Renormalization of Two-Dimensional Quantum Electrodynamics

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Abstract

The Schwinger model, when quantized in a gauge non-invariant way exhibits a dependence on a parameter a (the Jackiw-Rajaraman parameter) in a way which is analogous to the case involving chiral fermions (the chiral Schwinger model). For all values of $a \neq 1$, there are divergences in the fermionic Green's functions. We propose a regularization of the generating functional $Z[\eta, \overline{\eta}, J]$ and we use it to renormalize the theory to one loop level, in a semi-perturbative sense. At the end of the renormalization procedure we find an implicit dependence of a on the renormalization scale μ.

1 Introduction

The Schwinger model [1] is Quantum Electrodynamics with a single massless Dirac fermion in two-dimensional space-time. Since it was shown to be exactly solvable, many people studied the model trying to gain intuition to deal with various problems which are present in particle physics. Breakdown of global chiral symmetry through the $U(1)$ anomaly, charge shielding, quark trapping and the existence of θ vacua are among the phenomena which are present [1]-[5]. Both operator and functional methods were successfully employed to study various aspects of the model [2], [6]-[16].

All these studies were performed in a gauge invariant way, that is, the quantization of the fermionic degrees of freedom (the computation of the fermionic determinant) was performed in such a way that gauge symmetry was preserved at the quantum level.

However, when we consider Weyl instead of Dirac fermions (the chiral Schwinger model), a consistent and unitary quantum theory emerges [17] in spite of the fact that gauge invariance is lost (because of the gauge anomaly [18]). The resulting quantized model is dependent on a parameter a (Jackiw-Rajaraman parameter) which is introduced at quantum level, and can not be fixed *a priori* to any value. The fermionic Green's functions are divergent [19] (for any value of a) and it is readly seen that a fermionic wave function renormalization is sufficient to turn the theory finite. After renormalization, we still expect that Green's functions depend on a. The situation is quite disturbing, as it would indicate that the quantization of the theory could not be done in a unique way.

On the other side, it is well known that the ambiguities on the Green's functions of a renormalizable theory are not present at the S matrix level [20]. These ambiguities are parameterized by a massive parameter μ, which controls renormalization group equations. The Green's functions are μ dependent, but this dependence is cancelled in the S matrix by a compensating behaviour of the residues of the propagators of the theory.

At this point, the central idea of this paper emerges: if we could relate the parameter a with μ, a possibility would appear that the dependence on a could be cancelled in the computation of physical quantities. If this could be possible, the quantization of the theory would give always the same result, thus allowing us to choose the value of a according to our convenience.

Back to the case of the Schwinger model we see that there is a privileged value of a for which we do not need to perform any renormalization, because the whole theory is finite. This value is $a = 1$, and it corresponds to regularizations which preserve gauge invariance. However, if we do not require gauge invariance, *we have the same divergence structure that is present in the chiral Schwinger model.* So, the Schwinger model turns out to be the perfect scenario to test the conjectures above. For $a = 1$ we have access, in this case, to a complete non-perturbative solution of the model. This solution can be compared to the case $a \neq 1$, where we have only approximate solutions, obtained by semi-perturbative techniques (this happens because we can only solve the model in configuration space, while renormalization is performed in momentum space). This situation is better than the one that we find in the context of the chiral Schwinger model, where there is no exact renormalized solution, up to now.

The paper is organized as follows: in section 2 we present a calculation of the most relevant Green's functions and we show the structure of the divergences. In section 3 we give a regularization for them, in the gauge non-invariant formalism. In section 4 we study the Ward identities and in section 5 we perform the renormalization of the model. Finally, in section 6 we present our conclusions.

2 The Schwinger model

The Schwinger model is defined by the following Lagrangian density

$$\mathcal{L}[\psi, \overline{\psi}, A] = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(\, i \partial\!\!\!/ + e A\!\!\!/ \,)\psi. \tag{1}$$

The effective action $W[A_\mu]$ is [1]

$$\begin{aligned} e^{iW[A_\mu]} &= \int d\psi d\overline{\psi} \, \exp\left[\, i \int dx \, \overline{\psi}(\, i\partial\!\!\!/ + eA\!\!\!/ \,)\psi \, \right] \\ &= \det iD \quad , \quad D = i\partial\!\!\!/ + eA\!\!\!/. \end{aligned} \tag{2}$$

