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DIMENSIONAL REGULARIZATION AND FINITE TEMPERATURE DIVERGENT DETERMINANTS

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Abstract

Dimensional regularization is used to evaluate divergent functional determinants in some specific examples.

Key-words: Finite-temperature

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Introduction

In the formulation of Quantum field theory by means of functional integrals it often appears the need to compute determinants of differential operators which are divergents for the relevant boundary conditions.

The appearances of divergences even in classical physics is old dated. See for example refs. [1] [2] and, in particular, ref. [3]. In renormalization theory explicit divergences are avoided by some regularization procedure. See for instance ref. [4].

Due to gauge invariance the use of dimensional regularization has been of particular interest.

The purpose of this paper is to use the same method of analytic continuation in the number of dimensions to solve the above mentioned problems of divergent functional determinants in finite temperature field theories.

§1. As an introductory example, let us take the case discussed in ref. [5] the evaluation of the determinant of the laplacian operator in a finite temperature field theory. In this theory, the partition function is given by:

$$Z(\phi) = \int D\phi \, \exp\left[-i \, \frac{1}{2} \int_0^\beta dt \int d^3x \phi A \phi\right] \tag{1}$$

$$z(\phi) = \det A^{-1/2} \qquad A = \sum_{i=1}^{4} \partial_i^2 \tag{2}$$

$$\ell og\ z(\phi) = -\frac{1}{2}\,\mathrm{Tr}\ \ell og\ A \tag{3}$$

The eigenvalues of the laplacian operator take the form.

$$\lambda = k^2 + \left(\frac{2\pi n}{\beta}\right)^2 = k^2 + \omega_n^2 \quad \text{with} \quad \omega_n^2 = \frac{2\pi n}{\beta} \tag{4}$$

$\beta = \frac{1}{T}$ being the period in time.

Then:

$$\ell og\ z = -\frac{V}{2}\sum_{n=-\infty}^{+\infty}\int \frac{d^3k}{(2\pi)^3}\,\ell og(k^2+\omega_n^2) = -\frac{V}{2}\sum_{n=-\infty}^{+\infty} I_n. \tag{5}$$

where

$$I_n = \int \frac{d^\nu k}{(2\pi)^\nu}\,\ell og(k^2+\omega_n^2) = \frac{2\pi^{\frac{\nu}{2}}}{(2\pi)^\nu\,\Gamma(\frac{\nu}{2})}\int_0^\infty dk\ k^{\nu-1}\,\ell og(k^2+\omega_n^2) \tag{6}$$

where the number of space dimensions has been replaced by an analytic variable (ν)

According to ref. [6], p.563

$$\int_0^\infty dk\ k^{\nu-1}\ \ell n(1+k^2) = \frac{\pi}{\nu \sin\frac{\nu\pi}{2}} \tag{7}$$

So, we get:

$$I_n = \frac{2\omega_n^\nu\ \pi^{\frac{\nu}{2}+1}}{\nu\Gamma(\frac{\nu}{2})\sin\frac{\nu\pi}{2}.(2\pi)^\nu} \tag{8}$$

From (5) we obtain:

$$\ell og\ z = -V\sum_{n=1}^{\infty} 2\,\frac{(\omega_n^2)^{\frac{\nu}{2}}\,\pi^{\frac{\nu}{2}+1}}{\nu\Gamma(\frac{\nu}{2})\,sin\frac{\nu\pi}{2}\,(2\pi)^{\nu}}$$

As $\sum_{n=1}^{\infty}(\omega_n^2)^{\frac{\nu}{2}} = (2\pi T)^{\nu}\zeta(-\nu)$

$\zeta(-\nu)$ being the Rieman ζ-function, we have:

$$\ell og\ z = -\frac{V.\pi^{\frac{\nu}{2}+1}}{\Gamma(\frac{\nu}{2}+1)\,sin\frac{\nu\pi}{2}}\,T^{\nu}\zeta(-\nu) \tag{9}$$

