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A CHIRAL QUANTUM BARYON *

by

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Abstract

We show that a classical soliton for the non-linear $SU(2)$ sigma model in the hedgehog configuration admits a stable solution, when quantized through collective coordinates, which may be identified with the nucleon. The whole approach depends on a single, dimensional and arbitrary constant. Numerical results seem to converge for the mass and for the right value of the weak axial coupling.

Key-words: Chiral soliton; Non-linear sigma model; Skyrmeon.

It is currently admitted after the work of several authors [1], who revived the argument by Skyrme [2], that a baryon is a soliton of a chiral theory.

Classical stability arguments seemed to require, however, the introduction of an additional term to the non-linear sigma model lagrangean (in the non-relativistic limit),

$$\mathcal{L} = -\frac{1}{2} f_\pi^2 \int d^3x \, \mathrm{Tr} \sum_{k=1}^{3} (\partial_k U^\dagger)(\partial_k U), \qquad (.1)$$

where U is a unitary operator:

$$UU^\dagger = 1$$

and f_π is the usual pion-decay constant.

The additional term introduced by Skyrme,

$$-\frac{1}{32e^2} \int d^3x \, \mathrm{Tr} \, [U^\dagger(\partial_k U), U^\dagger(\partial_l U)]^2, \qquad (.2)$$

incorporated a dimensional parameter, e.

Several works [3] dealt with the phenomenology of this classically stable theory, and showed, after quantization, a reasonable agreement for physical quantities when the hedgehog form for U was used (spherically-symmetric ansatz):

$$U_0 = \exp\left(i\tau \cdot \mathbf{n} F(r)\right), \qquad (.3)$$

where τ_k represent the usual Pauli matrices for $SU(2)$ and

$$\mathbf{n} = \mathbf{r}/|\mathbf{r}| \qquad (.4)$$

$$r = \sum_{k=1}^{3} (x^k)^2. \qquad (.5)$$

There are several point which deserve further attention. First, since it is assumed that the effective chiral lagrangean must result from some more fundamental theory, namely, from QCD, it is hard to see how to generate a term like (.2). Second, it is difficult to ascribe a physical meaning to the new dimensional constant in the game, e. Some recent work attempts to relate it to the pion-decay constant, f_π [4]. Third, if one uses the full Skyrme lagrangean, the formal results for the description of chiral dynamics at low energies do not seem to depend on e [5].

We have lately addressed ourselves to the question of the meaning of a theory without a Skyrme term [6]. In particular, we have stressed the point that the classical Euler-Lagrange equation for $F(r)$ is singular and introduces a dimensional constant in the formalism. This constant carries, in the classical domain, the instability of the non-linear classical sigma model soliton. It seems that former work overlooked this constant. In fact, some feeling about it present in the work by Balachandran *et al.* [7], who introduced a kind of variational "shape" parameter, accounting for the size of the soliton.

In fact, as we showed [6], this constant appears naturally when one intends to solve the classical equation of motion for the lagrangean (.1) using the hedgehog $SU(2)$ solution (.3):

$$\frac{d^2F(r)}{dr^2} + \frac{2}{r}\frac{dFr}{dr} = \sin 2F(r). \qquad (.6)$$

To eliminate the first derivative, one uses

$$F(r) = \frac{\chi(r)}{r} \qquad (.7)$$

and, calling

$$r = 2x, \tag{.8}$$

we arrive finally to

$$\frac{d^2\chi(x)}{dx^2} = \frac{2}{x}\sin\left(\frac{\chi(x)}{x}\right). \tag{.9}$$

