

CLASSIFYING N -EXTENDED 1-DIMENSIONAL SUPERSYMMETRIC SYSTEMS *

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1. Introduction

In this talk I will report some results obtained in a joint collaboration with A. Pashnev, concerning the classification of the irreducible representations of the N -extended Supersymmetry in 1 dimension and which find applications to the construction of Supersymmetric Quantum Mechanical Systems [1].

This mathematical problem finds immediate application to the theory of dimensionally (to one temporal dimension) supersymmetric $4d$ theories, which gets 4 times the number of supersymmetries of the original models (the $N = 8$ supergravity being e.g. associated with the a $N = 32$ Supersymmetric Quantum Mechanical theory). Due to a lack of superfield formalism for $N > 4$, only partial results are known [2] and [3].

More recently, Supersymmetric and Superconformal Quantum Mechanics have been applied in describing e.g. the low-energy effective dynamics of a certain class of black holes, for testing the AdS/CFT correspondence in the case of AdS_2 , in investigating the light-cone dynamics of supersymmetric theories.

In this report of the work with Pashnev, two main results will be presented. At first a peculiar property of supersymmetry in one dimension is exhibited, namely that any finite dimensional multiplet containing d bosons and d fermions in different spin states are put into classes of equivalence individuated by irreducible multiplets of just two spin states, where all

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bosons and all fermions are grouped in the same spin. Later it is shown that all irreducible multiplets of this kind are in one-to-one correspondence with the classification of real-valued Clifford Γ matrices of Weyl type.

This classification refines (in the case of “non-Euclidean” supersymmetry, see below) the results obtained in [4] and [5]. Another reference where some aspects of the theory of the representation of 1-dimensional supersymmetry are discussed is given by [6].

The mathematical problem we are investigating can be stated as follows, finding the irreducible representation of the supersymmetry algebra

$$\{Q_i, Q_j\} = \omega_{ij}H, \quad (1)$$

where Q_i , $i = 1, 2, \dots, N$ are supercharges and

$$H = -i \frac{\partial}{\partial t} \quad (2)$$

is the Hamiltonian. The constant tensor ω_{ij} can be conveniently diagonalized and normalized in such a way to coincide with a pseudo-Euclidean metric η_{ij} with signature (p, q) . Usually the eigenvalues are all assumed being positive (i.e. $q = 0$), however examples can be given (see [7]), of physical systems whose supersymmetry algebra is characterized by an indefinite tensor. In the following I will discuss the simplest example of this kind.

Any given finite-dimensional representation multiplet of the above superalgebra can be represented in form of a chain of d bosons and d fermions

$$\Phi_{a_0}^0, \quad \Phi_{a_1}^1, \quad \dots, \quad \Phi_{a_{M-1}}^{M-1}, \quad \Phi_{a_M}^M \quad (3)$$

whose components $\Phi_{a_I}^I$, ($a_I = 1, 2, \dots, d_I$) are real and alternatively bosonic and fermionic ($d = d_0 + d_2 + d_4 + \dots = d_1 + d_3 + d_5 + \dots$). For such a multiplet the short notation $\{\mathbf{d}_0, \mathbf{d}_1, \dots, \mathbf{d}_M\}$ will also be employed.

Due to dimensionality argument the i -th supersymmetry transformation for the $\Phi_{a_I}^I$ components is given by

$$\delta_\varepsilon \Phi_{a_I}^I = \varepsilon^i (C_i^I)_{a_I}{}^{a_{I+1}} \Phi_{a_{I+1}}^{I+1} + \varepsilon^i (\tilde{C}_i^I)_{a_I}{}^{a_{I-1}} \frac{d}{d\tau} \Phi_{a_{I-1}}^{I-1}, \quad (4)$$

and it simplifies for the end-components (due to the absence of the $I = -1$ and $I = M + 1$ components).

