq. (18) reduces to

$$(1+2q^2)(-i\Omega+\gamma_p)V_p-(iq_xu+q_x^2D_{xx})V_0=0. \tag{33}$$

In addition, it follows from Eq. (25) that

$$-i\Omega V_\parallel+\gamma_\parallel V_p=0. \tag{34}$$

This equation couples the parallel velocity with the poloidal one.

Using Eq. (34), Eq. (2) yields

$$V_0=\left(\frac{i\Omega}{\gamma_\parallel}-\frac{\epsilon}{q}\right)V_\parallel. \tag{35}$$

Substituting Eqs. (34) and (35) into Eq. (33), we arrive at the zonal-flow dispersion relation

$$i\Omega(1+2q^2)(-i\Omega+\gamma_p)-q_x(iu+q_xD_{xx})\left(i\Omega-\frac{\epsilon}{q}\gamma_\parallel\right)=0. \tag{36}$$

It can be seen from Eq. (35) that, due to the neoclassical viscosity, the zonal-flow dispersion relation proves to be quadratic with respect to the perturbation frequency Ω. Appearance of the second root for Ω is an indication of a new type of zonal-flow instabilities. As will be shown below, just this instability leads to generation of considerable toroidal zonal flow.

The value $\gamma_\parallel$ can be considered as a small parameter of order ϵ/q [see Eq. (24)]. Therefore, the roots of Eq. (36) can be separated into "large" and "small" ones, i.e., $\Omega=(\Omega_1,\Omega_2)$, where Ω_1 and Ω_2 are the large and small roots, respectively.

IV. ANALYSIS OF RESONANT ZONAL-FLOW INSTABILITIES

A. Poloidal-flow instability

The large root of Eq. (36) is found by taking $\gamma_\parallel \to 0$. Then Eq. (36) reduces to

$$(1+2q^2)(-i\Omega_1+\gamma_p) - q_x(iu+q_xD_{xx}) = 0. \tag{37}$$

Hence, we obtain

$$\mathrm{Re}\,\Omega_1 = -q_xu/(1+2q^2), \tag{38}$$

$$\mathrm{Im}\,\Omega_1 = -\gamma_p + q_x^2D_{xx}/(1+2q^2). \tag{39}$$

One can see that these relations coincide with those found in Ref. 19 if one substitute $1+2q^2 \to 1$. In accordance with Ref. 19, they describe generation of poloidal flow due to negative perpendicular viscosity related to the drift-wave turbulence[1] and their radial transport with the group velocity proportional to the parameter of the radial propagation. Such a generation takes place for not too strong neoclassical viscosity[19]

$$\kappa_\theta < q_x^2D_{xx}. \tag{40}$$

In the opposite case of

$$\kappa_\theta > q_x^2D_{xx}, \tag{41}$$

the instability is suppressed by the neoclassical viscosity. Turning to Eq. (25), one can find that, in the presence of such instability, the toroidal velocity V_t proves to be small compared with the poloidal one, V_p,

$$V_t/V_p \simeq \epsilon/q. \tag{42}$$

Therefore, this instability can be called the poloidal-flow instability.

B. Toroidal-flow instability

The small root of Eq. (36) is obtained by the limiting transition

$$\Omega \ll \gamma_p. \tag{43}$$

Then Eq. (36) reduces to

$$i\Omega_2[\kappa_\theta - q_x(iu+q_xD_{xx})] + \frac{\epsilon}{q}\gamma_\parallel q_x(iu+q_xD_{xx}) = 0. \tag{44}$$

In accordance with previous discussion, one can see that $\Omega_2 \neq 0$ only in the presence of neoclassical viscosity, $\gamma_\parallel \neq 0$.

One has from Eq. (44)

$$\mathrm{Re}\,\Omega_2 = -\left(\frac{\epsilon}{q}\right)^2 \kappa_\theta^2 \frac{q_xu}{(\kappa_\theta - q_x^2D_{xx})^2 + (q_xu)^2}, \tag{45}$$

$$\mathrm{Im}\,\Omega_2 = \left(\frac{\epsilon}{q}\right)^2 \kappa_\theta q_x^2 D_{xx} \frac{\kappa_\theta - q_x^2D_{xx} - u^2/D_{xx}}{(\kappa_\theta - q_x^2D_{xx})^2 + (q_xu)^2}. \tag{46}$$

According to Eq. (46), the perturbations considered are unstable for

$$\kappa_\theta > q_x^2D_{xx} + u^2/D_{xx}. \tag{47}$$

The unstable perturbations correspond to the toroidal-flow instability. It can be seen from a comparison of Eq. (47) to Eq. (41) that such an instability takes place only in the condition when the poloidal-flow instability is suppressed. Thus, we conclude that neoclassical viscosity, suppressing the poloidal-flow instability, leads to the toroidal-flow instability. The growth rate of the latter one is small compared with that of the first one as $(\epsilon/q)^2$,

$$\mathrm{Im}\,\Omega_2/\mathrm{Im}\,\Omega_1 \simeq (\epsilon/q)^2. \tag{48}$$

It can be seen from Eq. (45) that, as in the case of the poloidal-flow instability [cf. Eq. (38)], the unstable toroidal-flow modes are radially transported with the group velocity proportional to the parameter of radial propagation u. Similarly to the growth rate, this transport is weakened in comparison with the case of poloidal-flow unstable modes as $(\epsilon/q)^2$.