We calculate the fermionic determinant using a prescription which is gauge non-invariant [18, 22]. Thus, we find $W[A_\mu]$, given by

$$W[A_\mu] = \int dx \, \frac{1}{2} A_\mu(x) \left[m^2(a) \, g^{\mu\nu} - \frac{e^2}{\pi} \frac{\partial^\mu \partial^\nu}{\Box} \right] A_\nu(x), \tag{3}$$

where $m^2(a)$ is

$$m^2(a) = \frac{e^2}{2\pi}(a+1). \tag{4}$$

The generating functional of the model is

$$Z[\eta, \overline{\eta}, J] = N \int dA_\mu d\psi d\overline{\psi} \, \exp\left[\, i \int dx \left(\mathcal{L}[\psi, \overline{\psi}, A] + \overline{\eta}\psi + \overline{\psi}\eta + J_\mu A^\mu \right) \right]. \tag{5}$$

As we computed $W[A_\mu]$, we can integrate over the fermion fields and get

$$\begin{aligned} Z[\eta, \overline{\eta}, J] &= \int dA_\mu \, \exp\left[\, i \int dx \left(\frac{1}{2} A_\mu \Gamma^{\mu\nu} A_\nu + J_\mu A^\mu \right) \right] \\ &\quad \times \exp\left[-i \int dx \, dy \, \overline{\eta}(x) G(x, y; A) \eta(y) \right], \end{aligned} \tag{6}$$

[1] Here dx means dx^2. Our conventions are

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = g^{\mu\nu} \quad , \quad \epsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\mu\nu}.$$

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2 \, g^{\mu\nu} \quad , \quad {\gamma_0}^\dagger = \gamma_0 \quad , \quad {\gamma_1}^\dagger = -\gamma_1 \, . \\ \{\gamma_\mu, \gamma_5\} &= 0 \qquad , \qquad \gamma_5 = \gamma_0 \gamma_1, \quad {\gamma_5}^\dagger = \gamma_5 \, . \end{aligned}$$

where $G(x, y; A)$ is the two-point fermion Green's function in the external field A_μ. This can be computed and is

$$\begin{aligned} G(x,y;A) &= \exp\left[-ie\int dz\; A_\mu(z) j_-^\mu(z,x,y)\right] P_- S_F(x-y) \; + \\ &+ \exp\left[-ie\int dz\; A_\mu(z) j_+^\mu(z,x,y)\right] P_+ S_F(x-y). \end{aligned} \tag{7}$$

Here S_F satisfies $i\not{\partial}_x S_F(x-y) = \delta(x-y)$, and $j_\pm^\mu$ and $\Gamma^{\mu\nu}$ are given by

$$j_\pm^\mu(z,x,y) = (\partial_z^\mu \pm \tilde{\partial}_z^\mu)[D_F(z-x) - D_F(z-y)]\;, \tag{8}$$

$$\Gamma^{\mu\nu}(x-y) \;=\; \left(g^{\mu\nu}(\Box + m^2(a)) - \partial^\mu\partial^\nu - \frac{e^2}{\pi}\frac{\partial^\mu\partial^\nu}{\Box}\right)\delta(x-y)\;. \tag{9}$$

Furthermore, $P_\pm$ denotes the projection operator on the right-(left-)handed chiralities

$$P_+ = \frac{1+\gamma_5}{2}\;,\quad P_- = \frac{1-\gamma_5}{2}\;, \tag{10}$$

while $\tilde{\partial}_\mu$ stands for

$$\tilde{\partial}_\mu = \epsilon_{\mu\nu}\partial^\nu\;. \tag{11}$$

All the correlation functions of the theory can in principle be exactly calculated in configuration space, but not in momentum space, where one does not know how to bosonize directly the theory.

The computation of the photon propagator from (6) is straightforward and yields

$$\begin{aligned} \langle 0|TA_\mu(x)A_\nu(y)|0\rangle &= -i\int\frac{dk}{(2\pi)^2}\,\tilde{G}_{\mu\nu}(k)\,e^{-ik\cdot(x-y)} \\ \tilde{G}_{\mu\nu}(k) &= \frac{1}{k^2-m^2(a)}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \frac{2\pi}{e^2(a-1)}\frac{k_\mu k_\nu}{k^2}\;. \end{aligned} \tag{12}$$

We observe that this propagator has a pole in $m^2(a)$, that is, the photon acquires mass after the quantization of the theory. We observe the explicit dependence on a of the photon mass, which leaves it indefinite. Moreover, the photon propagator is divergence free. Its high-momentum behavior is similar to the one in Proca's theory.

