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Non-relativistic q-gases with low critical temperatures

Ligia M.C.S. Rodrigues[*,a] *and Stenio Wulck*[**,b]

[*]Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brazil

[**]Instituto de Física,
Universidade Federal do Rio de Janeiro,
Cidade Universitária - Ilha do Fundão
21945-970 – Rio de Janeiro, RJ – Brazil

Abstract

We propose a deformed q-oscillator system that allows the study of its thermodynamic properties when a deformation parameter $q < 1$. Our analysis shows that deformation is somehow connected to λ-point transitions and that the different values of the deformation parameters can either favor criticality or render it more difficult to attain. We show that in the non-relativistic case the critical temperature of condensation of the system presented can be much smaller than in the non-deformed ideal bosonic gas.

[a]ligia@cbpf.br

[b]steniow@if.ufrj.br

1 Introduction

Quantum groups [1-3] are a mathematical structure, also called Quasitriangular Hopf algebras, that have attracted a great interest from physicists and mathematicians in the last two decades; they are connected to q-oscillators [4,5], which are objects that satisfy deformed Heisenberg algebras. The dynamical properties of q-oscillators and their relation to anharmonic oscillators [6] have been studied by means of a Lie-algebraic approach [7].

The application of deformed algebras to physics has attracted much interest and they have seen to be useful in several different areas and problems [8-24]. A particular feature of q-deformed systems, which began to be explored more recently [25-27], is that they present nonextensive properties and are consequently connected to nonextensive statistical mechanics [8].

The thermal properties of ideal qantum q-gases, which are systems described by deformed Hamiltonians made of bosonic q-oscillators, have been studied in the case of $q > 1$ [29-35]. As it is quite difficult to obtain exact expressions when studying the statistical properties of such Hamiltonians, most of the papers have considered approximations around the deformation parameter q. Those deformed systems have been analysed both in the fundamental [31,34,35] and in inequivalent representations [32,33] of a q-oscillator algebra and they have been shown to exhibit Bose-Einstein condensation phenomenon in all cases. They were applied to describe phonons in ^4He and results compatible with the experimentally proved stability of the phonon spectrum were obtained [18,36,37].

In this paper, our purpose is to analyse the thermodynamic properties of a deformed q-oscillator system when $q < 1$. As will be seen, for those values of the deformation parameter the Hamiltonian studied in the papers quoted above [29-36] leads to models lacking of interest, and a somewhat different Hamiltonian is proposed. The interesting result is that the values of the critical temperature of condensation of this system can be much smaller than in the non-deformed ideal bosonic gas. This is a new result as in the quantum q-gas models so far analysed $T_c^q > T_c$ [31-35]. We note that the measured temperature of λ-point in ^4He is also smaller than T_c.

This paper is organised as follows: in Section 2 we make a brief review of q-oscillators; in section 3 propose a q-deformed Hamiltonian that is more adequate for $q < 1$; in Section 4 we study the Bose-Einstein condensation and the behaviour of the specific heat for our deformed system; finally, in section 5 we discuss our results and present some conclusions.

2 Brief review on q-oscillators

Let us consider the q-oscillator algebra generated by the elements A, A^+ and N described by the relations [4,5]

$$\begin{aligned}&[A, A^+]_A = AA^+ - q^2 A^+ A = 1, \qquad [N, A^+] = A^+, \\ \text{and} \qquad &[N, A] = -A \ ,\end{aligned} \tag{2.1}$$

where q is a real parameter, $-\infty < q < \infty$, and $N^+ = N$. Also, with a pair of independent q-oscillators it is possible to realize the $su_q(2)$ algebra analogously to the Schwinger construction of $su(2)$.

A representations of the relations (2.1) in the Fock space F spanned by the normalised eigenstates $|n>$ of the number operator N is given by

$$\begin{aligned}&A|0>= 0 \quad , \quad N|n>= n|n> \quad , \\ &|n>= \frac{1}{([n]!)^{1/2}}(A+)^n|0> \quad , \quad n = 0, 1, 2, \cdots\end{aligned} \tag{2.2}$$

where $[n]! \equiv [n][n-1]...[1]$, with $[n] = \frac{q^{2n}-1}{q^2-1}$, known as Gauss number. Note that $[n]\underset{q\to 1}{\longrightarrow} n$. In the Fock space F, it is possible to express the deformed oscillators in terms of the standard bosonic ones b, b^+ as [38]

$$A = \frac{[N+1]^{1/2}}{(N+1)^{1/2}}b \qquad , \qquad A^+ = b^+\frac{[N+1]^{1/2}}{(N+1)^{1/2}} \ , \tag{2.3}$$

where $b^+b = N$; it can easily been shown in F that

$$AA^+ = [N+1] \qquad , \qquad A^+A = [N] \ , \tag{2.4}$$

and, as expected, the standard bosonic algebra is obtained in the $q \to 1$ limit.

It is very well known that Heisenberg algebra describes the algebraic structure of the harmonic oscillator. It is possible to define the Hamiltonian

$$H = \hbar\omega A^+A = \hbar\omega[N] \tag{2.5}$$

that recovers the harmonic oscillator Hamiltonian for $q \to 1$ and whose algebraic structure is the q-oscillator algebra (2.1): the creation and annihilation operators A^+, A generate the spectrum of the system described by (2.3), as $A^+|n>\to|n+1>$ and $A|n+1>\to|n>$.

It is interesting to note that the energy eigenvalues ϵ_n of Hamiltonian (2.3) follow a Fibonacci-type relation [39]. By definition,

$$H|n>=\epsilon_n|n>, \tag{2.6}$$

where $\epsilon_n = \hbar\omega[n]$ and from (2.2), (2.4) and the definition of the Gauss number, it is simple to show that ϵ_n satisfies

$$\epsilon_{n+1} = (1+q^2)\epsilon_n - q^2\epsilon_{n-1}. \tag{2.7}$$

3 A model for non-relativistic ideal q-gases with q<1

The Hamiltonian of an ideal deformed system has in general [30-36] been defined as

$$H_0 = \sum_i A_i A_i^+ = \sum \omega_i [N_i]_A \ , \tag{3.1}$$

where A_i, A_i^+ and N_i are interpreted respectively as independent annihilation, creation and occupation number operators of particles in level i, with energy ω_i. These operators satisfy algebra (2.1) and commute for different levels.