Meanwhile, the parameter $(\epsilon/q)^{-2}$ is well known in the theory of neoclassical resistive modes.[10] It describes the above-mentioned neoclassical inertia renormalization predicted in Ref. 6 (see also Ref. 7) and studied in detail in Refs. 8 and 10 and a series of other papers cited in Ref. 10.

For sufficiently large κ_θ, $\kappa_\theta \gg (q_x^2D_{xx}, q_xu)$, we find from Eq. (44)

$$\Omega_2 = q_x\left(\frac{\epsilon}{q}\right)^2(u - iq_xD_{xx}). \tag{49}$$

Hence we obtain the following simplified version of Eqs. (45) and (46):

$$\mathrm{Re}\,\Omega_2 = -(\epsilon/q)^2 q_xu, \tag{50}$$

$$\mathrm{Im}\,\Omega_2 = (\epsilon/q)^2 q_x^2 D_{xx}. \tag{51}$$

One can see that formally these expressions do not contain the neoclassical viscosity. This is explained by that they are obtained in the limit of infinite neoclassical viscosity. Equation (50) coincides formally with Eq. (38) if one makes the substitution given by Eq. (4), whereas Eq. (51) transits to Eq. (39) if one makes the same substitution and omits γ_p.

Turning to Eqs. (39) and (46), we conclude that there is a "stability gap"

$$q_x^2D_{xx} < \kappa_\theta < q_x^2D_{xx} + u^2/D_{xx}. \tag{52}$$

If the neoclassical viscosity coefficient κ_θ is in this gap, neither the poloidal-flow instability nor the toroidal-flow one do not take place. It follows from Eq. (52) that the stability gap is due to the radial transport of the perturbations.

V. NONLINEAR SATURATION OF RESONANT TOROIDAL-FLOW INSTABILITY

In accordance with Ref. 19, in order to study nonlinear saturation of zonal flow instabilities, we shall keep in the expansion of the function Λ, given by Eq. (30), terms squared with respect to zonal-flow velocities. Then, using Eq. (17), Eq. (30) is modified as follows:

$$\hat{\Lambda} \to \hat{\Lambda} + \Lambda^{(2)}. \tag{53}$$

Here $\Lambda^{(2)}$ is given by

$$\Lambda^{(2)} = bV_0^2, \tag{54}$$

where

$$b = \frac{1}{2}\left(\frac{c}{B_0}\right)^2 \int \frac{V_g^{-1}k_y^2 k_x}{(1+k_\perp^2\rho_s^2)^2}\frac{\partial}{\partial k_x}\left(V_g^{-1}\frac{\partial N_{\mathbf{k}}^{(0)}}{\partial k_x}\right)d\mathbf{k}. \tag{55}$$

Respectively, Eq. (33), linear with respect to the velocities, is substituted by a nonlinear equation of the form

$$(1+2q^2)\frac{\partial V_p}{\partial t} - u\frac{\partial V_0}{\partial x} - b\frac{\partial V_0^2}{\partial x} + D_{xx}\frac{\partial^2 V_0}{\partial x^2} = -\kappa_\theta V_p. \tag{56}$$

This equation is complemented by Eq. (25), which remains linear with respect to zonal flow velocities.

In the limit of vanishing neoclassical viscosity, $\kappa_\theta \to 0$, $\gamma_\parallel \to 0$, Eq. (34) yields $V_\parallel = 0$, whereas Eq. (56) reduces to

$$(1+2q^2)\frac{\partial V_0}{\partial t} - u\frac{\partial V_0}{\partial x} - b\frac{\partial V_0^2}{\partial x} + D_{xx}\frac{\partial^2 V_0}{\partial x^2} = 0. \tag{57}$$

This equation describes nonlinear stage of the poloidal-flow instability. One can see that the terms with b and D_{xx} in Eq. (57) are of the same order for

$$V_0 \simeq q_x D_{xx}/b. \tag{58}$$

This relation can be used for estimating the saturated poloidal velocity. Then, using Eqs. (31) and (55), we find the estimate

$$V_0 \simeq V_g \frac{q_x}{\langle k_y\rangle}\frac{\omega_*}{\gamma_{\mathbf{k}}}, \tag{59}$$

where $\langle k_y\rangle$ is an effective k_y of the drift-wave turbulence.

