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On the Generalized Bose-Einstein Condensation

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Abstract

Bose-Einstein condensation is considered within the Tsallis generalized thermostatistics as an illustrative case of a situation where interactions are inexistent in the Hamiltonian and the system evolves in a fractal space. The $(1-q)/kT \to 0$ asymptotic behavior is studied.

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The Boltzmann-Gibbs statistics and its connection with thermodynamics are powerful tools in theoretical physics to study situations where the following conditions are satisfied (i) the effective microscopic interaction is short-ranged or inexistent, (ii) the microscopic memory is short-ranged or inexistent and (iii) the system evolves in a nonfractal space-time. This is to say, whenever the extensive (additive) properties of thermodynamics hold.

Whenever an Euclidean-like (nonfractal) space is involved, the ideal Bose-Einstein gas (no interaction exists in the Hamiltonian) is the simplest exactly solvable continuous system that has a phase transition. Therefore, it has its place in any course on statistical mechanics including quantum statistics [1]. An elementary, but rigorous, calculation is made in ref. [2] by controlling the difference between sums and integrals in the thermodynamic limit. This phase transition is the well known Bose-Einstein condensation (most of particles go to the zero momentum state). The Bose-Einstein condensation is a physical situation in vogue nowadays because of important experimental advancements have that ocurred recently [3].

Let us start by considering an ideal gas of particles that obey to the Bose-Einstein statistics at temperature T and chemical potential μ, for a large (hyper)volume V and a large number of particles N_1. The average number of particles is given by

$$N_1 = n_1^{(0)} + V\left(\frac{mkT}{2\pi\hbar^2}\right)^{D/2} g_{D/2}(e^{\mu/k_BT}). \tag{1}$$

Here D is the dimension of the system and we assume $D > 2$ in order to have a non-vanishing critical temperature; k_B is the Boltzmann constant and $\hbar = h/2\pi$ (h is the Planck constant). The function $g_{D/2}$ is defined as $g_{D/2}(y) = \sum_{n=1}^{\infty} y^n/n^{D/2}$. The quantity $n_1^{(0)} \equiv 1/(\exp(-\mu/k_BT) - 1)$ is called the zero momentum state and can be macroscopically occupied. The phase transition happens as $\mu \to 0$.

The occupation of the zero momentum state is of course the Bose-Einstein condensation. It occurs at temperature $T \leq T_{c1}$. The critical temperature T_{c1} is given by

$$T_{c1} = \frac{2\pi\hbar^2}{mk_B}\left(\frac{N_1}{\zeta(D/2)V}\right)^{2/D}. \tag{2}$$

where $\zeta(x)$ is the Riemann function. It is worthy to recall that, for $T \geq T_{c1}$, $n_1^{(0)} = 0$; for $T \leq T_{c1}$, $\mu = 0$; and that, for $T = T_{c1} + \epsilon$, where $\epsilon \to 0$, $n_1^{(0)} \propto \epsilon$, which could be compared with the mean field approximation, namely $\epsilon^{1/2}$.

The fraction of particles in the zero momentum state can be written

$$\frac{n_1^{(0)}}{N_1} = 1 - \left(\frac{T}{T_{c1}}\right)^{D/2}, \qquad \text{for } T \leq T_{c1}. \tag{3}$$

Eqs.(1)-(3) is the more interesting set of results on the problem within the Boltzmann-Gibbs thermostatistics. Some thermodynamic properties of an ideal gas in D dimensions have been discussed in ref. [4]. Nevertheless, this formalism fails whenever the physical system includes (i) long-range force and/or (ii) long-memory effect and/or (iii) a (multi)fractal space-time. In any of these cases, the system is expected to violate the standard extensive properties.

More precisely [5], the difficulties and their consequences are classified as follows:

(i) For a relevant Euclidean-like space-time and if either the forces or the memory (or both) are long-ranged, but we are interested in an equilibrium state, the Boltzmann-Gibbs statistics is weakly violated, the formalism can be used to obtain an approximate description. However, if we are interested in a meta-equilibrium state [6], the Boltzmann-Gibbs description is strongly violated. Other formalism must be used.

(ii) For a relevant (multi)fractal space, the Boltzmann-Gibbs formalism is strongly violated again and other formalism is needed.

The explicit need for a nonextensive thermodynamics has been well known in cosmology, gravitation and astrophysics [7], magnetic systems [8], Lévy-like anomalous diffusion [9], etc.

As a possible solution, Tsallis proposed a nonextensive thermostatistics in his paper [10]. This formalism has already received some applications. Among them, let us mention: Self-gravitating systems, Stellar polytropes, Vlasov equation [11, 6]; Lévy-like anomalous diffusion [9, 12]; Simulated annealing [13]. Furthermore, its connection within quantum statistics [14], quantum groups [15], quantum uncertainty [16], fractals [17, 18], etc., has been established.