The grand-canonical partition function is given by

$$Z = Tr\exp\left[-\beta(H-\mu N)\right] = e^{-\beta\Omega}, \tag{3.2}$$

where $\beta = (k_B T)^{-1}$, k_B the Boltzman constant; N is the total number operator

$$N = \sum_i N_i \ , \tag{3.3}$$

μ is the chemical potential, and Ω is the grand canonical potential. For the above system, Z factorises and the grand canonical potential is given by a sum over single-level partition functions

$$\Omega = -\frac{1}{\beta}\sum_i \log Z_i^0(\omega_i, \beta, \mu) \ , \tag{3.4}$$

where

$$Z_i^0(\omega_i, \beta, \mu) = \sum_{n=0}^{\infty} e^{-\beta(\omega_i[n]-\mu n)} \ . \tag{3.5}$$

As we will be interested in the non-relativistic q-boson, the dispersion law is

$$\omega_i = \frac{p^2}{2m} \tag{3.6}$$

and the general expression for the grand canonical potential (4) is

$$\Omega = -\frac{V}{h^3\beta}\int d^3p \ln \sum_{n=0} e^{-\beta(\frac{p[n]}{2m}-\mu n)} \ . \tag{3.7}$$

Integrating over the angular variables, defining the new variable $x = \beta(p/2m)$, and integrating by parts, Ω can be rewritten as

$$\Omega = -\frac{2\pi^{3/2}V(2m)^{3/2}}{2\Gamma(3/2)h^3\beta^{5/2}}\int dx x^{3/2}\ \frac{\sum_{n=0}^{\infty}[n]_A z^n e^{-[n]_A x}}{\sum_{n=0}^{\infty} z^n e^{-[n]_A x}}, \tag{3.8}$$

where the thermal wavelength Λ,

$$\Lambda^{-3} = \frac{(2m\pi)^{3/2}}{(\beta h^2)^{3/2}}\ , \tag{3.9}$$

is the relevant expansion parameter in the thermodynamic functions.

For the q-oscillator in 3-spatial dimensions and energy spectrum given by (12), the pressure $P = -\Omega/V$ and the density $n = \partial P/\partial\mu|_{T,V}$ are then:

$$P(T,z) \ = \ kT\Lambda^{-3}Y_q(z) \tag{3.10}$$

$$n(T,z) \ = \ \Lambda^{-3}y_q(z)\ , \tag{3.11}$$

where [35]

$$Y_q(z) = \frac{1}{\Gamma(5/2)}\int dx x^{3/2}\ \frac{\sum_{n=0}^{\infty}[n]_A z^n e^{-[n]_A x}}{\sum_{n=0}^{\infty} z^n e^{-[n]_A x}} \tag{3.12}$$

and

$$y_q(z) = z\partial_z Y_q(z) = \frac{3/2}{\Gamma(5/2)}\int dx x^{1/2}\ \frac{\sum_{n=0}^{\infty} n z^n e^{-[n]_A x}}{\sum_{n=0}^{\infty} z^n e^{-[n]_A x}}. \tag{3.13}$$

Let us consider functions $Y_q(z)$ and $y_q(z)$ given by (3.12) and (3.13). It has been shown [35] that these functions always converge when $q > 1$. We have now analysed their behaviour for $q < 1$ and verified numerically that they do not converge for those values. This is a consequence of the fact that for $q < 1$ $[n]_q$ goes to the asymptotic value $\frac{1}{1-q^2}$ and this asymptotic level is infinitely degenerate. Therefore the exponential in the sums in (3.12) and (3.13) converge to a finite value which contributes to the series an infinite number of terms. Another consequence is that no other state constributes when the mean values of physical quantities are calculated and what we have is effectively a one-system state. In this paper we propose a Hamiltonian that breaks this degeneracy, allowing us to analyse the thermodynamic behaviour of the system.

Let us then consider that our system is described by the deformed Hamiltonian

$$H_1 = \sum_i \omega_i \frac{A_i A_i^+ + \gamma \bar{A}_i \bar{A}_i^+}{1+\gamma} = \sum_i \omega_i \frac{[N_i]_A + \gamma [N_i]_{\bar{A}}}{1+\gamma}, \tag{3.14}$$

where γ is a real constant, $0 < q < 1$ and $\bar{q} \geq 1$.

We have now two sets of operators A, A^+, N_A and $\bar{A}, \bar{A}^+, N_{\bar{A}}$ each satisfying relations (2.1) for two different values of the deformation parameter, q and $\bar{q}$. For each of these sets we construct a Fock space representation of algebra (2.1), according to (2.2). The Hamiltonian operator(3.14), that has to be written in a more complete way as

$$H_1 = \sum_i \omega_i \ \frac{A_i A_i^+ \otimes I_{\bar{A}} + \gamma \bar{A}_i \bar{A}_i^+ \otimes I_A}{1+\gamma} = \sum_i \omega_i \ \frac{[N_i]_A \otimes I_{\bar{A}} + \gamma [N_i]_{\bar{A}} \otimes I_A}{1+\gamma}, \tag{3.15}$$

where $I_A (I_{\bar{A}})$ is the identity operator on the Fock space generated by $|n>_q$ ($|n>_{\bar{q}}$), acts on the Fock space generated by the normalized eigenstates $|n>_q \otimes |n>_{\bar{q}}$ according to

$$H_1 (|n>_q \otimes |n>_{\bar{q}}) = \sum_i \omega_i \frac{[n_i]_A + \gamma [n_i]_{\bar{A}}}{1+\gamma} (|n>_q \otimes |n>_{\bar{q}}) \ . \tag{3.16}$$

For the Hamiltonian above expressions (3.12) and (3.13) for Y_q (z) and $y_q(z)$ will be replaced by

$$Y_(q, \bar{q})(z) = \frac{1}{\Gamma(5/2)} \int dx x^{3/2} \ \frac{\sum_{n=0}^{\infty} E_n z^n e^{-E_n x}}{\sum_{n=0}^{\infty} z^n e^{-E_n x}} \tag{3.17}$$

and

$$y_(q, \bar{q})(z) = z \partial_z Y_q(z) = \frac{3/2}{\Gamma(5/2)} \int dx x^{1/2} \ \frac{\sum_{n=0}^{\infty} n z^n e^{-E_n x}}{\sum_{n=0}^{\infty} z^n e^{-E_n x}}. \tag{3.18}$$

where

$$E_n = \frac{[n]_A + \gamma [n]_{\bar{A}}}{1+\gamma}. \tag{3.19}$$

We have analysed the behaviour of functions $Y_{q,\bar{q}}(z)$ and $y_{q,\bar{q}}(z)$ when $\bar{q} = 1$ and $\bar{q} > 1$, in both cases keeping $q < 1$, for different values of gamma. We found that they always converge, which means that the degeneracy presented by Hamiltonian (3.1) was broken.

4 Specific heat for $q < 1$ non-relativistic q-gases

We are now able to study the Bose-Einstein condensation for the case $q < 1$. Following the usual path [40], when $z \to 1$ (or $T \to T_c$, T_c being the critical temperature) we have to take into account the zero-point energy and single out its contribution in the functions $Y_{q,\bar{q}}(z)$ and $y_{q,\bar{q}}(z)$. If we keep n constant and decrease the temperature, $n\Lambda^3$ and, consequently, z increase, until $z=1$, which happens when $T=T_c^{q,\bar{q}}$, by definition

$$T_c^{q,\bar{q}} = \frac{h^2 n^{2/3}}{\kappa y_{q,\bar{q}}^{2/3}(1)}. \tag{4.1}$$

The deformed critical temperature above is related to the critical temperature T_c for the non-deformed non-relativistic ideal gas of the same density n, according to

$$\frac{T_c^{q,\bar{q}}}{T_c} = \left[\frac{2.61}{y_{q,\bar{q}}(1)}\right]^{2/3} \tag{4.2}$$

where

$$T_c = \frac{h^2 n^{2/3}}{2.612\kappa}. \tag{4.3}$$

As in the vicinity of $T_c^{q,\bar{q}}$ we have to take into account the zero-point energy and single out its contribution in (3.16)and (3.17) [40], the expressions for P and n become

$$P(T,z) = \beta^{-1}\Lambda^{-3}Y_{q,\bar{q}}(z) \quad , \quad n(T,z) = \frac{z}{V(1-z)} + \Lambda^{-3}y_{q,\bar{q}}(z), \tag{4.4}$$

where the first term on the right side of $n(T,z)$ is relevant only for $T \leq T_c^{q,\bar{q}}$, due to the contribution of the zero energy.