It is easy to verify that for the second derivative we arrive to an identity, so it remains a dimensional free parameter. In order to solve (.9), we must require

$$\chi(0) = 0 \tag{.10}$$

$$\chi'(0) = 0,\ \pm 2n\pi, \qquad n = 1, 2, \ldots \tag{.11}$$

To have a soliton solution with winding number n,

$$F(0) = -n\pi \tag{.12}$$

$$\chi'(0) = -2n\pi, \tag{.13}$$

provided $F(r)$ is zero at infinity, and we have at the end,

$$\chi(x) = -2n\pi x + \frac{1}{2}\chi''(0)x^2 X((\chi''(0))^2), \tag{.14}$$

where

$$X(s) = \sum_{n=1}^{\infty} f_n s^{2(n-1)} \tag{.15}$$

$$\chi''(0) = \left.\frac{d^2\chi(x)}{dx^2}\right|_{x=0} \tag{.16}$$

and $s = \chi''(0)x$ is a dimensionless variable. The first coefficients in the expansion of $X(s)$ are

$$
\begin{aligned}
f_1 &= 1 \\
f_2 &= -\frac{1}{5!} = -\frac{1}{120} \\
f_3 &= \frac{1}{6!}\frac{3^2}{2^4 \cdot 7} = \frac{1}{8960} \\
f_4 &= -\frac{1}{8!}\frac{17}{2^4 \cdot 3 \cdot 5} = -\frac{17}{9,676,800} \\
f_5 &= \frac{1}{10!}\frac{3 \cdot 7 \cdot 73}{2^8 \cdot 5 \cdot 11} = \frac{73}{2,433,024,000} \\
f_6 &= -\frac{3337}{6,199,345,152,000}
\end{aligned}
$$

The appearence of the dimensional parameter $\chi''(0)$ for the solution of the soliton has remained unnoticed to the authors of previous work. It seems, however, as we mentioned earlier, that Balachandran and coworkers [1] were somewhat aware of its necessity, when they introduced a variational *ad hoc* shape parameter. Besides, notice that this parameter should even be included with the Skyrme term (Eq. (.2)), since it does not contribute to the singularity at the origin.

It turns out that the chiral angle itself, $F(r)$, is in fact a function of the dimensionless variable s, as seen replacing (.14) in (.7):

$$F(r) = F(s) = -n\pi + \frac{1}{4}sX(s). \qquad (.17)$$

This new dimensional parameter, which, we stress, comes from the consistency of the series solution at the origin for the chiral angle, is intimately connected to the usual stability argument against the soliton solution for the non-linear sigma model lagrangean. If we write the expression for the mass of the soliton,

$$M_0 = 4\pi f_\pi^2 \int_0^\infty dr' \left[r'^2 \left(\frac{dF}{dr'}\right)^2 + 2\sin^2 F(r') \right] \qquad (.18)$$

in terms of Eq. (.17) above, we find

$$M_0 = 2\pi f_\pi^2 \frac{1}{\chi''(0)} \int_0^\infty ds' \left[\frac{1}{4} s'^2 \mathcal{F}'^2(s') + 8\sin^2\left(\frac{1}{4}\mathcal{F}(s')\right)\right] \qquad (.19)$$

putting

$$\mathcal{F}(s) = sX(s) \qquad (.20)$$

and $\mathcal{F}'(s)$ being its first derivative. The integral over the dimensionless variable s' in Eq. (.19) is a pure number, and the usual argument for the instability of the soliton, coming from the replacement

$$r' \longrightarrow \lambda r'$$

in Eq. (.18) translates into the instability under a variation of $\chi''(0)$.

It is well known, though, that when quantizing with the help of collective coordinates

$$\begin{aligned} U(\mathbf{r},t) &= A(t)U_0(\mathbf{r})A^\dagger(t) \\ &= \cos F(r) + i\tau_j D_{jk}(t) n_k \sin F(r), \qquad (.21) \end{aligned}$$

where $D_{jk}(t)$ are rotation matrices, the expression for the energy of the quantized system becomes the one for a rotating top (see, for instance, the lecture notes by Balachandran [7] or the article by Adkins, Nappi and Witten in Ref. [3]),

$$M = M_0 + \frac{1}{2\lambda}\mathbf{J}^2, \qquad (.22)$$

where the "momentum of inertia", λ, is

$$\lambda = \frac{16}{3} f_\pi^2 \int_0^\infty dr'\, r'^2 \sin^2 F(r'). \qquad (.23)$$

