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Quantum Algebraic Nature of the Phonon Spectrum in ^4He

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Abstract

We propose that the phonons in ^4He obey a q-deformation of the Heisenberg algebra and give an algebraic interpretation for the polynomial expansion of the small momenta phonon dispersion relation. Comparison with C_V experimental data shows that our spectrum reproduces the experimental one within less than 5% of discrepancy. .

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The superfluid properties of ^4He [1] are well described by Landau theory [2]; nevertheless, even for temperatures as low as 1^0K there are still unsolved discrepancies between theory and experiment. In Landau theory, the superfluidity follows from phonon and roton elementary excitations [3]. The anomalous dispersion of phonon spectrum in ^4He, $\omega(p) = c_0 p(1-\gamma p^2)$ (c_0 is the sound velocity), was theoreticaly derived [4] and γ estimated to be positive. On the other hand, data from ^4He specific heat measurements fit to a different expression for the dispersion of phonon spectrum and give a negative γ for most values of the pressure [5–6]. Negative γ leads to an unstable phonon spectrum, which is confirmed by experimental measurements of phonon lifetime in scattering of neutrons [7]. In this letter we show that this difficulty can be overcome if we treat the phonons as bosonic q-oscillators [8]. Using A, B and D values experimentally determined by fitting the low-temperature phonon specific heat

$$C_V^{phonon} = AT^3 + BT^5 + DT^7 \ , \tag{1}$$

with measured specific heat data of ^4He[6] at the temperature range $0.14 \leq T \leq 0.86$, our model leads to unstable phonons for all the analysed values of the pressure.

Bosonic q-oscillators [8] are a generalization of the Heisenberg algebra obtained by introducing a deformation parameter q. For $q > 1$ [9], an ideal q-gas presents Bose-Einstein condensation and the specific heat exhibits a λ-point discontinuity [10], two features connected to superfluidity [11]. On the other hand there have been interesting indications that the continuum descriptions of physical quantities break down both in a convergent fluid flow [12] and, more recently, in superfluid ^4He [13]. As a similar breakdown has been observed in connection to deformed algebras [14], we are led to think that they might have a role to play in the study of superfluidity.

Let us then consider the algebra generated by a, a^+ and N satisfying

$$\begin{array}{l} [N, a^+] = a^+ \quad , \quad [N, a] = -a \\ aa^+ - q^{-1}a^+a = q^N \quad (q \in \mathbb{R}) \quad . \end{array} \tag{2}$$

Assuming that a and a^+ are mutually adjoint, $N = N^+$ and the spectrum is non-degenerate, the following representations of (2) were obtained [15] for $q > 1$:

$$\begin{aligned} a^+|n\rangle &= q^{\nu_0/2}[n+1]^{1/2}|n+1\rangle \ , \\ a|n\rangle &= q^{\nu_0/2}[n]^{1/2}|n-1\rangle \ , \\ N|n\rangle &= (\nu_0 + n)|n\rangle \ , \end{aligned} \tag{3}$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$ and ν_0 is a real free parameter which goes to zero when (2) becomes the usual Heisenberg algebra ($q \to 1$). Note that only when $\nu_0 = 0$, N is the usual particle number operator for the normalized vector state $|n\rangle$; otherwise the particle number operator is $\hat{N} = N - \nu_0$ and ν_0 is a parameter that classifies the inequivalent representations of the algebra (2) [15, 16, 17].

Generalizing previous results obtained for $\nu_0 = 0$ [18], in the Fock space spanned by the vectors $|n\rangle$, we can express the above deformed oscillators in terms of the standard bosonic ones, b and b^+, according to

$$a = q^{\nu_0/2}\left(\frac{[N+1-\nu_0]}{N+1-\nu_0}\right)^{1/2} b \ \ , \ a^+ = q^{\nu_0/2}\ b^+\left(\frac{[N+1-\nu_0]}{N+1-\nu_0}\right)^{1/2} \tag{4}$$

and it can be easily shown that

$$aa^+ = q^{\nu_0}[N+1-\nu_0] \quad , \quad a^+a = q^{\nu_0}[N-\nu_0] \; , \tag{5}$$

where $N - \nu_0 = b^+b$. This shows that bosonic q-oscillators, in arbitrary representations ν_0 and for real $q > 1$, can be reinterpreted as standard bosonic oscillators.