By definition, C_V, the specific heat per particle, is

$$\frac{C_V}{k_B} = \frac{1}{kn}\frac{\partial e}{\partial T}|_n; \tag{4.5}$$

e is the internal energy per volume. We obtain C_V in the two regimes:

$$\begin{array}{ll} T > T_c^{q,\bar{q}} & \dfrac{C_V}{k_B} = \dfrac{15Y_{q,\bar{q}}(z)}{4\Lambda^3 n} - \dfrac{9y_{q,\bar{q}}(z)}{4y'_{q,\bar{q}}(z)} \\ T < T_c^{q,\bar{q}} & \dfrac{C_V}{k_B} = \dfrac{15Y_{q,\bar{q}}(1)}{4\Lambda^3 n}, \end{array}$$

with

$$y'_{q,\bar{q}}(z) = \frac{\partial y_{q,\bar{q}}(z)}{\partial z}. \tag{4.6}$$

The existence of a λ-point kind of transition depends on the value of Δ, by definition,

$$\frac{9y_(q,\bar{q})(z)}{4y'_(q,\bar{q}(z)} = \Delta. \tag{4.7}$$

From (4.2) and (4.6), we see that the thermodynamic behaviour of our system depends on the functions $Y_{q,\bar{q}}(z)$ and $y_{q,\bar{q}}(z)$, which are given by expressions (3.16)and (3.17), and on

$$y'_{q,\bar{q}}(z) \;=\; \frac{3/2}{\Gamma(5/2)}\int dx\; x^{1/2}\left[\frac{\sum_{n=0}^{\infty} z^n n^2 e_n^{-xE_n}}{\sum_{n=0}^{\infty} z^n e_n^{-xE_n}} - \frac{\sum_{n=0}^{\infty} z^n n e_n^{-xE_n}}{[\sum_{n=0}^{\infty} z^n e_n^{-xE_n}]^2}\right], \tag{4.8}$$

which are all convergent, well behaved functions. Our interest is to study the behaviour of the specific heat and the existence of λ-point transitions. We will then analyse equations (4.2) and (4.6) for $q < 1$ in two cases: (a) $\bar{q} = 1$ and (b) $\bar{q} > 1$.

(a) When $q < 1$ and $\bar{q} = 1$, $Y_{q,\bar{q}}(z)$ can be trivially rewritten as

$$Y_{q,\bar{q}}(z) = \frac{1}{\Gamma(5/2)} \int dx x^{3/2} \frac{\sum_{n=0}^{n_0} E_n z^n e^{-E_n x} + \sum_{n_0+1}^{\infty} E_n z^n e^{-E_n x}}{\sum_{n=0}^{n_0} z^n \ e^{-E_n x} + \sum_{n_0+1}^{\infty} z^n e^{-E_n x}} = \frac{1}{\Gamma(5/2)} \int dx x^{3/2} \frac{I_1 + I_2}{I_3 + I_4}, \tag{4.9}$$

where now

$$E_n = \frac{[n]_A + \gamma n}{1 + \gamma}. \tag{4.10}$$

If $n_0{+}1$ is the lowest value of n for which E_n above reaches the asymptotic value $\frac{1}{(1-q^2)(1+\gamma)}$, then the four terms of the integrand in (13) become respectively

$$I_1 = \sum_{n=0}^{n_0} \frac{[n]_A + \gamma n}{1+\gamma} z^n e^{-\frac{[n]_A + \gamma n}{1+\gamma} x},$$
$$I_2 = \sum_{n_0+1}^{\infty} \frac{\frac{1}{1-q^2} + \gamma n}{1+\gamma} z^n e^{-\frac{\frac{1}{1-q^2}+\gamma n}{1+\gamma} x},$$
$$I_3 = \sum_{n=0}^{n_0} z^n e^{-\frac{[n]_A + \gamma n}{1+\gamma} x},$$
$$I_4 = \sum_{n_0+1}^{\infty} z^n e^{-\frac{\frac{1}{1-q^2}+\gamma n}{1+\gamma}} x. \tag{4.11}$$

The terms I_2 and I_4 can be easily summed. For the sake of simplicity, let us define

$$\begin{aligned} A &= \frac{1}{(1-q^2)(1+\gamma)}; \\ \bar{\gamma} &= \frac{\gamma}{1+\gamma}; \\ B &= A + \bar{\gamma}(n_0 + 1). \end{aligned} \tag{4.12}$$

After some straightforward calculations, recalling that

$$\sum_{n=0}^{\infty} z^n e^{-anx} = \frac{1}{1 - z e^{-ax}} \ , \tag{4.13}$$

we rewrite I_2 and I_4 as

$$I_2 = \frac{z^{n_0+1} e^{-Bx}}{(1 - z e^{-\bar{\gamma}x})^2} \left[B(1 - z e^{-\bar{\gamma}x}) + \bar{\gamma} z e^{-\bar{\gamma}x} \right],$$
$$I_4 = \frac{e^{-Bx} z^{n_0+1}}{1 - z e^{-\bar{\gamma}x}}. \tag{4.14}$$

Therefore, our $Y(q,\bar{q})(z)$ becomes

$$Y(q,\bar{q})(z) = \frac{1}{\Gamma(5/2)} \int dx x^{3/2} \left[\frac{(\sum_{n=0}^{n_0} \frac{[n]_A+\gamma n}{1+\gamma} z^n e^{-\frac{[n]_A+\gamma n}{1+\gamma}x})(1-ze^{-\bar{\gamma}x})^2}{(\sum_{n=0}^{n_0} z^n e^{-\frac{[n]_A+\gamma n}{1+\gamma}x})(1-ze^{-\bar{\gamma}x})^2 + z^{n_0+1}e^{-Bx}(1-ze^{-\bar{\gamma}x})} + \frac{e^{-Bx}z^{n_0+1}(B(1-ze^{-\bar{\gamma}x})+\bar{\gamma}ze^{-\bar{\gamma}x})}{(\sum_{n=0}^{n_0} z^n e^{-\frac{[n]_A+\gamma n}{1+\gamma}x})(1-ze^{-\bar{\gamma}x})^2 + z^{n_0+1}e^{-Bx}(1-ze^{-\bar{\gamma}x})} \right] \tag{4.15}$$