The slab-geometry version ($q \to 0$) of Eq. (57) has been studied in Ref. 19 for obtaining the spatial form of the saturated poloidal velocity profile. In contrast to Ref. 19, we will use Eq. (56) for studying nonlinear stage of toroidal-flow instability considered in Sec. IV.

Taking κ_θ to be sufficiently large, Eq. (56) in the case of toroidal-flow instability reduces to

$$\frac{q^2}{\epsilon^2}\frac{\partial V_0}{\partial t} - u\frac{\partial V_0}{\partial x} - b\frac{\partial V_0^2}{\partial x} + D_{xx}\frac{\partial^2 V_0}{\partial x^2} = 0. \tag{60}$$

This equation leads to the same estimate for the saturated cross-field drift velocity V_0 as Eq. (58). Meanwhile, according to Eq. (35), in the case considered

$$V_\parallel = -(q/\epsilon)V_0. \tag{61}$$

Therefore, we arrive at the estimate

$$V_\parallel \simeq \frac{q}{\epsilon}V_g \frac{q_x}{\langle k_y\rangle}\frac{\omega_*}{\gamma_{\mathbf{k}}}. \tag{62}$$

Equation (60) is of the same structure as that of Eq. (57). Therefore, it can be analyzed similarly to Ref. 19.

Instead of x, we introduce the variable

$$\hat{x} = x + (\epsilon/q)^2 u_0 t, \tag{63}$$

where u_0 is a constant. Then Eq. (60) reduces to

$$\left(\frac{q}{\epsilon}\right)^2\frac{\partial V_0}{\partial t} + (u-u_0)\frac{\partial V_0}{\partial \hat{x}} - D_{xx}\frac{\partial^2 V_0}{\partial \hat{x}^2} + b\frac{\partial V_0^2}{\partial \hat{x}} = 0. \tag{64}$$

The instability is saturated for $\partial V_0/\partial t \to 0$. In this case Eq. (64) yields

$$(u-u_0)V_0 - D_{xx}\frac{\partial V_0}{\partial \hat{x}} + bV_0^2 = C, \tag{65}$$

where C is an integration constant.

Similarly to Ref. 19, Eq. (65) has the solution

$$V_0 = \frac{1}{2}\left[V_{01} + V_{02} + (V_{01} - V_{02})\tanh\left(-\frac{\hat{x}}{\delta}\right)\right], \tag{66}$$

where V_{01} and V_{02} are the values of V_0 for $x \to \mp\infty$, respectively,

$$\delta = 2D_{xx}/[b(V_{01} - V_{02})]. \tag{67}$$

In terms of V_{01} and V_{02}, the constant C is given by

$$C = -bV_{01}V_{02}, \tag{68}$$

whereas the difference $u-u_0$ is involved in the problem by

$$V_{02} + V_{01} = -(u-u_0)/b. \tag{69}$$

One can see that, allowing for Eq. (61), the estimate for $V_\parallel$ given by Eq. (62) is compatible with Eq. (66) for $\delta \simeq 1/q_x$, $V_{01} - V_{02} \simeq V_0$.

VI. NEOCLASSICAL ZONAL-FLOW INSTABILITIES OF MONOCHROMATIC DRIFT-WAVE PACKET

A. Starting equations

Studying instabilities of the monochromatic drift-wave packet, one can start from Eqs. (33)–(35) neglecting the coefficient of radial diffusion. Then Eq. (33) reduces to

$$(1+2q^2)(-i\Omega + \gamma_p)V_p - iq_x u V_0 = 0. \tag{70}$$

In addition, similarly to Ref. 24, the parameter of radial propagation of perturbations u should be modified by substituting in the integrand of the right-hand sides of Eq. (32)

$$\frac{1}{V_g} \to -\frac{q_x}{\Omega - q_x V_g}. \tag{71}$$

Then Eq. (32) is transformed to

$$u = -q_x\frac{c^2}{B_0^2}\int \frac{k_x k_y^2}{(1+k_\perp^2\rho_s^2)^2}\frac{\partial N_{\mathbf{k}}^{(0)}/\partial k_x}{\Omega - q_x V_g}d\mathbf{k}. \tag{72}$$

Now, similarly to Ref. 24, we integrate here by parts over k_x and take

$$N_{\mathbf{k}}^{(0)} = N_{\mathbf{k}_0}\delta(\mathbf{k} - \mathbf{k}_0). \tag{73}$$

Then Eq. (72) reduces to

$$u = \frac{k_y q_x}{2V_*\rho_s^2}\frac{c^2 N_{\mathbf{k}_0}}{B_0^2}\frac{\Omega}{(\Omega - q_x V_g)^2}\left(\frac{\partial V_g}{\partial k_x}\right)_{\mathbf{k}=\mathbf{k}_0}. \tag{74}$$