We propose that the phonons in ^4He are described by a q-gas. Considering that our model will be compared with the experimental results at the temperature range $0.14 \leq T \leq 0.86$ [6], where the rotons contribution is at most 0.5% of the total specific heat [6], they will be treated as usual [3]. We take for the phonon gas the Hamiltonian

$$H = \sum_i \; \omega_i a_i^+ a_i = \sum_i \; \omega_i \left([N_i] - q^{N_i}\mathcal{C}\right) \; , \tag{6}$$

where $\mathcal{C} = q^{-N}([N] - a^+a)$ is a Casimir operator of the algebra (2) and in the representations (3) one has

$$\mathcal{C}|n\rangle = q^{\nu_0}[\nu_0]|n\rangle \; . \tag{7}$$

In (6) a_i and a_i^+ are the annihilation and creation operators of particles in levels i with energy ω_i and N_i is the number operator of particles in levels i plus ν_0^i, which we are assuming level dependent.

As the partition function factorizes for the above system the canonical potential is

$$\Omega = -\frac{1}{\beta} \; \sum_i \; \ln \sum_{n=o}^{\infty} \; e^{-\beta\omega_i q^{\nu_o^i}[n]} \; . \tag{8}$$

where $\beta = (k_B T)^{-1}$, with k_B the Boltzmann constant.

The phonon anomalous dispersion relation is $\omega(p) = c_0 p(1-\alpha p^2)$, with c_0 the velocity of sound, and we propose the dispersion relation

$$\nu_0(p) = \frac{\delta^2}{\theta} \; p^2 = \frac{p^2/2m}{E_\lambda} \tag{9}$$

with δ an algebraic dimensional constant, $[\delta] = gr^{-1}cm^{-1}sec$, and $q = e^\theta$. As a consequence of the dimensionlesness of $\nu_0(p)$ it appears in (9) an energy scale, E_λ, that we take as $E_\lambda = k_B T_\lambda$, where T_λ is the temperature at which liquid ^4He undergoes a transition and becomes superfluid. Moroever, it seems natural to take $m = m_{He^4}$ since we have for $\nu_0(p)$ the non-relativistic classical dispersion law. For small phonon momenta we can expand our energy-momentum relation as

$$q^{\nu_0(p)}\omega(p) = e^{\delta^2 p^2} c_0 p(1-\alpha p^2) = c_0 p[1 - (\alpha - \delta^2)p^2 - (\alpha\delta^2 - \frac{1}{2}\;\delta^4)p^4 - \cdots] \; . \tag{10}$$

We are thus presenting an algebraic interpretation to the usually ad-hoc introduced small momenta phonon dispersion relation [3, 5, 6].

It follows from a straightforward calculation that the low-temperature q-phonon specific heat per mole is given by

$$C_{V,q}^{phonon} = \tilde{A}T^3 + \tilde{B}T^5 + \tilde{D}T^7 + \tilde{G}T^9 + \cdots \; , \tag{11}$$

where

$$\tilde{A} = \frac{2k_B^4 V}{\pi^2\hbar^3 c_0^3}\,\omega^{(3)} \;\; ; \;\; \tilde{B} = \frac{15k_B^6(\alpha-\delta^2)V}{\pi^2\hbar^3 c_0^5}\,\omega^{(5)} \;\; , \;\; \tilde{D} = \frac{28k_B^8\left(\frac{7}{2}\,\alpha^4+4\delta^2-7\alpha\delta^2\right)V}{\pi^2\hbar^3 c_0^7}\,\omega^{(7)}$$

$$\tilde{G} = \frac{15k_B^{10}(-81\delta^6+110\alpha^3+243\alpha\delta^4-270\delta^2\alpha^2)V}{2\pi^2\hbar^3 c_0^9}\,\omega^{(9)} \qquad (12)$$

with V the molar volume and

$$\omega^{(m)} = \int_0^\infty dy\; y^{m-1}\,\frac{\sum_{n=0}^{\infty}[n]e^{-y[n]}}{\sum_{n=0}^{\infty} e^{-y[n]}}\;. \qquad (13)$$