Following a similar procedure, we obtain for $y(q,\bar{q})$ the expression

$$y(q,\bar{q})(z) = \frac{1}{\Gamma(5/2)} \int dx x^{3/2} \left[\frac{(\sum_{n=0}^{n_0} n z^n e^{-\frac{[n]_A+\gamma n}{1+\gamma}x})(1-ze^{-\bar{\gamma}x})^2}{(\sum_{n=0}^{n_0} z^n e^{-\frac{[n]_A+\gamma n}{1+\gamma}x})(1-ze^{-\bar{\gamma}x})^2 + z^{n_0+1}e^{-Bx}(1-ze^{-\bar{\gamma}x})} + \frac{e^{-Bx}z^{n_0}((n_0+1)(1-ze^{-\bar{\gamma}x})+ze^{-\bar{\gamma}x})}{(\sum_{n=0}^{n_0} z^n e^{-\frac{[n]_A+\gamma n}{1+\gamma}x})(1-ze^{-\bar{\gamma}x})^2 + z^{n_0+1}e^{-Bx}(1-ze^{-\bar{\gamma}x})} \right]. \tag{4.16}$$

From (4.15) and (4.16) we can compute the specific heat per particle (4.6) for different values of γ and q. In figure I, we show some results for $\bar{q}=1$ and four different values of q. (b) When $q<1$ and $\bar{q}>1$, the specific heat (4.6) can be numerically calculated from (3.16), (3.17) and (4.8). We note that for $q<1$, $y(q,\bar{q})(1)$ is larger than the non-deformed value $y(1)=2.61$ and increases as q decreases. On the contrary, for $q>1$, $y_q(1)$ was smaller than 2.61 and the deformed critical temperatures were larger than the non-deformed value T_c, obtained from the non-relativistic ideal bosonic gas [35].

The results for $q<1$ and $\bar{q}>1$ are presented in figure II .

5 Conclusions

The deformed Hamiltonian here presented enabled us to complete the study of the non-relativistic deformed bosonic gas for deformation parameters ranging from 0 to ∞. Our analysis showed that deformation is somehow connected to the existence of λ-type transitions. Besides, different critical temperatures are obtained and criticality can either be favored or rendered more difficult to attain, depending on the values of deformation parameters taken.

When $\bar{q}=1$, we have analysed the results of numerical calculations of the specific heat (4.6), for four values of q smaller than 1 and two values of γ. As can be seen from figures

I, the Δ term is zero and the specific heat passes by the critical value of the temperature continuously. No λ-point type of discontinuity appears in those cases.

We have also studied the behaviour of C_V for four sets of values of q smaller than 1 and $\bar{q}$ larger than 1 and two values of γ. In these cases, $\Delta \neq 0$ and the specific heat does present a λ-point transition, as shown in figures II.

We note that in all cases, there is, for $1 < \bar{q} < \infty$, the deformed critical temperatures $T^(q, \bar{q})_c$ are smaller than the usual T_c.

Therefore, we found that the presence of the Bose-Einstein condensation phenomenon in the non-relativistic q-gas only happens when one of the q-oscillators terms in Hamiltonian (3.15) has a deformation parameter larger than one. Recalling that all the q-gases so far studied have a critical temperature larger than the usual one, which is obtained from the non-deformed bosonic gas, we can also conclude that the presence of a deformation smaller than 1 is a necessary condition to have lower critical temperatures.

Analysing the results shown in Table I, we see that the critical temperature always decreases when any of the two parameters q or γ decrease and $\bar{q}$ approaches 1.0, indicating that we can get very low critical temperatures by choosing sufficiently small values of q and γ. This means that the critical temperature goes down as the weight of the $q < 1$ term in the Hamiltonian that describes our deformed system gets larger.

In the model here presented, the possibility of decreasing the critical temperatures indicates that quantum symmetries might as well play some role in the study of ^3He, in which the phenomenon of superfluidity is associated to a very low critical temperature of the order of $10^{-3}K$.

6 Acknowledgements

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Table I

(a) Comparison of the values of $T_c^{q,\bar{q}}$ for the same values of $\bar{q}$ and q and two values of γ

$$\bar{q} = 1.01,\ q = 0.3,\ \gamma = 1.0,\ T_c^{q,\bar{q}} = 0.791 T_c$$

$$\bar{q} = 1.01,\ q = 0.3,\ \gamma = 0.1,\ T_c^{q,\bar{q}} = 0.280 T_c$$

(b) Comparison of the values of $T_c^{q,\bar{q}}$ for the same values of q and γ and two values of $\bar{q}$

$$\bar{q} = 1.1,\ q = 0.9,\ \gamma = 0.1,\ T_c^{q,\bar{q}} = 0.987 T_c$$

$$\bar{q} = 1.01,\ q = 0.9,\ \gamma = 0.1,\ T_c^{q,\bar{q}} = 0.532 T_c$$

(c) Comparison of the values of $T_c^{q,\bar{q}}$ for the same values of $\bar{q}$ and γ and two values of q

$$\bar{q} = 1.01,\ q = 0.9,\ \gamma = 0.1,\ T_c^{q,\bar{q}} = 0.532 T_c$$

$$\bar{q} = 1.01,\ q = 0.3,\ \gamma = 0.1,\ T_c^{q,\bar{q}} = 0.280 T_c$$

Figure I

Specific heat per particle C_v/k_B (4.6) for the system described by the Hamiltonian (3.15), $\theta = T_c^q/T_c$, with $\bar{q} = 1$ and:

(a) $q = 0.1$, $\gamma = 0.1$ and $T_c^{\bar{q}} = 0.178 T_c$;

(b) $q = 0.3$, $\gamma = 0.1$ and $T_c^{\bar{q}} = 0.181 T_c$;

(c) $q = 0.9$, $\gamma = 0.1$ and $T_c^{\bar{q}} = 0.270 T_c$;

(d) $q = 0.1$, $\gamma = 1.0$ and $T_c^{\bar{q}} = 0.630 T_c$.

Figure II

Specific heat per particle C_v/k_B (4.6) for the system described by the Hamiltonian (3.15), $\theta = T_c^{q,\bar{q}}/T_c$, with:

(a) $q = 0.3$, $\bar{q} = 1.01$, $\gamma = 0.1$ and $T_c^{q,\bar{q}} = 0.280T_c$;

(b) $q = 0.9$, $\bar{q} = 1.01$, $\gamma = 0.1$ and $T_c^{q,\bar{q}} = 0.532T_c$;

(c) $q = 0.3$, $\bar{q} = 1.01$, $\gamma = 1.0$ and $T_c^{q,\bar{q}} = 0.791T_c$;

(d) $q = 0.9$, $\bar{q} = 1.1$, $\gamma = 0.1$ and $T_c^{q,\bar{q}} = 0.987T_c$;