Substituting Eq. (74) into Eq. (70) leads to

$$(\Omega - q_x V_g)^2 \left(1 + \frac{i\gamma_p}{\Omega}\right) V_p = -\Gamma^2 V_0, \tag{75}$$

where

$$\Gamma^2 = \Gamma_0^2/(1 + 2q^2), \tag{76}$$

$$\Gamma_0^2 = -q_x^2 \frac{k_{y0}}{2V_* \rho_s^2} \frac{c^2 N_{\mathbf{k}_0}}{B_0^2} \frac{\partial V_g}{\partial k_x}. \tag{77}$$

Neglecting the toroidal inertia renormalization, $1+2q^2 \to 1$, neoclassical viscosity, $\gamma_p \to 0$, and the parallel flow velocity, $V_\parallel \to 0$, Eq. (75) reduces to that found in Ref. 24. Then the value Γ_0 means the growth rate of instability of hydrodynamic type leading to generation of poloidal flow, pointed out in Ref. 24. Allowing for the toroidal inertia renormalization and neglecting the neoclassical viscosity, one reveals physically the same instability described by the dispersion relation

$$(\Omega - q_x V_g)^2 = -\Gamma^2. \tag{78}$$

According to analysis of Ref. 24, the instability condition, $\Gamma_0^2 > 0$, means

$$1 - 3k_{x0}^2 \rho_s^2 + k_{y0}^2 \rho_s^2 > 0. \tag{79}$$

Therefore, the instability is possible only for not too large k_{x0}.

Using Eqs. (34) and (35), Eq. (75) results in the general zonal-flow dispersion relation of the monochromatic drift-wave packet of the form

$$(\Omega - q_x V_g)^2 \left(1 + \frac{i\gamma_p}{\Omega}\right) = -\Gamma^2 \left(1 + \frac{i\epsilon}{q\Omega}\gamma_\parallel\right). \tag{80}$$

In contrast to Eq. (36), squared with respect to Ω, Eq. (81) is the cubic one. The cubicity is due to allowing for the term with $\gamma_\parallel$, i.e., the parallel plasma motion.

B. Understanding the mechanism of zonal-flow instabilities of monochromatic drift-wave packet

1. Transformation of driving force of zonal-flow instabilities

Turning to Eqs. (77) and (78), the expression Γ_0^2 can be called the driving force of zonal-flow instabilities of the monochromatic drift-wave packet. Meanwhile, Eq. (77) has a rather complicated form. Moreover, it contains in the denominator the factor ρ_s^2, so that there is a mathematical problem to transit to the limit $\rho_s^2 \to 0$. In order to find out this problem, we should express the value $\partial V_g / \partial k_x$ in its explicit form. Turning to Eq. (5), we obtain

$$V_g \equiv \frac{\partial \omega_{\mathbf{k}}^{(0)}}{\partial k_x} = -\frac{2V_* \rho_s^2 k_x k_y}{(1 + k_\perp^2 \rho_s^2)^2}. \tag{81}$$

The subscript "0" at the wave vector is omitted for simplicity. Therefore,

$$\frac{\partial V_g}{\partial k_x} = -2V_* \rho_s^2 \frac{k_y[1 + \rho_s^2(k_y^2 - 3k_x^2)]}{(1 + k_\perp^2 \rho_s^2)^3}. \tag{82}$$

Using Eq. (82), one can see that, for $\rho_s^2 \to 0$, the right-hand side of Eq. (77) looks as the expression of type "0/0," so that ρ_s^2 in the denominator and the nominator are mutually cancelled. Then, for arbitrary $k_\perp^2 \rho_s^2$, Eq. (77) takes the form

$$\Gamma_0^2 = \frac{c^2 q_x^2 k_y^2}{B_0^2} \frac{1 + \rho_s^2(k_y^2 - 3k_x^2)}{(1 + k_\perp^2 \rho_s^2)^3} N_{\mathbf{k}}. \tag{83}$$

In particular, in the approximation $\rho_s^2 \to 0$ corresponding to the case of nondispersive primary wave and nondissipative zonal flow, Eq. (83) reduces to

$$\Gamma_0^2 = c^2 q_x^2 k_y^2 N_{\mathbf{k}} / B_0^2. \tag{84}$$

In the same approximation, one has $\Omega - g_x V_g \to \Omega$. Then, in the slab-geometry approximation, $1+2q^2 \to 1$, Eq. (78) transits to

$$\Omega^2 = -\Gamma_0^2 \equiv -c^2 q_x^2 k_y^2 N_{\mathbf{k}} / B_0^2. \tag{85}$$

This relation can be called the standard zonal-flow dispersion relation of hydrodynamic type.