Using its usual dispersion relation, $\omega_r(p) = \Delta + (p-p_0)^2/2\mu$, where Δ is the energy gap, p_0 is the position of the energy minimum and μ is the effective mass of the roton, the roton contribution to the molar specific heat is:

$$C_V^{roton} = \frac{2V\mu^{1/2}p_0^2\Delta^2}{(2\pi)^{3/2}\hbar^3 k_B^{1/2}T^{3/2}}\left(1+k_BT/\Delta+\frac{3}{4}(k_BT/\Delta)^2\right)e^{-\Delta/k_BT} \qquad (14)$$

and the total specific heat is

$$C_V = C_{V,q}^{phonon} + C_V^{roton}\;. \qquad (15)$$

Taking for the coefficients $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$ the least-squares fits for A, B, and D in (1) [6] of the measured specific heat data (analysis 2 in ref. [6]) and $\tilde{G}=0$, we obtain for q, α, δ and c_0 the results listed in table I. The values of q are derived from

$$\ln q = 2m_{He^4}\delta^2 k_B T_\lambda\;, \qquad (16)$$

which is a consequence of (9), and c_0, α and δ from relations (12). As the very large errors in the T^7 coefficients for the samples 10-16 [6] lead to a high inaccuracy in the derivation of expression (16), we restrict our analysis to the samples 6-9 [6].

In table I we see that the values of q increase with the pressure, and that the values of c_0 are around 4% lower than the directly measured sound velocities [19]. These results are obtained by least-squares fits of the specific heat data [6] with the expression (15), considering terms up to T^7 in $C_{V,q}^{phonon}$. Since in our model higher powers of T are relevant, in the second row of table II we show the results obtained considering terms up to T^9 in (11) and taking $\tilde{A} = 80\times10^4\;erg/mol\;K^4$, $\tilde{B} = -21\times10^4\;erg/mol\;K^6$, $\tilde{D} = 83\times10^4\;erg/mol\;K^8$ and $\tilde{G} = -67.5\times10^4\;erg/mol\;K^{10}$. These values reproduce, within 5% of accuracy, the curve resulting from least-square fit of C_V data for sample 6 [6], with $C_V = AT^3+BT^5+DT^7+C_V^{roton}$ (see Fig. 1). We see that the c_0 value is then more in accordance with the experimental one. We note that for a given value of q, c_0 and the parameters α and δ are obtained directly from the values of $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$ through relations (12). The coefficient $\tilde{G}$ of T^9 is crucial to show the consistency of our model. In fact, with the values of α, δ and c_0 in the lower row of table II, the coefficient $\tilde{G}$ calculated from the last relation (12) is equal to -67.5×10^4 $erg/mol\ K^{10}$.

In summary, considering the phonons in ^4He as being described by a quantum q-gas in a special representation of the Heisenberg algebra, we have shown the q-algebraic nature of the polynomial expansion of the small momenta phonon dispersion relation. Moreover, our estimated values of c_0 are in good agreement with the directly measured sound velocities. To test the present model we have compared it with the available experimental data: our spectrum reproduces the experimental one for the entire $0.14 \leq T \leq 0.86$ range, within less than 5% of discrepancy. Finally, we would like to stress that as a consequence of the proposed dispersion relation (10), with only two free parameters (q, ν_0) we have been able to fit the experimental data with the three coefficients $\tilde{B}$, $\tilde{D}$ and in $\tilde{G}$ in the specific heat expansion (11).