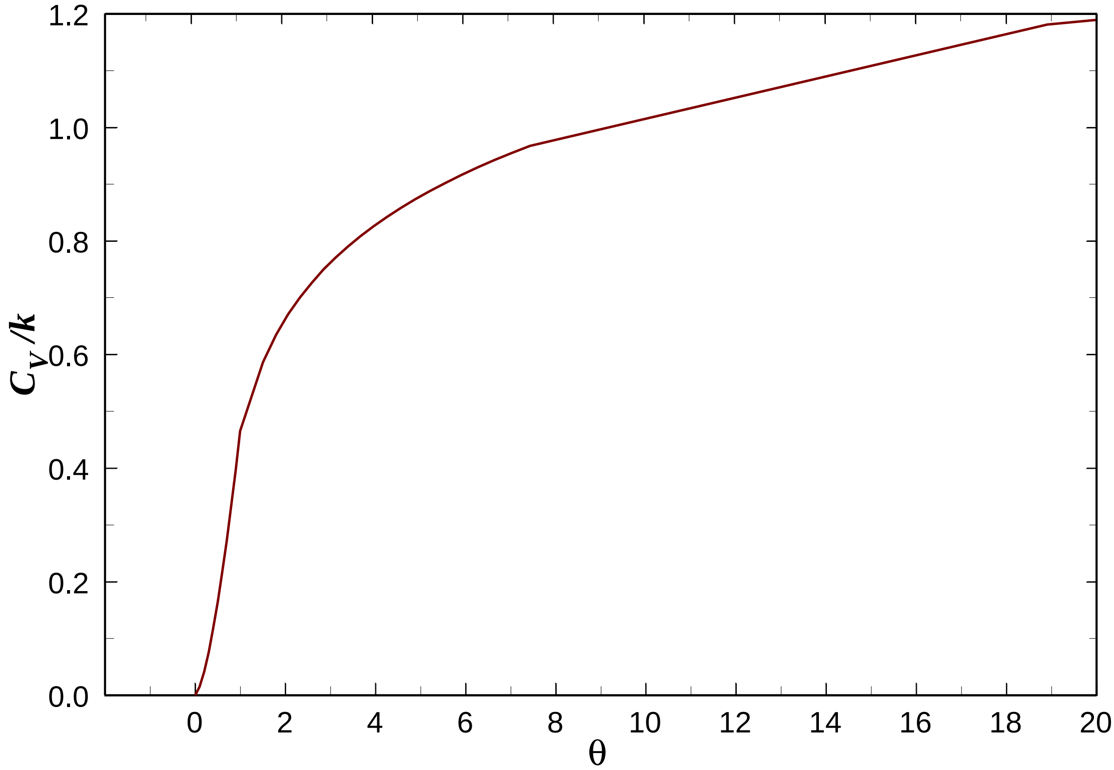

Fig. 1a

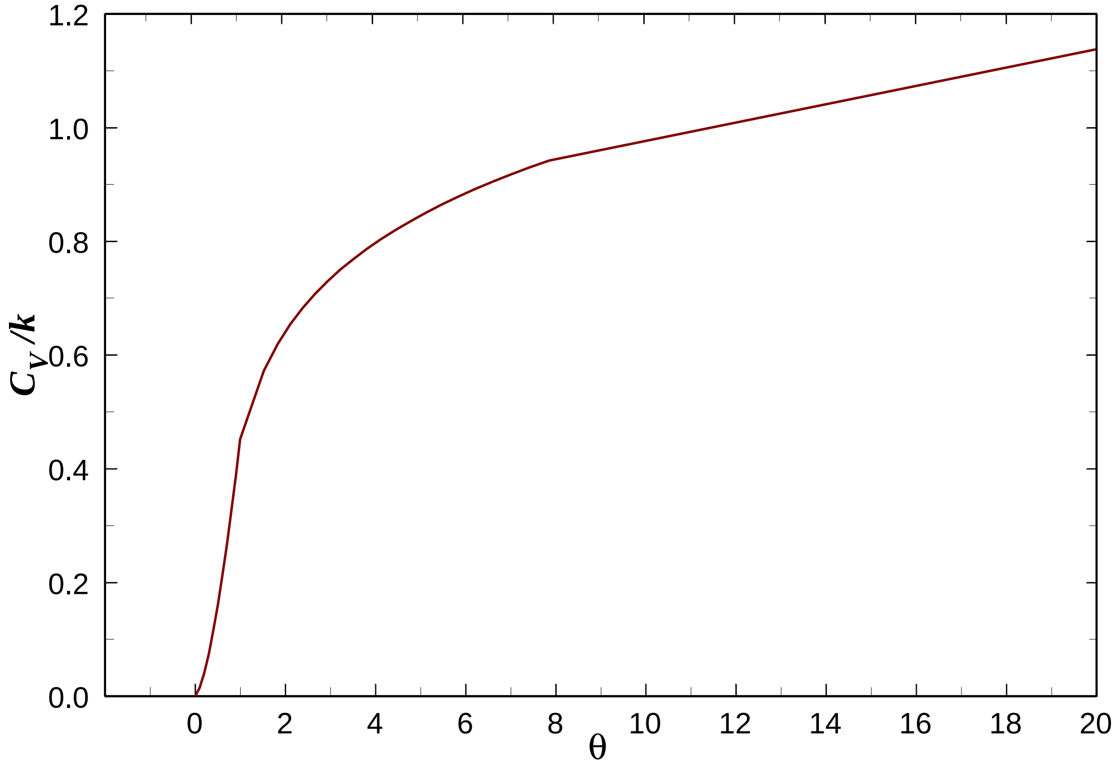

Fig. 1b

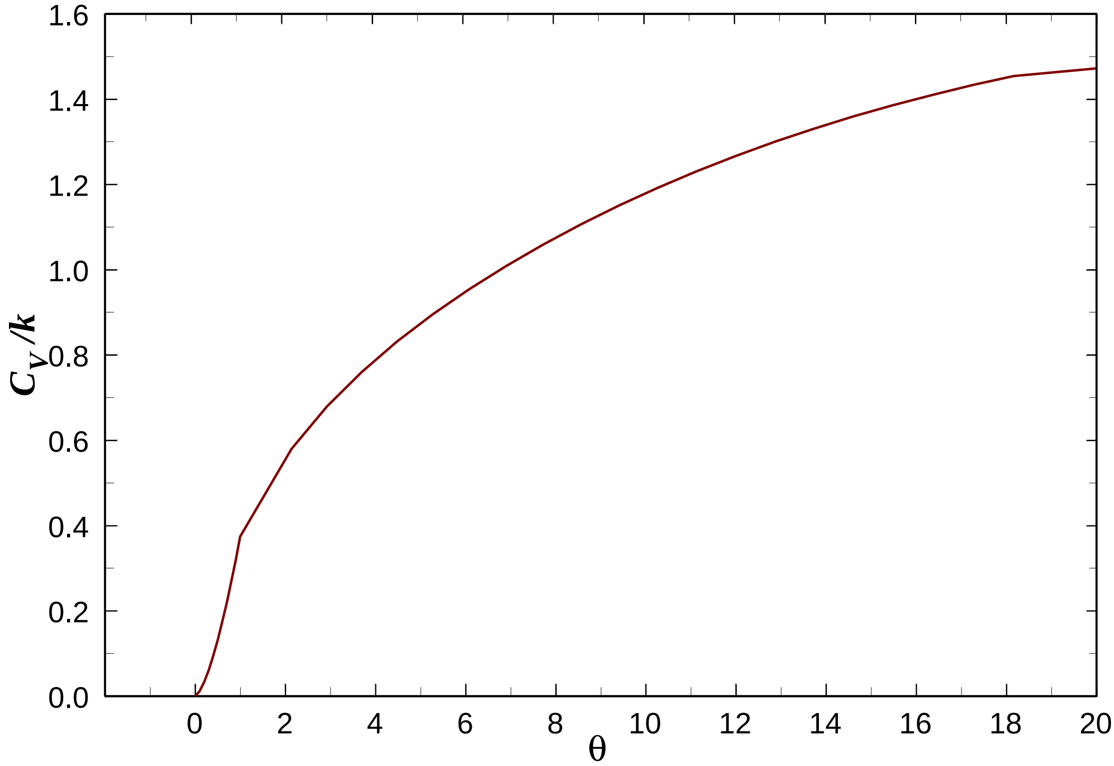

Fig. 1c

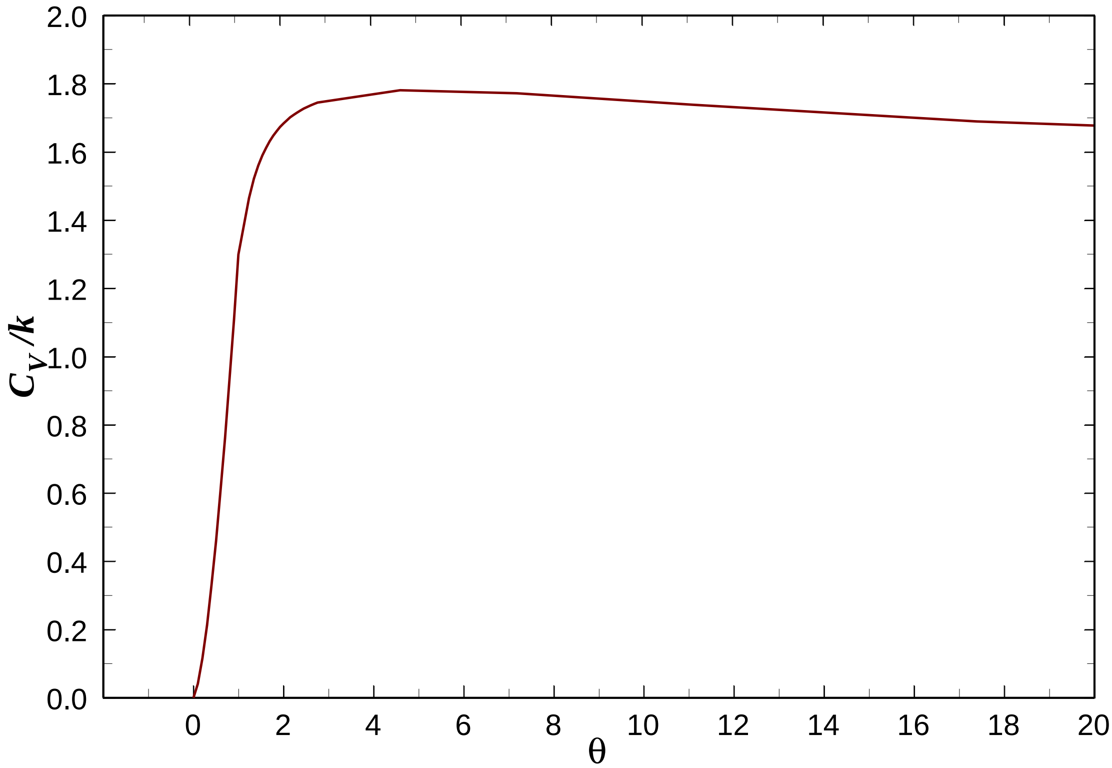

Fig. 1d

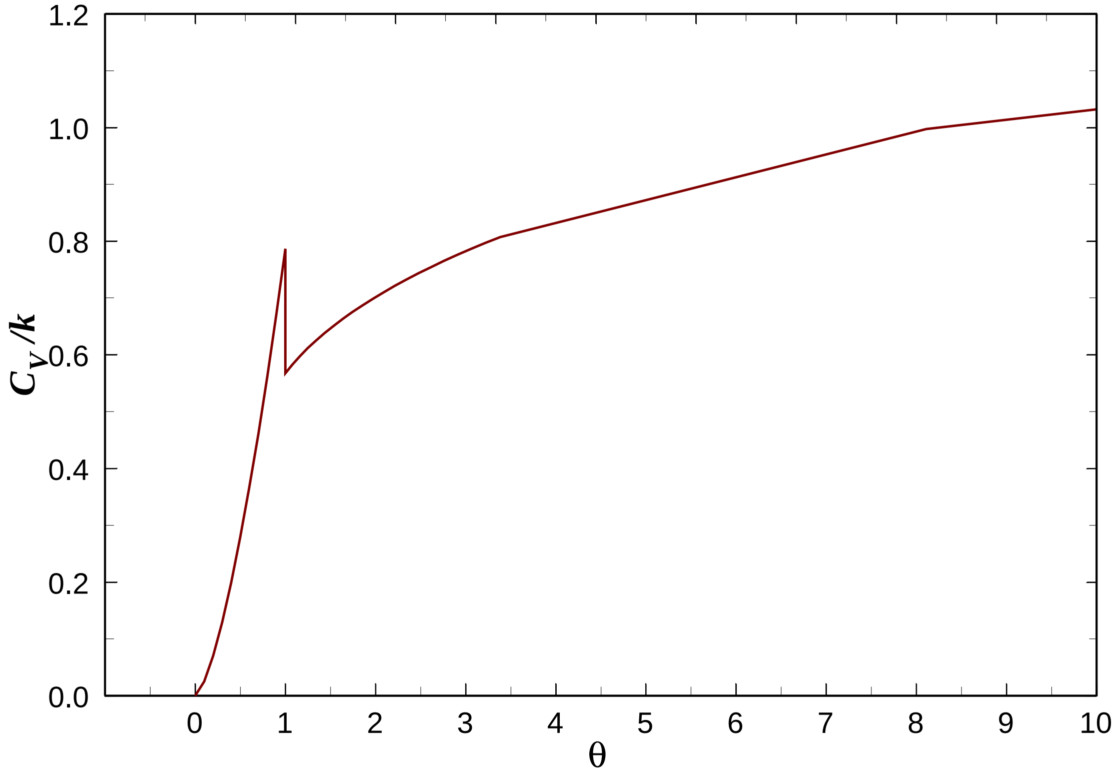

Fig. 2a

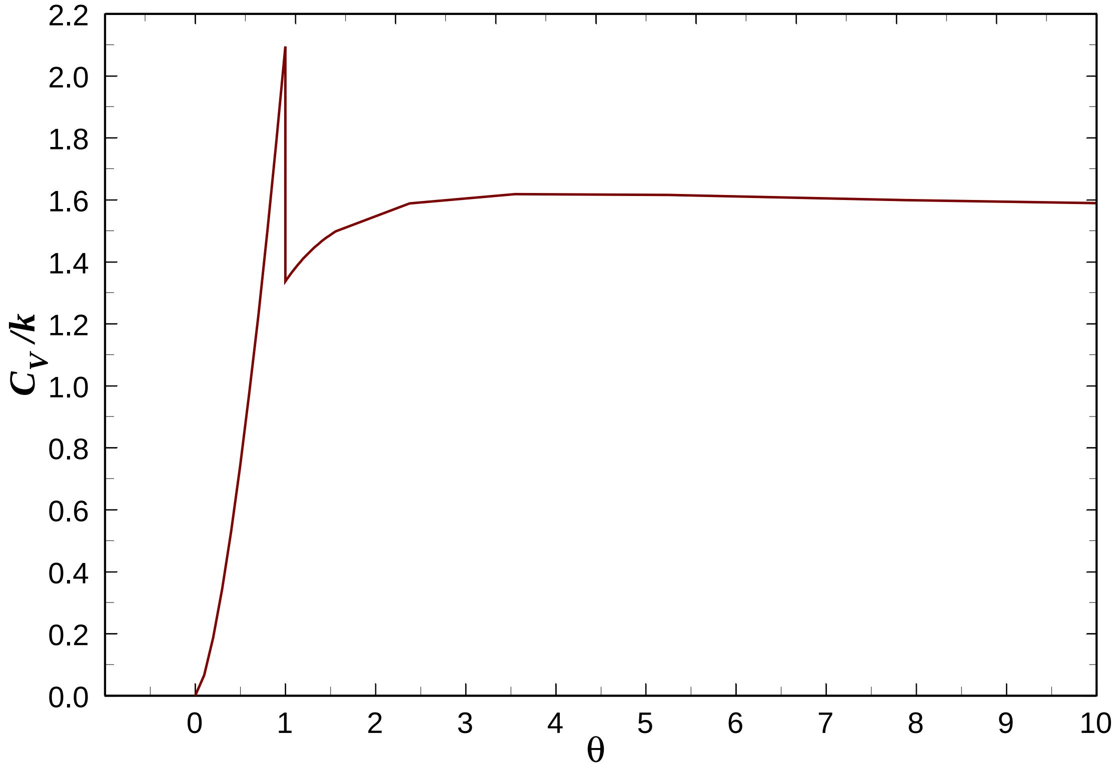

Fig. 2b

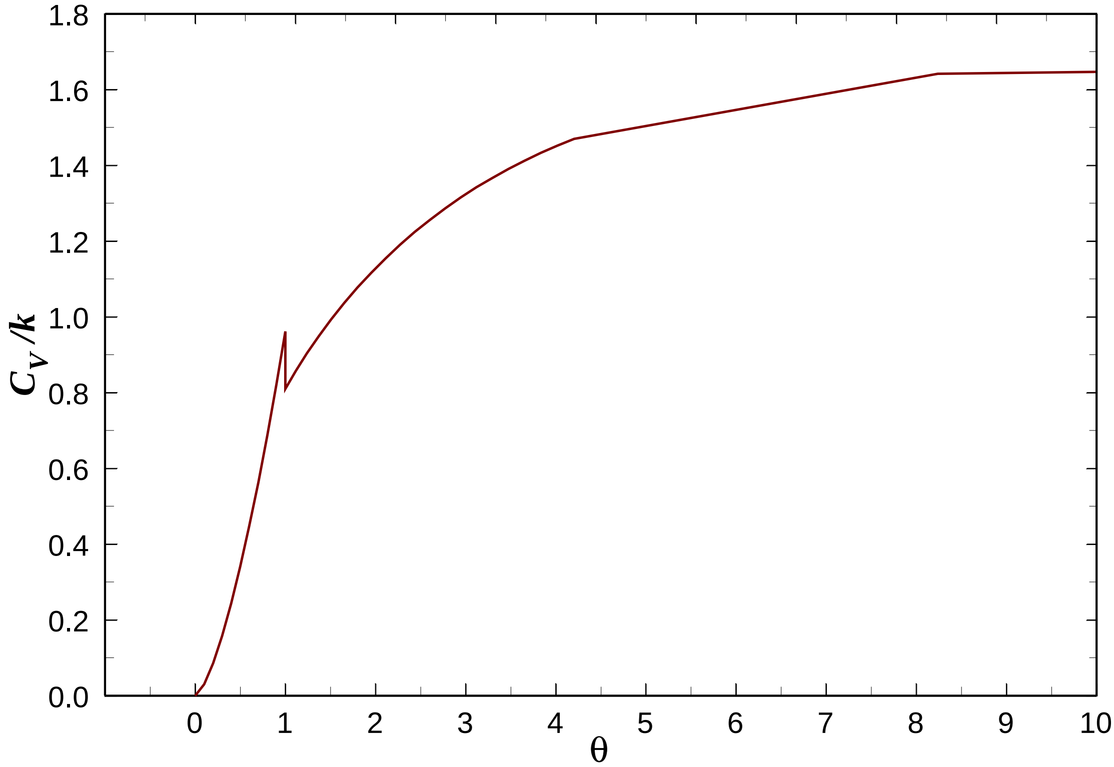

Fig. 2c

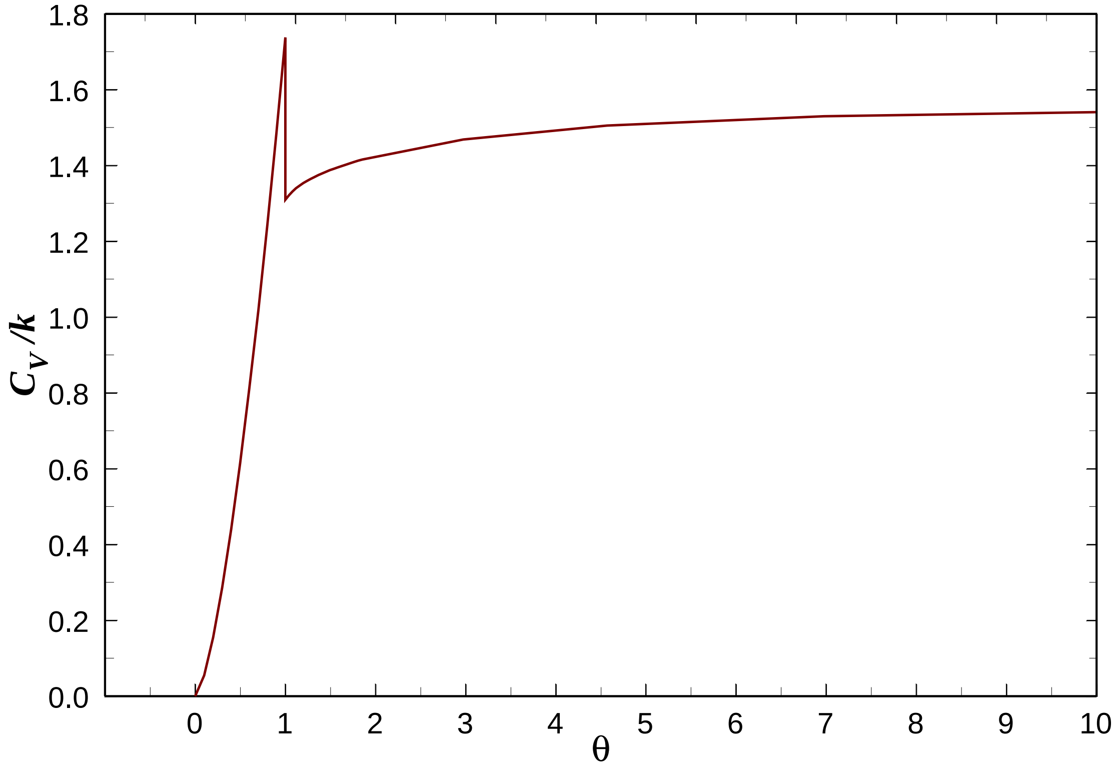

Fig. 2d

REFERENCES

[1] V.G. Drinfeld, Sov. Math. Dokl. **32** (1985) 254; M.Jimbo, Lett.Math.Phys. 10 (1985) 63

[2] L.D.Faddeev, N.Yu.Reshetikin and L.A. Takhtadzhyan, Algebra and Analysis **1** (1987) 178

[3] For reviews, see for instance: S.Majid, Int.J.Mod.Phys. **A45** (1990) 1; P.Aschieri and L.Castellani, Int.J.Mod.Phys. **A8** (1993) 1667

[4] A.J.Macfarlane, J.Phys. **A22** (1989) 4581; L.C. Biedeharn, J.Phys. **A22** (1989) L873

[5] M.Chaichian and P.Kulish, Phys.Lett. **B234** (1990) 72

[6] J.L. Birman, Phys.Lett. A167 (1992) 363; M.Artoni, J.Zang, and J.L. Birman, Phys.Rev. **A47** (1993) 2555

[7] J.L. Gruver, Phys.Lett. **A254** (1996) 1