(9) can also be written, using

$$\frac{\zeta(-\nu)}{sin\frac{\nu\pi}{2}} = -2\,\frac{\Gamma(1+\nu)}{(2\pi)^{1+\nu}}\,\zeta(1+\nu), \tag{10}$$

as

$$\ell og\ z = \frac{V\ T^{\nu}}{\pi^{\frac{\nu+1}{2}}}\,\Gamma(\frac{\nu+1}{2})\ \zeta(1+\nu) \tag{11}$$

which, for $\nu = 3$ coincides with the result quoted in ref. [5], namely:

$$\ell og\ z = \frac{\pi^2\ V\ T^3}{90}$$

Another interesting example can be found in App. D. ref. [7]. One is led to compute

$$I = \frac{V}{(2\pi)^3}\sum_{n=-\infty}^{+\infty}\int d^3k\ \ell n(k^2+W_n^2) \tag{12}$$

where

$$W_n^2 = \omega_n^2 - \frac{\pi q}{\beta} \qquad q \text{ being some constant number.} \tag{13}$$

(12) can be written according to (6), (7) and (8)

$$I = \frac{V}{(2\pi)^\nu} \frac{\pi^{\frac{\nu}{2}+1}}{\Gamma(\frac{\nu}{2}+1)\, sin\frac{\nu\pi}{2}} \sum_{n=0}^{\infty} (\omega_n - \frac{\pi q}{\beta})^\nu$$

$$I = \frac{V}{\Gamma(\frac{\nu}{2}+1)} \frac{\pi^{\frac{\nu}{2}+1}}{sin\frac{\nu\pi}{2}\beta^\nu} \zeta(-\nu, -\frac{q}{2}) \tag{14}$$

with $\zeta(z,q) = \sum_{n=\nu}^{\infty} \frac{1}{(n+q)^z}$

In particular, for $\nu = 3$, we have $\zeta(-3,q) = -\frac{1}{4} B_4(q)$ with $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$ and (14) can be compared with ref. [7], form D.3.

§2. We now want to discuss the calculation of the determinant related with the consideration of domain walls in $\lambda\phi^4$ theory (Ref. [8]).

For the calculation of the partition function, one is led to the following expression (Ref. [8], form 2.15)

$$I = \ell n \frac{\prod_{n=-\infty}^{n=+\infty} \prod_J (\omega_n^2 + E_s^2(J))}{\prod_{n=-\infty}^{+\infty} \prod_j (\omega_n^2 + E_V^2(j))} \tag{15}$$

where the eigenvalues E^2 are given by (see ref. [8], App A).

$$E_s^2(0) = k_T^2$$

$$E_s^2(1) = k_T^2 + \frac{3}{2} M^2$$

$$E_s^2(k_L) = k_T^2 + k_L^2 + 2M^2 + \frac{2}{L}\,\delta(k_L) \tag{16}$$

$$E_v^2(k_L) = k_T^2 + k_L^2 + 2M^2$$

where k_T, k_L are transversal and longitudinal with respect to the wall. L^3 is the volume of the space and $\delta(k_L)$ is the phase shift produced by the soliton wall

Eq. 15 reduces to

$$I = L^2 I_1 + L^2 I_2 + L^3 \int I_3(k_L)\,dk_L \tag{17}$$

$$I_1 = \sum_u d^2k_T \;\ell n\,(k_T^2+\omega_n^2) \qquad I_2 = \sum_u d^2k_T \;\ell n\,(k_T^2+\tfrac{3}{2}M^2+\omega_n^2)$$

$$I_3 = \sum_u \int d^2k_T \;\ell n\, \frac{k_T^2+k_L^2+\frac{2\delta}{L}k_L+2M^2\omega_n^2}{k_T^2+k_L^2+2M^2+\omega_n^2} \tag{18}$$

Using (5) to (10) we easily obtain:

$$I_1 = -\,2^{\nu+1}\,\pi^{\frac{\nu-1}{2}}\,\Gamma\!\left(\frac{\nu+1}{2}\right) T^{\nu}\,\zeta(\nu+1) \tag{19}$$