Using Eq. (.17),

$$\lambda = 2\pi f_\pi^2 \frac{1}{\chi''(0)^3} \int_\infty^0 ds'\, \frac{64}{3} s'^2 \sin^2\left(\frac{1}{4}\mathcal{F}(s')\right). \qquad (.24)$$

With this, Eq. (.22) takes the form

$$M = \frac{2\pi f_\pi^2}{\chi''(0)} a + \frac{1}{2}\frac{\chi''(0)^3}{2\pi f_\pi^2 b}\mathbf{J}^2. \qquad (.25)$$

The quantization for the symmetric top as a fermion brings that the possible values for $\mathbf{J}^2$ (and for the isotopic spin, $\mathbf{T}^2 = \mathbf{J}^2$) are half integer.

It is easily seen that Eq. (.25) has a minimum in terms of $\chi''(0)$. The only remaining fixed scale parameter in Eq. (.25) is f_π, the pion-decay constant. The values for $\chi''(0)$ and the mass at the minimum are

$$\chi''(0) \;=\; \left[\frac{2}{3}\frac{(2\pi)^2}{\mathbf{J}^2} ab\right]^{1/4} f_\pi \qquad (.26)$$

$$M \;=\; \frac{4}{3}\left[\frac{3}{2} 92\pi)^2 \mathbf{J}^2 \frac{a^3}{b}\right]^{1/4} f_\pi. \qquad (.27)$$

We have immediately a prediction for the mass ratio of the lowest states:

$$\frac{M(J=\frac{3}{2})}{M(J=\frac{1}{2})} = 5^{1/4} \simeq 1.495\ldots, \qquad (.28)$$

which agrees rather well with the known experimental ratio for the Δ resonance and the nucleon:

$$\frac{M(\Delta)}{M(N)} \simeq 1.32\ldots. \qquad (.29)$$

It may seem that we have lost any trace of the value of the "baryon number", or winding number, as it appears in the first term of Eq. (.17). This is not the case, since asymptotically the expression for $X(s)$ is well determined.

In order to see this, let us go back to the solution for the chiral angle at infinity, looking for the solution of Eq. (.6). It is readily seen that, with $y = 1/x$, we have

$$\begin{aligned} \chi(x) &= \psi(y) \\ K(y) &= y\psi(y) \end{aligned}$$

and, finally,

$$K''(y) = \frac{2}{y^2}\sin K(y), \qquad (.30)$$

with the relation

$$F(x) = \frac{1}{2}K(y). \qquad (.31)$$

The series solution of Eq. (.30) gives

$$K(y) = 2n_\infty\pi + \frac{1}{2}K''(0)y^2Y(y), \qquad (.32)$$

with

$$
\begin{aligned}
Y(y) &= \sum_{j=1}^{\infty} k_j (K''(0) y^2)^{2(j-1)} \\
K''(0) &= \left. \frac{d^2 K(y)}{dy^2} \right|_{y=0} \\
k_1 &= 1 \\
k_2 &= = -\frac{1}{6!}\frac{15}{7} = -\frac{1}{336} \\
k_2 &= \frac{1}{11!} 2 \cdot 3^4 \cdot 5 = \frac{1}{9280} \\
k_5 &= -\frac{1}{6,209,280}
\end{aligned}
$$

The winding number of the soliton is given by the difference

$$N = n - n_\infty,$$

so, if n=1, in order to have N=1, n_∞ must be zero. The dimensional parameter $\chi''(0)$ translates at infinity in the dimensional parameter $K''(0)(\sim -\chi''(0)^{-2})$. Then, as the radial coordinate grows to infinity,

$$F \sim -\frac{K''(0)}{r^2}. \qquad (.33)$$

Comparing Eq. (.33) with Eq (.17), we see that at infinity,

$$X(s) \sim +\frac{4\pi n}{s} + O(s^{-2}). \qquad (.34)$$

The behavior at infinity resulting from Eq. (.32) allows one to have information about the axial current coefficient, g_A, as shown by Adkins, Nappi and Witten [3],

$$g_A = \frac{8}{3} 24\pi f_\pi^2 K''(0). \qquad (.35)$$

We have begun to work out numerical results for the $SU(2)$

chiral theory. They are at the moment not complete, but we think they deserve some consideration.