2. Analogy with linear ideal MHD instabilities

According to Ref. 10, the standard linear ideal MHD (magnetohydrodynamic) instabilities are characterized by the dispersion relation

$$\omega^2 = -\gamma_{\mathrm{MHD}}^2, \tag{86}$$

where ω is the mode frequency and γ_{MHD} is the growth rate defined by equilibrium plasma parameters and the concrete type of instability. Comparing Eq. (85) to Eq. (86), we conclude that the zonal-flow instability of hydrodynamic type is an analog of the ideal MHD instabilities.

3. Deriving the standard zonal-flow dispersion relation of hydrodynamic type by the approach of convective-cell theory

Turning to Eq. (4.44) of Ref. 28, one obtains

$$\left(\frac{\partial}{\partial t} + V_* \frac{\partial}{\partial y}\right)\tilde{\Phi} + \frac{c}{B_0} \frac{\partial \bar{\Phi}}{\partial x} \frac{\partial \tilde{\Phi}}{\partial y} = 0, \tag{87}$$

$$\frac{\partial \bar{\Phi}}{\partial t} - \frac{c}{B_0} \left\langle \frac{\partial \tilde{\Phi}}{\partial x} \frac{\partial \tilde{\Phi}}{\partial y} \right\rangle = 0. \tag{88}$$

Here $\bar{\Phi}$ is the zonal-flow electrostatic potential related to the velocity V_0 by

$$V_0 = (c/B_0)\, \partial \bar{\Phi} / \partial x, \tag{89}$$

whereas $\tilde{\Phi}$ is the drift-wave electrostatic potential in the presence of the zonal flow. The angular brackets $\langle \cdots \rangle$ mean averaging over the drift-wave oscillations.

We analyze Eqs. (87) and (88) by analogy with Ref. 2. Then we take the function $\bar{\Phi}$ in the form

$$\bar{\Phi} = \bar{\Phi}_0 \exp(-i\Omega t + iq_x x) + \text{c.c.}, \tag{90}$$

where c.c. is the complex conjugative. The function $\tilde{\Phi}$ is represented as follows:

$$\widetilde{\Phi} = \widetilde{\Phi}_0 \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r}) + \Phi^{+}\exp[-i(\omega+\Omega)t + i(\mathbf{k}\cdot\mathbf{r} + q_x x)] + \Phi^{-}\exp[-i(\Omega-\omega)t + i(q_x x - \mathbf{k}\cdot\mathbf{r})] + \text{c.c.} \tag{91}$$

Here $\widetilde{\Phi}_0$ is the amplitude of the unperturbed drift-wave electrostatic potential, whereas the values Φ^+ and Φ^- are the upper and lower side-band harmonics of the perturbed electrostatic potential, respectively.

We find from Eq. (88)

$$i\Omega\overline{\Phi}_0 + \frac{c}{B_0}k_y[(2k_x+q_x)\widetilde{\Phi}_0^{*}\Phi^{+} + (2k_x-q_x)\widetilde{\Phi}_0\Phi^{-}] = 0. \tag{92}$$

Similarly, Eq. (87) results in

$$\Phi^{+} = i\frac{c}{B_0}\frac{q_x k_y \overline{\Phi}_0 \widetilde{\Phi}_0}{\omega + \Omega - k_y V_*}, \tag{93}$$

$$\Phi^{-} = -i\frac{c}{B_0}\frac{q_x k_y \overline{\Phi}_0 \widetilde{\Phi}_0^{*}}{\Omega - \omega + k_y V_*}. \tag{94}$$

Substituting Eqs. (93) and (94) into Eq. (92) yields the zonal-flow dispersion relation

$$\Omega + \frac{1}{2}\left(\frac{c}{B_0}\right)^2 k_y^2 q_x N_{\mathbf{k}}\left(\frac{2k_x+q_x}{\omega+\Omega-k_yV_*} - \frac{2k_x-q_x}{\Omega-\omega+k_yV_*}\right) = 0. \tag{95}$$

Here, in accordance with Eqs. (8) and (91),

$$N_{\mathbf{k}} = 2\widetilde{\Phi}_0^{*}\widetilde{\Phi}_0. \tag{96}$$

Now we allow for that, in the approximation $\rho_s^2 \to 0$, the drift-wave dispersion relation, Eq. (5), reduces to

$$\omega = k_y V_*. \tag{97}$$

Using Eq. (97), Eq. (95) transits to Eq. (85).