In one dimension it is therefore possible to redefine the last components according to

$$\Phi_{a_M}^M = \frac{d}{d\tau} \Psi_{a_M}^{M-2} \quad (5)$$

in terms of some functions $\Psi_{a_M}^{M-2}$. The initial supermultiplet of length $M + 1$ is now re-expressed as the $\{\mathbf{d}_0, \mathbf{d}_1, \dots, \mathbf{d}_{M-2} + \mathbf{d}_M, \mathbf{d}_{M-1}, \mathbf{0}\}$ supermultiplet of length M . By repeating M times the same procedure the shortest

supermultiplet $\{\mathbf{d}, \mathbf{d}\}$ of length 2 can be reached. The above argument outlines the proof of the statement that all supermultiplets are classified according to the irreducible representations of supermultiplets of length 2.

2. Extended supersymmetries and real valued Clifford algebras

The main result of the previous Section is that the problem of classifying all N -extended supersymmetric quantum mechanical systems is reduced to the problem of classifying the irreducible representations of length 2. Having this in mind let us simplify the notations. Let the indices $a, \alpha = 1, \dots, d$ number the bosonic (and respectively fermionic) elements in the SUSY multiplet. All of them are assumed to depend on the time coordinate τ ($X_a \equiv X_a(\tau)$, $\theta_\alpha \equiv \theta_\alpha(\tau)$).

In order to be definite and without loss of generality let us take the bosonic elements to be the first ones in the chain $\{\mathbf{d}, \mathbf{d}\}$, which can be conveniently represented also as a column

$$\Psi = \begin{pmatrix} X_a \\ \theta_\alpha \end{pmatrix}, \quad (6)$$

the supersymmetry transformations are reduced to the following set of equations

$$\begin{aligned} \delta_\varepsilon X_a &= \varepsilon^i (C_i)_a^\alpha \theta_\alpha \equiv i(\varepsilon^i Q_i \Psi)_a \\ \delta_\varepsilon \theta_\alpha &= \varepsilon^i (\tilde{C}_i)_\alpha^b \frac{d}{d\tau} X_b \equiv i(\varepsilon^i Q_i \Psi)_\alpha \end{aligned} \quad (7)$$

where, as a consequence of (1),

$$C_i \tilde{C}_j + C_j \tilde{C}_i = i\eta_{ij} \quad (8)$$

and

$$\tilde{C}_i C_j + \tilde{C}_j C_i = i\eta_{ij} \quad (9)$$

Since $\varepsilon_i, X_a, \theta_\alpha$ are real, the matrices C_i 's, \tilde{C}_i 's have to be respectively imaginary and real. If we set (just for normalization)

$$\begin{aligned} C_i &= \frac{i}{\sqrt{2}} \sigma_i \\ \tilde{C}_i &= \frac{1}{\sqrt{2}} \tilde{\sigma}_i \end{aligned} \quad (10)$$

and accommodate $\sigma_i, \tilde{\sigma}_i$ into a single matrix

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i & 0 \end{pmatrix}, \quad (11)$$

they form a set of real-valued Clifford Γ -matrices of Weyl type (i.e. block antidiagonal), obeying the (pseudo-) Euclidean anticommutation relations

$$\{\Gamma_i, \Gamma_j\} = 2\eta_{ij}. \quad (12)$$

Therefore the classification of irreducible multiplets of representation of a (p, q) extended supersymmetry is in one-to-one correspondence with the classification of the real-valued Clifford algebras $C_{p,q}$ with the further property that the Γ matrices can be realized in Weyl (i.e. block antidiagonal) form.

Real-valued Clifford algebras have been classified in [8] for compact ($q = 0$) case, and in [9] for the non-compact one. I follow here the exposition in [10].

Three cases have to be distinguished for real representations, specified by the type of most general solution allowed for a real matrix S commuting with all the Clifford Γ_i matrices, i.e.