The generalized statistics relies on the so called Tsallis entropy, namely

$$S_q \equiv -k\frac{1-\sum_R p_R^q}{1-q}, \tag{4}$$

where $q \in \Re$; k is a positive constant and S_q recovers the standard form $-k_B\sum_R p_R \ln p_R$, in the limit $q \to 1$.

Expression (4) has enabled various (nontrivial, though mathematically simple and natural) generalizations of important properties such as ref. [14]

(i) The grand-canonical equilibrium distribution now becomes

$$p_R = \frac{[1-\beta(1-q)(\varepsilon_R-\mu N)]^{\frac{1}{1-q}}}{\Xi_q}, \tag{5}$$

with the generalized grand-partition function consistently given by

$$\Xi_q(\beta,\mu) = \sum_R [1-\beta(1-q)(\varepsilon_R-\mu N)]^{\frac{1}{1-q}}, \tag{6}$$

where $\beta \equiv 1/kT > 0$ and ε_R is the spectrum (R represents a set of given real numbers).

(ii) The thermodynamics associated with Eq.(4) is invariant under Legendre transformations and preserves thermodynamic stability [19]; in particular, the fundamental equation for open system is

$$\Omega_q = -kT\frac{\Xi_q^{1-q}-1}{1-q}. \tag{7}$$

The q-expectation value of the particle number is given by,

$$N_q = \sum_R p_R^q N = \frac{\partial\Omega_q}{\partial\mu} = -kT\frac{1}{(\Xi_q)^q}\frac{\partial\Xi_q}{\partial\mu}. \tag{8}$$

(iii) The corresponding Hilhorst integral transformation of the grand-partition function was obtained. In the same manner, the generalized distribution function as well as the q-expectation values of some thermodynamic quantities were established. The Hilhorst transformation (as discussed by Prato [20]) for the q-expectation value of the particle number can be obtained in a closed form as

$$N_q = \frac{\Gamma(\frac{1}{1-q})}{[\Xi_q(\beta)]^q}\frac{i}{2\pi}\oint_C dz(-z)^{\frac{-1}{1-q}}e^{-z}\Xi_1(-\beta(1-q)z,\mu)N_1(-\beta(1-q)z,\mu). \quad (9)$$

Now, within this formalism, we study the Bose-Einstein system (no interactions in the Hamiltonian) in a (non Euclidean) fractal space. The last fact makes necessary to use a nonstandard Boltzmann-Gibbs statistics. The Tsallis formalism will be used, where the information about the fractal dimension is kept in the parameter q [17].

The generalization of Eq.(1) is

$$N_q = n_q^{(0)} + V\left(\frac{mkT}{2\pi\hbar^2}\right)^{D/2} G_q(D/2,\mu), \quad (10)$$

defining

$$K_q(m,x,\mu) = \frac{(1-q)^{m-x}\Gamma(1/(1-q))}{\Gamma(1/(1-q)+x-m)} < (kT-(1-q)(\mathcal{H}-\mu\mathcal{N}))^{x-m} >_q,$$

where, $< \mathcal{O} >_q$ is the q-expectation values of the operator $\mathcal{O}$. As before, $n_q^{(0)}$ is the generalized occupation number of the zero momentum state and it is given by

$$n_q^{(0)} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(n\mu)^m}{m!}K_q(m,0,\mu), \quad (11)$$

and

$$G_q(D/2,\mu) = \sum_{n=1}^{\infty}\left(\frac{kT}{n}\right)^{D/2}\sum_{m=0}^{\infty}\frac{(n\mu)^m}{m!}K_q(m,D/2,\mu). \quad (12)$$

According to Eq.(10) and Eq.(12), the critical temperature is obtained by requiring the following expression to be satisfied:

$$< (kT_{cq}-(1-q)\mathcal{H})^{D/2} >_q = \left(\frac{2\pi\hbar^2}{m}\right)^{D/2}\frac{N_q}{\zeta(D/2)V}, \quad (13)$$

which is the generalization of Eq.(2). We can also obtain the generalization of Eq.(3), this is:

$$\frac{n_q^{(0)}}{N_q} = 1 - \frac{< (kT-(1-q)\mathcal{H})^{D/2} >_q}{< (kT_{cq}-(1-q)\mathcal{H})^{D/2} >_q}. \quad (14)$$

Furthermore, it is convenient to remark that the Eq.(13) for the critical temperature is satisfied whenever the generalized fraction of particles in the zero momentum state vanishes ($n_q^{(0)}/N_q = 0$).