In order to exploit our knowledge of the solutions by power series expansion of Eqs. (.9) and (.30), we propose a systematic approximation using Padé approximants [8]. They are in this case of a particular type, since we need to enforce the conditions fixing the soliton solution to be of winding number one. Defining

$$f[N,M](\alpha)=\frac{n_1+n_2\alpha+n_3\alpha^2+\cdots+n_M\alpha^{M-1}}{1d_1\alpha+d_2\alpha^2+\cdots+d_N\alpha^N}, \qquad (.36)$$

we find that the only approximants satisfying the conditions

$$\begin{aligned} X[N,M](0) &= 1 \\ N[N,M](\infty) &\sim \frac{4\pi}{s}+O(s^{-2}) \end{aligned}$$

are those with $N=2j+1$, $M=2j$, $j=1,2,\ldots$, *i.e.*, [3,2], [5,4], [7,6],.... For instance, [3,2] for $X(s)$ uses the first coefficient only, and is particularly simple:

$$X[3,2](s)=\left(1+\frac{s}{4\pi}\right)\left(1+\frac{s}{4\pi}+\left(\frac{s}{4\pi}\right)^2\right)^{-i}. \qquad (.37)$$

After determining the coefficients in the Padé approximant (.36), we calculate the integrals a and b in (.25) and find the values for $\chi''(0)$ and M. To have the axial vector coupling, we use the fact that the asymptotic form for the Padé approximants is

$$F(\sigma)\sim\pi\, c_1[N]\sigma^2 \qquad (\sigma\sim 0), \qquad (.38)$$

with $\sigma=1/s$ and

$$c_1[N]=-\pi\left(\frac{d_{N-2}}{d_N}-\frac{n_{N-2}}{4\pi d_N}\right). \qquad (.39)$$

The first results are (with f_π=0.067 GeV):

	$\chi''(0)(f_\pi)$	$\chi''(0)$(GeV)	$M_N(f_\pi)$	M_N(GeV)	g_A
[3, 2]	193.2	13.00	9.964	0.6676	0.891
[5, 4]	371.1	24.87	8.693	0.5824	1.087
[7, 6]	624.1	41.82	8.166	0.5471	1.162

We see that the above results show a systematic trend, and further work is currently being done increasing the order of the approximants (that is, using more information about the soliton solution) and enlarging the flavor group. The dimensional parameter $\chi''(0)$ is rather large, showing the importance of short distance behavior. The value for the mass is rather low, and seems to converge to a value around .50 GeV for our chosen value for f_π. Interestingly, the results for the axial weak coupling look nice, and may converge to the right value.

We think, however, that the above results strongly indicate that it is possible to obtain a consistent description of low-energy hadronic physics with the information available from current algebra, summarized by the non-linear sigma model lagrangean, Eq.(1).

The need to use a minimum of the quantum energy for a description of baryons does not seem to be quite extravagant. It arises from the exact behavior of the hedgehog classical solution.

If, on the other hand, one expects to describe low-energy hadron physics from a dynamical quantum theory like QCD through an effective lagrangean, experience with two dimensions

[9] seem to indicate that quantum (loop) effects are relevant.

One may also recall that the simple hydrogen atom is classically unstable, and the crudest quantum condition makes it into a stable, quantized system. The comparison may look exagerated, but it is worth to remember that not always the quantum system follows the paths suggested by classical dynamics.

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When the first draft of this article was complete, we received through the library of CBPF a copy of a preprint by Jain, Schechter and Sorkin from Syracuse Unversity (SU-4228-394), who agree with the general framework of our own work.

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