- i)* the normal case, realized when S is a multiple of the identity,
- ii)* the almost complex case, for S being given by a linear combination of the identity and of a real $J^2 = -\mathbf{1}$ matrix,
- iii)* finally the quaternionic case, for S being a linear combination of real matrices satisfying the quaternionic algebra.

Real irreducible representations of normal type exist whenever the condition

$p - q = 0, 1, 2 \pmod{8}$ is satisfied (their dimensionality being given by $2^{\lfloor \frac{N}{2} \rfloor}$, where $N = p + q$), while the almost complex and the quaternionic type representations are realized in the $p - q = 3, 7 \pmod{8}$ and in the $p - q = 4, 5, 6 \pmod{8}$ cases respectively. The dimensionality of these representations is given in both cases by $2^{\lfloor \frac{N}{2} \rfloor + 1}$.

We further require the extra-condition that the real representations should admit a block antidiagonal realization for the Clifford Γ matrices. This condition is met for $p - q = 0 \pmod{8}$ in the normal case (it corresponds to the standard Majorana-Weyl requirement), $p - q = 7 \pmod{8}$ in the almost complex case and $p - q = 4, 6 \pmod{8}$ in the quaternionic case. In all these cases the real irreducible representation is unique.

It is therefore possible to furnish the dimensionality of the irreducible representations of the of the supersymmetry algebra or, conversely, the allowed (p, q) signatures associated to a given dimensionality of the bosonic and fermionic spaces. The latter result is conveniently expressed by introducing the notion of maximally extended supersymmetry. The $C_{p,q}$ ($p - q = 6 \pmod{8}$) real representation for the quaternionic case can be recovered from the $7 \pmod{8}$ almost complex $C_{p+1,q}$ representation by deleting one of the Γ matrices; in its turn the latter representation is recovered from the $C_{p+2,q}$ normal Majorana-Weyl representation by deleting another Γ

matrix. The dimensionality of the three representations above being the same, the normal Majorana-Weyl representation realizes the maximal possible extension of supersymmetry compatible with the dimensionality of the representation. In search for the maximal extension of supersymmetry we can therefore limit ourselves to consider the normal Majorana-Weyl representations, as well as the quaternionic ones satisfying the $p-q = 4 \pmod{8}$ condition.

Let us therefore introduce a parameter ϵ , which assumes two values and is used to distinguish the Majorana-Weyl ($\epsilon = 0$) with respect to the quaternionic case ($\epsilon = 1$). A space of $d = 2^t$ bosonic and $d = 2^t$ fermionic states can carry the following set of maximally extended supersymmetries

$$(p = t - 4z + 5 - 3\epsilon, q = t + 4z + \epsilon - 3) \quad (13)$$

where the integer $z = k - l$ must take values in the interval

$$\frac{1}{4}(3 - t - \epsilon) \leq z \leq \frac{1}{4}(t + 5 - 3\epsilon) \quad (14)$$

in order to guarantee the $p \geq 0$ and $q \geq 0$ requirements.

3. An application and conclusions.

One of the most significant application of extended supersymmetric quantum mechanics concerns the 1-dimensional σ models evolving in a target spacetime manifold presenting both bosonic and fermionic coordinates. In general such models present a non-linear kinetic term and the extended supersymmetries put constraints on the metric of the target. In this section let us present here a very simplified model, which however is illustrative of how invariances under pseudo-Euclidean supersymmetry can arise. Let us in fact consider a model of d bosonic fields X_a and d spinors ψ_α freely moving in a flat d -dimensional target manifold, not necessarily Minkowskian or Euclidean, endorsed of a pseudo-euclidean η_{ab} . Let us furthermore introduce the free kinetic action being given by

$$S_K = \int dt \mathcal{L} = \frac{1}{2} \int dt \left(\dot{X}_a \dot{X}_b \eta^{ab} + i\delta \dot{\psi}_\alpha \psi_\beta \eta^{\alpha\beta} \right), \quad (15)$$

where the metric $\eta^{\alpha\beta}$ for the spinorial part is assumed to have the same signature as the metric η^{ab} , and δ is just a sign normalization ($\delta = \pm 1$).