To compute I_2 we are led to the series

$$S = \sum_{n=-\infty}^{+\infty} (\omega_n^2+b^2)^{\nu/2} \quad , \quad \text{which we write:} \tag{20}$$

$$S = b^{\nu} + 2\left(\frac{2}{\beta}\right)^{\nu} \sqrt{\pi}\, \frac{b^{\frac{\nu+1}{2}}}{\Gamma(-\frac{\nu}{2})} \int_0^{\infty} \frac{dt\ t^{-\frac{\nu+1}{2}}}{e^{\pi t} - 1}\, J_{-\frac{\nu+1}{2}}\left(\frac{bt}{2}\right) \tag{21}$$

(See ref. [6] pag. 713)

If we want to compute (21) for $\nu = 2$ (form (18)) the behaviour of the integrand for $t \to 0$ makes the integral divergent, so we shall add and substract the first two terms (the dangerous ones for $\nu \Rightarrow 2$) of the series for the Bessel functions i.e., in (21) we replace

$$J_{-\frac{\nu+1}{2}}(x) \Rightarrow \left[J_{-\frac{\nu+1}{2}}(x) - \frac{x^{-\frac{\nu+1}{2}}}{\Gamma(\frac{1-\nu}{2})} + \frac{x^{\frac{3-\nu}{2}}}{\Gamma(\frac{3-\nu}{2})} \right] + \frac{x^{-\frac{\nu+1}{2}}}{\Gamma(\frac{1-\nu}{2})} - \frac{x^{\frac{3-\nu}{2}}}{\Gamma(\frac{3-\nu}{2})} \tag{22}$$

The bracket gives now a convergent integral.

The other two terms can be computed using

$$\int_0^{\infty} \frac{x^{\alpha-1} dx}{e^{\pi x} - 1} = \frac{1}{\pi^{\alpha}}\, \Gamma(\alpha)\, \zeta(\alpha) \tag{23}$$

From (20) to (23), we get for $I_2(\nu \to 2)$ with $b^2 = \frac{3}{2}M^2$

$$I_2 = -\frac{4\pi}{\beta^2}\zeta(3) - \frac{16\pi}{\beta}\sqrt{\frac{3}{2}}\, M \int_0^{\infty} \frac{dt\ t^{-2}}{e^{\pi t} - 1} \left\{ -\sin\sqrt{\frac{3}{2}}\,\frac{M\beta t}{2} - \frac{\cos\sqrt{\frac{3}{2}}\,\frac{M\beta t}{2}}{\sqrt{\frac{3}{2}}\, M\,\frac{\beta t}{2}} + \frac{2}{\sqrt{\frac{3}{2}}\frac{M\beta t}{1}} + \sqrt{\frac{3}{2}}\,\frac{M\beta t}{4} \right\} + \frac{\pi^{\frac{\nu}{2}+1}}{\Gamma(\frac{\nu}{2}+1)}\, \frac{(\frac{3}{2}M^2)^{\frac{\nu}{2}}}{\sin\frac{\nu\pi}{2}} + \left(\frac{2}{\beta}\right)^{\nu} 2^{\nu-2}\, \pi^{-\frac{\nu+3}{2}} \left(\sqrt{\frac{3}{2}}M\beta\right)^{2} \frac{\Gamma(2-\nu)}{\Gamma(\frac{3-\nu}{2})}\, \zeta(2-u) \tag{24}$$

If we make a series development of $\sin x$ and $\cos x$, use form (23) and take into account (See [6] p. 46)

$$\ln \sin x = \ln x + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1}}{k} \frac{B_{2k}}{(2k)!} x^{2k}$$

and $\zeta(2k) = \frac{2^{2k-1}}{(2k)!} \pi^{2k} |B_{2k}|$ (See [6] p. 1074).