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Sample	$\tilde{A}/10^4$ $(erg/mol\ K^4)$	$\tilde{B}/10^4$ $(erg/mol\ K^6)$	$\tilde{D}/10^4$ $(erg/mol\ K^8)$	V (cm^3)	$c_0/10^4$ (cm/sec)	$\alpha/10^{38}$ $(gr^{-2}cm^{-2}sec^2)$	$\delta/10^{19}$ $(gr^{-1}cm^{-1}sec)$	q
6	84.42	-49.8	83	27.5790	2.2854	2.1	1.7745	3.5090
7	69.3	-36.10	67	26.9650	2.4177	2.7178	1.9361	4.3956
8	57.77	-25.4	49	26.4240	2.5501	3.0873	2.0134	4.8778
9	49.85	-18.8	38	25.9760	2.6625	3.3821	2.0700	5.2490

Table I - Values of c_0, α, δ and q resulting from the least-square fits of the specific heat data with the expression $C_V = \tilde{A}T^3 + \tilde{B}T^5 + \tilde{D}T^7 + C_V^{roton}$, for samples 6-9 in analysis 2 of ref. [6]; the roton data are those of [6].

Sample	$\tilde{A}/10^4$ $(erg/mol\ K^4)$	$\tilde{B}/10^4$ $(erg/mol\ K^6)$	$\tilde{D}/10^4$ $(erg/mol\ K^8)$	$\tilde{G}/10^4$ (erg/mol^{10})	V (cm^3)	$c_0/10^4$ (cm/sec)	$\alpha/10^{38}$ $(gr^{-2}cm^{-2}sec^2)$	$\delta/10^{19}$ $(gr^{-1}cm^{-1}sec)$	q
6	84.42	-49.8	83	0	27.5790	2.2854	2.1	1.7745	3.5090
6	80	-21	83	-67.5	27.5790	2.3209	3.4484	1.9815	4.7839

Table II - In the upper row, we repeat the values of table I for sample 6. In the lower one, we have the values for c_0, α, δ and q obtained taking for $\tilde{A}$, $\tilde{B}$, $\tilde{D}$ and $\tilde{G}$ values that reproduce, within 5% of accuracy, the curve resulting from least-square fit of C_V data [6] with the expression $C_V = AT^3 + BT^5 + DT^7 + C_V^{roton}$.

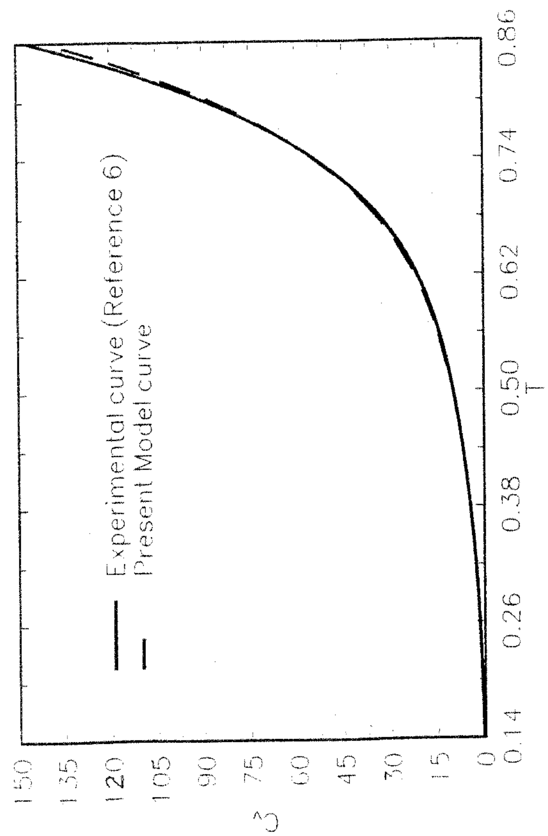

Fig. 1 -- Specific heat of ^{4}He. Comparison of the curve obtained in our model with the one resulting from least–square fit of C_V data for sample 6 of the analysis 2 — Reference 6.

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