A natural question to be asked is which supersymmetries are invariances of the above free kinetic action. The answer is furnished by accommodating the d bosonic and d fermionic coordinates into a (maximally extended) irreducible representation of the extended supersymmetries, and later counting how many such transformations survive as invariances of the action. The

first non-trivial example concerns a 2-dimensional target ($d = 2$), whose two bosonic and two fermionic degrees of freedom carry the $\{\mathbf{2}, \mathbf{2}\}$ representation of $(2, 2)$ extended supersymmetry. However, only half of these supersymmetries are realized as invariances of the action. The action indeed is invariant under either the $(2, 0)$ or the $(1, 1)$ extended supersymmetries, whether the target space is respectively Euclidean or Minkowskian. Therefore already in the 2-dimensional Minkowskian case we observe the arising of a pseudo-Euclidean supersymmetry invariance. The next simplest example is realized by a 4-dimensional target. The four bosonic and four fermionic coordinates can be accommodated into three irreducible representations of maximally extended supersymmetry, according to formula (13), namely the $(4, 0)$, the $(0, 4)$ and the $(3, 3)$ extended supersymmetries. The action (15) turns out to be invariant, for Euclidean $(4 + 0)$, Minkowskian $(3 + 1)$ and $(2 + 2)$ signature for the metric η , according to the following table

	$(4, 0)$	$(0, 4)$	$(3, 3)$	
$(4 + 0)$	$(4, 0)$	$(0, 0)$	$(3, 0)$	$\delta = +1$
$(4 + 0)$	$(0, 0)$	$(0, 4)$	$(0, 3)$	$\delta = -1$
$(3 + 1)$	$(1, 0)$	$(0, 0)$	$(1, 0)$	$\delta = +1$
$(3 + 1)$	$(0, 0)$	$(0, 1)$	$(0, 1)$	$\delta = -1$
$(2 + 2)$	$(2, 0)$	$(0, 2)$	$(2, 1)$	$\delta = +1$
$(2 + 2)$	$(2, 0)$	$(0, 2)$	$(1, 2)$	$\delta = -1$

which should be understood as follows. The central entries denote how many supersymmetries are realized as invariances of the (15) action for each one of the three irreducible representations of maximally extended supersymmetry, in correspondence with the given signature of spacetime and sign for δ . In this particular case invariance under pseudo-Euclidean supersymmetry is guaranteed for the target of signature $(2 + 2)$.

In this talk I have presented some results concerning the representation theory for irreducible multiplets of the one-dimensional $N = (p, q)$ extended supersymmetry. A peculiar feature of the one-dimensional supersymmetric algebras consists in the fact that the supermultiplets formed by d bosonic and d fermionic degrees of freedom accommodated in a chain with $M + 1$ ($M \geq 2$) different spin states uniquely determines a 2-chain multiplet of the form $\{\mathbf{d}, \mathbf{d}\}$ which carries a representation of the N extended supersymmetry. Furthermore, it is shown that all such 2-chain irreducible multiplets of the (p, q) extended supersymmetry are fully classified; when e.g. the condition $p - q = 0 \pmod{8}$ is satisfied, their classification is equivalent to that one of Majorana-Weyl spinors in any given space-time, the number $p + q$ of

extended supersymmetries being associated to the dimensionality D of the spacetime, while the $2d$ supermultiplet dimensionality is the dimensionality of the corresponding Γ matrices. The more general case for arbitrary values of p and q has also been fully discussed.

These mathematical properties can find a lot of interesting applications in connection with the construction of Supersymmetric and Superconformal Quantum Mechanical Models. These theories are vastly studied due to their relevance in many different physical domains, to name just a few it can be mentioned the low-energy effective dynamics of black-hole models, the dimensional reduction of higher-dimensional superfield theories, which are a laboratory for the investigation of the spontaneous breaking of the supersymmetry, and so on.

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