C. Effect of neoclassical viscosity on poloidal-flow instability

1. Dissipative neoclassical poloidal-flow instability

Allowing for neoclassical viscosity and neglecting the parallel velocity, Eq. (80) yields

$$(\Omega - q_x V_g)^2(1 + i\gamma_p/\Omega) = -\Gamma^2. \tag{98}$$

For sufficiently large γ_p, $\gamma_p \gg \Omega$, Eq. (98) reduces to

$$(\Omega - q_x V_g)^2 - i\Omega\Gamma_0^2/\kappa_\theta = 0. \tag{99}$$

Then we find that in the case of sufficiently large radial transport, $q_x V_g \gg \Gamma_0^2/\kappa_\theta$, there is a zonal-flow instability characterized by the relations

$$\text{Re}\,\Omega = q_x V_g, \tag{100}$$

$$\text{Im}\,\Omega = \frac{|\Gamma_0|}{\sqrt{2}}\left(\frac{|q_x V_g|}{\kappa_\theta}\right)^{1/2}. \tag{101}$$

In the case of negligibly weak radial transport, $q_x V_g \ll \Omega$, Eq. (98) reduces to

$$\Omega^2 + i\gamma_p\Omega = -\Gamma^2. \tag{102}$$

For $\gamma_p \gg \Omega$, instead of Eqs. (100) and (101), one has

$$\Omega = i\Gamma_0^2/\kappa_\theta. \tag{103}$$

Equations (100), (101), and (103) show that for strong neoclassical viscosity the hydrodynamic instability, pointed out in Ref. 24, transits to a dissipative one. In other words, the strong neoclassical viscosity does not completely suppress the instability studied in Ref. 24, but leads to decreasing its growth rate.

2. Analogy with linear ideal-viscous instabilities

According to Ref. 10, in the presence of neoclassical viscosity, the dispersion relation for the linear MHD instabilities, Eq. (86), is substituted by

$$\omega^2 + i\gamma_p\omega = -\gamma_{\text{MHD}}^2, \tag{104}$$

which is similar to Eq. (102). For $\gamma_p \gg \gamma_{\text{MHD}}$ Eq. (104) reduces to the dispersion relation

$$\omega = i\gamma_{\text{MHD}}^2/\gamma_p, \tag{105}$$

describing the linear ideal-viscous instabilities. Comparing Eq. (103) with Eq. (105), we conclude that the nondispersive dissipative zonal-flow instability is an analogue of the linear ideal-viscous instabilities.

D. Neoclassical toroidal-flow instability of hydrodynamic type

Allowing for toroidal velocity, it follows from Eqs. (75), (34), and (35) that for sufficiently strong neoclassical viscosity:

$$(\Omega - q_x V_g)^2 - i\frac{\Gamma_0^2}{\kappa_\theta}\left(\Omega + i\frac{\epsilon^2}{q^2}\kappa_\theta\right) = 0. \tag{106}$$

Hence it can be seen that dissipative zonal-flow instability described by Eqs. (100), (101), and (103) takes place for

$$\gamma_\parallel \ll \Gamma_0. \tag{107}$$

In the opposite case, when

$$\gamma_\parallel \gg \Gamma_0, \tag{108}$$

Eq. (106) reduces to

$$(\Omega - q_x V_g)^2 = (\epsilon/q)^2\Gamma_0^2. \tag{109}$$

This dispersion relation transits formally to Eq. (78) in the substitution given by Eq. (4). It describes the neoclassical toroidal-flow instability of the hydrodynamic type. This instability corresponds to generation of considerable toroidal velocity.

E. Generalization to the case of arbitrary nondispersive drift-wave spectra and nondispersive zonal flows

The main physical results of the present section are valid not only in the case of monochromatic drift-wave packet but also in the case of arbitrary nondispersive drift-wave spectra and nondispersive zonal flow, i.e., for the case $\rho_s^2 \to 0$. In order to show this fact, we note that, in the case $\rho_s^2 \to 0$ and arbitrary drift-wave spectra, one has, instead of Eq. (74),

$$u = -\frac{c^2 q_x}{B_0^2 \Omega} \int k_y^2 N_{\mathbf{k}} d\mathbf{k}. \tag{110}$$

Using Eq. (110), one can obtain, instead of Eq. (84),

$$\Gamma_0^2 = \frac{c^2 q_x^2}{B_0^2} \int k_y^2 N_{\mathbf{k}} d\mathbf{k}. \tag{111}$$

By means of Eq. (111) one can generalize all preceding relationships containing Γ_0^2 for the case of arbitrary spectra.

VII. DISCUSSION

The analysis of this article shows that existing theory of zonal flows in tokamaks was incomplete since it ignored the combined effect of neoclassical viscosity and toroidal flow velocity. We have generalized it for the case of the standard drift waves. Let us summarize the main points of our approach.