Due to the mathematical difficulties associated with a generic values of q, let us from now focus the $q \approx 1$ case. By using Eq.(7) from the ref. [21], Eq.(13) asymtotically becomes

$$kT_{cq}\Xi_1(T_{cq})^{2(1-q)/D}\left(1+(1-q)(D/2-1)\frac{U_1(T_{cq})}{kT_{cq}}\right)^{2/D}=k_BT_{c1}. \quad (15)$$

When $T \leq T_{c1}$, the internal energy U_1 for a Bose-Einstein system is written as a function of the temperature, namely

$$U_1(T)=\frac{D}{2}V\left(\frac{m}{2\pi\hbar^2}\right)^{D/2}(kT)^{D/2+1}\zeta(D/2+1); \quad (16)$$

for the same case, the thermodynamic potential is given by

$$\Omega_1=-V\left(\frac{m}{2\pi\hbar^2}\right)^{D/2}(kT)^{D/2+1}\zeta(D/2+1).$$

So, as $\Xi_1=\exp(-\beta\Omega_1)$, the grand partition function is written as

$$\Xi_1=\exp\left(V\left(\frac{mkT}{2\pi\hbar^2}\right)^{D/2}\zeta(D/2+1)\right). \quad (17)$$

Evaluating Eq.(15), we obtain an equation for T_{c1}, and by inverting for T_{cq}, we have

$$T_{cq}=\left[1-(1-q)\left(\frac{D}{2}+\frac{2}{D}-1\right)\frac{\zeta(D/2+1)}{\zeta(D/2)}N_1\right]T_{c1}, \quad (18)$$

whenever $k=k_B$ by the first order correction in $(1-q)$. This approximation shows that, if $q<1$ the critical temperature decreases. According to Eq.(2) there exists an apparent modification on the density (*i.e.,* if the average number of particles N_1 is constant the volume increases as $q<1$) in the $q \rightarrow 1^-$ limit . Now, we calculate

$$\frac{1}{N_1}\frac{\partial}{\partial q}\left(\frac{T_{cq}}{T_{c1}}\right)=\frac{(D-2)^2}{2D}\frac{\zeta(D/2+1)}{\zeta(D/2)}; \quad (19)$$

so, we remark that, the first derivative of T_{cq} with respect to q is positive for all values of $D>2$. In Fig.1 is depicted (solid line) the profile of the Eq.(19) as a function of the D parameter; (if $V=\ell^D$, where ℓ is the side of a box in D dimensions) the quantity $N_1^{2/3-2/D}T_{c1}(D)/T_{c1}(3)$ versus D (dashed line).

Summarizing, the standard results are recovered from the generalized results when $q=1$, as it should be. It is shown that, in the present approximation, the critical temperature for the phase transition of the Bose-Einstein condensation is modified whenever the nonextensive thermostatistics is considered. Furthermore, the critical temperature in other problems change as well if $q \neq 1$; for instance, in the canonical ensemble, the self-dual planar lattice Ising ferromagnet within renormalization group calculation [22] and a z-coordinated spin-$\frac{1}{2}$ Ising ferromagnet within molecular field approximation [23] suffer modifications perfectly consistent to our result. In particular, all these theoretical approaches provide a critical temperature which increases with q.

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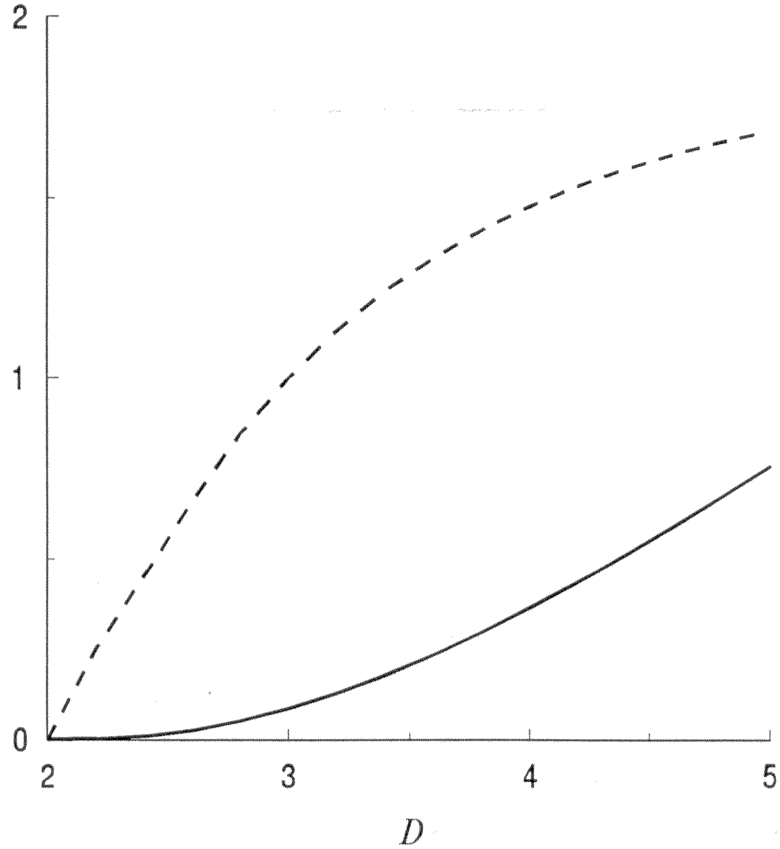

Fig.1: Profile of the Eq.(19) (solid line) and the quantity $N_1^{2/3-2/D}T_{c1}(D)/T_{c1}(3)$ (dashed line) as a function of D.

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