We finally get:

$$- \frac{16\pi}{\beta} (\sqrt{\tfrac{3}{2}}M) \int_0^{\infty} \frac{dt\ t^{-2}}{e^{\pi t}-1} \left[- \sin (\frac{\sqrt{\frac{3}{2}}M\beta t}{2}) - \frac{\cos (\sqrt{\frac{3}{2}} \frac{M\beta t}{2})}{\sqrt{\frac{3}{2}} \frac{M\beta t}{2}} + \frac{2}{\sqrt{\frac{3}{2}} M\beta T} + \frac{\sqrt{\frac{3}{2}} M\beta t}{4} \right] = - \frac{16\pi^3}{\beta^2} \int_0^{\frac{1}{2\pi}\sqrt{\frac{3}{2}}\beta M} z \ln \frac{Sh \pi z}{\pi z} dz$$

The last term in (24) has a pole at $\nu = 2$. We take finite part in the usual way i.e, multiply by $(\nu-2)$ and take the derivative with respect to ν. As we know that the theory is renormalizable (see ref. [8]) there is still an arbitrary constant multiplying the residue, as an extra term, but this constant will be taken care of by the renormalization prescriptions.

We then get for I_2:

$$I_2 = - \frac{16\pi^3}{\beta^2} \int_0^{\frac{1}{2\pi}\sqrt{\frac{3}{2}}MB} dz\ z \ln \frac{Sh\pi z}{\pi z} - \frac{4\pi}{\beta^2} \zeta(3) + C \frac{3}{2}M^2 - 3\pi M^2 \ln \beta^2 \frac{3}{2} M^2 \qquad (26)$$

Where the arbitrary constant is fixed by renormalization conditions.

We now go to I_3. (eq. (18)) we just write the following result which can be proved on exactly parallel lines to those used in deriving (26).

$$\sum_n \int d^2k \; \ell n(k^2+\omega_n^2+N^2) = -\frac{16\pi^3}{\beta^2}\int_0^{N\beta/2\pi} dz\; z\; \ell n\frac{\mathrm{Sh}\pi z}{\pi z} - \frac{4\pi}{\beta^2}\zeta(3) + \pi N^2(1-\ell n(\beta N)^2) \tag{27}$$

Note that, in order to compute I_3, we need (27) for

$$N_1 = 2M^2 + k_L^2 + \frac{2k_L\delta}{L} \quad , \quad N_2^2 = k_L^2 + 2M^2$$

and then take the difference of both (infinitesimal for $L\to\infty$).

The result is:

$$I_3(k_L) = -\frac{4\pi k_L}{L}\,\delta(k_L)\;\ell n\; 2\,\mathrm{Sh}\,\frac{N_2\beta}{2} \tag{28}$$

As $\delta(k_L) = -2\, arctg \dfrac{\frac{3\sqrt{2}}{M}k_L}{2-2\frac{k_L^2}{M^2}}$ (see ref. [8])

The integral over k_L will diverge for $k_L \to \infty$

In order to take care of this divergence, we renormalize by substracting the divergent part at $T = 0$ $(\beta\to\infty)$

$$\frac{1}{\beta}\sum_n \Rightarrow \frac{1}{2\pi}\int dk.$$

with which we have, for instance:

$$\sum_n \int d^2k\; \ell n(k^2+N^2) \to \frac{1}{2\pi}\int_{\alpha=3} d^\alpha k\; \ell n(k^2+N^2) = -\frac{2}{3}\pi N^3$$

We must add two counter terms corresponding to N_1 and N_2

$$I_{3R} = \frac{2\pi}{3}(N_2^3-N_1^3) \cong \pi\Delta(N^2) = 2\pi k_L\frac{\delta(k_L)}{L}(k_L^2+2M^2)^{1/2} \tag{29}$$

We are finally left with:

$$\int dk_L I_3(k_L) \Rightarrow \int (I_3+I_{3R}) dk_L = \int dk_L \frac{4\pi kL}{L} 2\, artg \frac{\frac{3\sqrt{2}}{M} kL}{2-2\frac{k^2}{M^2}} \ln\left[1-e^{-(k_L^2+2M^2)\beta}\right] \quad (3\ )$$

which is a convergent integral and can be computed numerically. Forms (19), (26) and (30) solve the problem of computing (15).

We have seen in the previous paragraph that although we started from ill defined divergent integrals, we were able to arrive through analytic continuation to well defined expressions. In particular, the combination of this analytic continuation, with form (21) provides a convenient way to attack problems in finite temperature field theories arriving at the expression of the partition function in such a way that it is in many cases, more conveniently obtained than with the use of other methods. (See also refs. [8], [9]).

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