We have used the standard wave kinetic equation for the broad-spectrum drift-wave turbulence in the assumptions that the wave frequency is shifted due to the mean (averaged) cross-field drift velocity [see Eq. (6)]. In description of mean flows we have turned to the motion equations in the presence of neoclassical viscosity derived in Ref. 5. One of them is the vorticity equation, integrated over the radial coordinate and the second is the toroidal motion equation. Then we have assumed that the external forces entering the approach of Ref. 5 are the Reynolds stress forces due to the drift-wave turbulence. These forces are expressed in terms of a single Reynolds stress function similar to that of the two-dimensional theory.[1,19] Similarly to Ref. 5, we have shown in Sec. II that, instead of the toroidal motion equation, it is more convenient in our problem to use the parallel motion equation: since the parallel Reynolds stress force vanishes, the latter proves the same as in the theory of equilibrium plasma rotation in the absence of the parallel external force. The effects of neoclassical viscosity, introduced in Sec. II, have been incorporated in Sec. III into the zonal-flow dispersion relation, Eq. (35), derived by the standard approach presented, in particular, in Refs. 1 and 19.

It is significant for our problem that the mean cross-field drift velocity differs from the poloidal velocity. This difference results in that the parallel velocity enters the vorticity equation. On the other hand, the parallel velocity is coupled with the poloidal velocity by the parallel equation of motion. The combined effect of the turbulence and the neoclassical viscosity leads to the new type of the zonal flow instabilities. Formally, it manifests itself in the zonal-flow dispersion relation, which, according to Eq. (36), becomes quadratic with respect to perturbation frequency, contrary to the theory neglecting the neoclassical viscosity, where it is linear.

Physically, the new type of the instability corresponds to generation of toroidal flow. Equations given in Sec. III show that this generation is a process slower than possible generation of the poloidal flow. Mathematically, this means that the roots of our zonal-flow dispersion relation are well separated in the amplitude, so that the results of the theory neglecting the neoclassical viscosity, presented in Ref. 19, remain unchanged. They are briefly summarized in Sec. IV where it is emphasized that the inclusion of the toroidal velocity into the analysis of Ref. 19 does not change the poloidal velocity and that the toroidal velocity remains to be sufficiently small. Therefore, the instability studied in Ref. 19 was called in Sec. IV the poloidal-flow instability. Subsequently, the main thrust of our analysis is the smaller root of our dispersion relation, given by Eqs. (45) and (46) or, for sufficiently strong neoclassical viscosity, by Eqs. (50) and (51), describing generation of the toroidal velocity. The previous equations show that the growth rate of the toroidal-flow instability of the broad-spectrum drift-wave turbulence is small compared with that of the poloidal-flow instability as the parameter $(\epsilon/q)^2$ governing the neoclassical inertia renormalization. In this context, the toroidal flow instability is similar to the linear neoclassical resistive modes.[6–10]

Studying in Sec. V nonlinear saturation of the toroidal-flow instability, we have used the fact that both the linear and nonlinear parts of the Reynolds stress depend only on the mean cross-field drift velocity, see Eqs. (53) and (54). Then the nonlinear evolution equation for this instability, similarly to that for the poloidal-flow instability, can be expressed in terms of the cross-field velocity. Such an equation has mathematically the same solution as that found in Ref. 19 with the neoclassical inertia renormalization, see Eq. (66). This solution shows that, though the evolution process is rather slow, the saturated toroidal velocity proves to be essentially larger than the saturated cross-field drift velocity. It can be estimated by means of Eq. (62).

A rather many-sided problem is the topic of zonal-flow instabilities of the monochromatic drift-wave packet considered in Sec. VI. The traditional approach to studying this topic is the approach of convective-cell theory.[2] Meanwhile, Ref. 24 studied it by means of the approach of wave kinetic equation.[25] Thereby, this topic lies at the junction of the two ideologies appealing to the above-mentioned approaches.

It is evident that the results of Ref. 24 can be found from those of Ref. 2 by means of corresponding limiting transition. At the same time, for a better understanding of the mechanism of the previous instabilities, it seemed to be reasonable to obtain these results directly by means of the convective-cell theory approach. A part of Sec. VI was addressed to such an obtaining. Then we have found the simplest version of zonal-flow dispersion relation given by Eq. (95), elucidating the so-called “driving force” of zonal-flow instabilities given by Eq. (84).

Appealing to the linear stability theory, we have explained that the poloidal-flow instability of hydrodynamic type is similar to the ideal MHD instabilities. As the latter, the poloidal-flow instability does not suppressed by neoclassical viscosity. Instead of this, if the viscosity increases, it transits to the dissipative poloidal-flow instability similar to the ideal-viscous instabilities. For sufficiently strong viscosity, the dissipative poloidal-flow instability is changed by the neoclassical toroidal-flow instability of hydrodynamic type. Its growth rate is small compared with that of the poloidal-flow instability of hydrodynamic type as ϵ/q, that is explained by the neoclassical inertia renormalization.