[8] C.Zachos, Contemporary Mathematics **134** (1992) 351 and references therein

[9] D.Bonatsos, C.Daskaloyannis and P.Kolokotronis, J.Chem.Phys. **106** (1997) 605; D.Bonatsos and C.Daskaloyannis, Phys. Rev. **A48** (1993) 3611

[10] D.Bonatsos and C.Daskaloyannis, Mod.Phys.Lett. **B21** (1996) 1011

[11] M.Chaichian, D.Ellinas and P.Kulish, Phys.Rev.Lett. **65** (1990) 980

[12] S.Mancini V.I.Mank'o and P.Tombesi, Mod.Phys.Lett. **B12** (1998) 403 13)

[13] S.Mancini, Phys.Lett. A233 (1997) 291

[14] E.Celeghini, M.Rasetti and G.Vitiello, Phys.Rev.Lett. 66 (1991) 2056

[15] J.L.Matheus-Valle and M.R.Monteiro, Mod.Phys.Lett. **A7** (1992) 3032; Phys.Lett. **B66** (1993) 330

[16] L.Castellani and M.R.Monteiro, Phys.Lett. **B314** (1993) 24

[17] A.Lerda and S.Sciuto, Nucl.Phys. **B401** (1993) 613; R.Caracciolo and M.R.Monteiro, Phys.Lett. **B308** (1993) 58; M.Frau, M.R.Monteiro and S.Sciuto, J.Phys. **A27** (1994) 801

[18] M.R.Monteiro, L.M.C.S. Rodrigues and S.Wulck, Phys.Rev.Lett. **76** (1996) 1098; Physica **A259** (1998) 245

[19] B.E. Palladino and P. Leal Ferreira, Il Nuovo Cimento **110 A**, (1997) 303; Braz. J. Phys. **28**, (1998) 444