We have explained that the main physical regularities pointed out for zonal-flow instabilities, driven by the monochromatic wave packet, remain in force for arbitrary drift-wave spectra if both the drift waves and the zonal flows are nondispersive. In this context, the approximation of the monochromatic wave packet can be considered as the simplest model for studying the nondispersive zonal-flow instabilities.

The drift waves described by Eq. (5) are the simplest particular variety of small-scale oscillations of magnetized inhomogeneous plasma.[28] Analysis of neoclassical zonal-flow instabilities driven by other types of drift waves in toroidal systems can be a subject of following investigations.

Since the neoclassical viscosity is a result of toroidicity and the linear instabilities in toroidal geometry are often studied by means of ballooning representation,[29,10] it seems to be of interest to formulate a theory of zonal flow instabilities in the basis of ballooning mode eigenfunctions. An important step in formulating such a theory has been made in Ref. 21, where the approach of convective-cell theory has been used, whose simplest version has been explained in Sec. VI B 3. Our results are in correspondence with those obtained in Ref. 21. At the same time, it seems to be of interest to include the elements of the ballooning-mode approach into the trend of zonal flow theory dealing with the wave kinetic equation.[1] Such a problem can be a subject of following studies.

As known, strong toroidal (and parallel) flow shear may trigger a negative compressibility instability [a variety of the Kelvin-Helmholtz (KH) instabilities] as originally studied in Ref. 30 and later in Ref. 31 and other papers. It is then of interest to estimate whether the toroidal zonal flows studied in our paper are subject to such an instability. In order to made such an estimation, let us turn to the instability criterion derived in Ref. 30:

$$\frac{\tau V_\parallel' L_s}{(\tau+1)v_{T_i}} > \frac{9}{8\pi}. \tag{112}$$

Here the prime is the radial derivative, L_s is the length of magnetic shear, v_{T_i} is the ion thermal velocity, $\tau = T_e/T_i$ is the ratio of electron and ion temperatures. According to Eq. (26), $(\cdots)' \simeq q_x$. Estimating $q_x \simeq 1/L_n$, where L_n is the characteristic scale length of density inhomogeneity and taking $L_s \simeq qR$ and $\tau \simeq 1$, Eq. (112) yields

$$V_\parallel / v_{T_i} > L_n / qR. \tag{113}$$

Now we turn to estimation for $V_\parallel$ given by Eq. (62). Then one can find

$$V_\parallel \simeq V_* q/\epsilon. \tag{114}$$

Substituting Eq. (114) into Eq. (113) and allowing for $V_* \simeq \rho_i v_{Ti}/L_n$, where ρ_i is the ion Larmor radius, we arrive at the KH instability condition

$$\frac{\rho_i}{r} > \frac{L_n^2}{(qR)^2}. \tag{115}$$

This condition can be satisfied in the edge plasma, where L_n is sufficiently small. This analysis shows that the KH instability of the toroidal zonal flows in the edge plasma can be a subject of following investigations.

Studying in Sec. V the nonlinear saturation of resonant toroidal-flow instability, we have taken into account the squared nonlinear effect described by the function $\Lambda^{(2)}$ given by Eq. (54), expressed in terms of the parameter b given by Eq. (55). Meanwhile, if the nonperturbed drift wave spectrum $N_{\mathbf{k}}^{(0)}$ is even function of the radial wave number k_r, the right-hand side of Eq. (55) vanishes, so that $b=0$ and, as a consequence, $\Lambda^{(2)}=0$. Then, in contrast to Eq. (53), one should take into account the cubic nonlinear effect, i.e., substitute

$$\widehat{\Lambda} \to \widehat{\Lambda} + \Lambda^{(3)}, \tag{116}$$

where $\Lambda^{(3)}$ is proportional to V_0^3. Then one arrives at an evolution equation similar to Eq. (55) with the substitution

$$b\frac{\partial V_0^2}{\partial x} \to \hat{c}\frac{\partial V_0^3}{\partial x}. \tag{117}$$

Here $\hat{c}$ is a parameter given by integral over $\mathbf{k}$, similar to Eq. (53), but with the integrand even with respect to k_r. This is the essence of the saturation mechanism of the poloidal zonal flow studied in Ref. 20, see also Ref. 32.

Let us explain how this saturation mechanism can be generalized by inclusion of toroidal zonal flow. The fact is that Refs. 32 and 20 have taken the mean cross-field drift velocity V_0 to be identically equal to the zonal-flow poloidal velocity V_p, $V_0 = V_p$. Therefore, such a generalization should consist in taking $V_0 \neq V_p$. Then one should complement the vorticity equation of these references by the equation of parallel motion given by Eq. (25). As a result, one could elucidate how Eqs. (64)–(69) are modified in the case of saturation mechanism of Refs. 32 and 20. Such an elucidation can be a subject of following investigations.

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