[20] S.S. Avancini, A.Eiras, D. Galetti, C.L. Lima and B.M. Pimentel, J.Phys. A: Math.Gen. **28** (1995) 4915; D.Galetti, B.M. Pimentel, C.L. Lima and J.T.Lunardi, Physica **A242** (1997) 501

[21] S.S. Avancini and J.C. Brunelli, Phys. Lett. **A174** (1993) 358; S.S. Avancini, F.F. de Souza Cruz, D. Menezes, M.M. Watanabe de Moraes, J.Phys. A: Math Gen. **27** (1994) 831

[22] P. Kulish and E. Sklyanin, Lect. Notes in Phys. **151** (1982) 61

[23] V.Man'ko, G.Marno, S.Solimeno and F.Zaccaria, Int.J.Mod.Phys. **A8** (1993) 3577; Phys.Lett. **A176** (1993) 173

[24] M.R. Kibler, in Symmetry and Structural Properties of Condensed Matter, eds. Florek, D. Lipinski and T. Lulek (World Scientific, Singapore, 1993); R. Barbier, J. Meyer and M. Kibler, Int.J.Mod.Phys. **E2** (1995) 385

[25] C.Tsallis, Phys. Lett **A195** (1994) 329

[26] S.Abe, Phys.Lett. **A244** (1998) 229

[27] M.Arik, J.Kornfilt and A.Yildiz, Phys.Lett. **A235** (1997) 318

[28] C. Tsallis, J.Stat.Phys. **52** (1988) 479

[29] M.Martin-Delgado, J.Phys. **A24** (1991) 1285; P.Neskovic and B.Urossevic, Int.J.Mod.Phys. **A7** (1992) 3379

[30] M.Chaichian, R.Gonzalez Felipe and C.Montonen, J.Phys. **A26** (1993) 4020

[31] M.R.Monteiro, I.Roditi and LMCS Rodrigues, Phys.Lett. **A188** (1994) 11

[32] M.R.Monteiro and LMCS Rodrigues, Mod.Phys.Lett. **B9** (1995) 883

[33] M.R.Monteiro and LMCS Rodrigues, Mod.Phys.Lett. **B10** (1996) 41

[34] M.R.Monteiro, I.Roditi and L.M.C.S. Rodrigues, Mod.Phys.Lett. **B7** (1993) 1897

[35] M.R.Monteiro, I.Roditi and L.M.C.S. Rodrigues, Int.J.Mod.Phys. **B8** (1994) 3281

[36] M.R.Monteiro and L.M.C.S. Rodrigues, Mod.Phys.Lett. **B10** (1996) 41

[37] LMCS Rodrigues and S.Wulck, Mod.Phys.Lett. **B11** (1997) 297

[38] P.Kulish and E.Damaskinsky, J.Phys. **A23** (1990) L415; A.Polychronakos, Mod.Phys.Lett. **A5** (1990) 2325

[39] J.V. Uspensky and M.A. Heaslet, Elementary Number Theory, McGraw Hill, New York and London, 1939.

[40] R.Balescu, Equilibrium and non-equilibrium statistical mechanics, Wiley, New York, 1975; R.Balian, Du microscopique au macroscopique, Tome 2, École Polytéchnique, Paris, 1982