Now we calculate the fermion propagator, starting from (6). We arrive at

$$\langle 0|T\psi(x)\overline{\psi}(y)|0\rangle \;=\; G(x-y)\;,$$

$$\begin{aligned} G(x-y) &= i\,\exp\left\{-\frac{2\pi\, i}{a-1}\int\frac{dk}{(2\pi)^2}\,\frac{1-e^{-ik\cdot(x-y)}}{k^2}\right\} \\ &\times\exp\left\{-i\,e^2\int\frac{dk}{(2\pi)^2}\,\frac{1-e^{-ik\cdot(x-y)}}{k^2(k^2-m^2(a))}\right\} S_F(x-y)\;. \end{aligned} \tag{13}$$

From (13) we see that the fermionic propagator has an UV logarithmic divergence, but is free of IR divergences. This divergence is better understood in momentum space. The Fourier transform is not possible to be calculated. However, we can see that the following differential equation is satisfied

$$\left(\partial\!\!\!/_x + e^2 \int \frac{dk}{(2\pi)^2}\, f(k) k\!\!\!/ e^{-ik\cdot(x-y)} \right) G(x-y) = \delta(x-y), \tag{14}$$

where $f(k)$ is given by

$$f(k) \;=\; -\frac{2\pi}{e^2(a-1)\,k^2} - \frac{1}{k^2(k^2-m^2(a))}. \tag{15}$$

This equation expressed in momentum space allows one to find a recursive equation for the fermion propagator $\tilde{G}(p)$

$$\tilde{G}(p) = \frac{i}{p\!\!\!/} - ie^2 \int \frac{dk}{(2\pi)^2}\, f(k)\, \frac{1}{p\!\!\!/}\, k\!\!\!/\, \tilde{G}(p-k)\ . \tag{16}$$

We expand the above equation for $\tilde{G}(p)$ in powers of e^2, and show that it gives a loopwise expansion by using the exact photon propagator, with the n loop order associated to $\hbar^n$ [21] as in the usual case. Writing $\hbar$ explicitly, we get

$$\begin{aligned} \frac{1}{\hbar}\tilde{G}(p) \;&=\; \frac{i}{p\!\!\!/} \;+\; \hbar e^2 \int \frac{dk}{(2\pi)^2}\, f(k)\, \frac{1}{p\!\!\!/}\, k\!\!\!/\, \frac{1}{p\!\!\!/ - k\!\!\!/} \;+ \\ &\quad - i\hbar^2 e^4 \int \frac{dk}{(2\pi)^2} \frac{ds}{(2\pi)^2}\, f(k) f(s) \frac{1}{p\!\!\!/} k\!\!\!/ \frac{1}{p\!\!\!/ - k\!\!\!/} s\!\!\!/ \frac{1}{p\!\!\!/ - k\!\!\!/ - s\!\!\!/} \;+\; \ldots \end{aligned} \tag{17}$$

A power counting analysis of expansion (17), shows the presence of an UV logarithmic divergence in the fermion propagator. This divergence is similar to the one present in a Proca theory with fermions, due to the bad high-momentum behavior of the exact photon propagator. This loopwise expansion will allow us to renormalize the theory.

We can calculate also the three-point Green's function

$$\langle 0|T\psi(x)\overline{\psi}(y)A_\mu(z)|0\rangle \;=\; G_\mu(x,y,z)$$

$$G_\mu(x,y,z) \;=\; i\,e \int \frac{dk}{(2\pi)^2}\, g_\mu(k) \left[\, e^{-ik\cdot(z-x)} - e^{-ik\cdot(z-y)} \,\right]\, G(x-y), \tag{18}$$

with $g_\mu(k)$ given by

$$g_\mu(k) = \frac{2\pi k_\mu}{e^2(a-1)\,k^2} - \frac{\gamma_5 \tilde{k}_\mu}{k^2(k^2-m^2(a))},$$

In momentum space $\tilde{G}_\mu(p,-p-q,q) \equiv \tilde{G}_\mu(p,q)$

$$\tilde{G}_\mu(p,q) = i\,e\, g_\mu(q) \left[\tilde{G}(p+q) - \tilde{G}(p) \right]. \tag{19}$$

We see that the divergence in this function is due to the fermionic propagator. It can be easily seen that only Green's functions with fermionic legs will have UV divergences. A careful analysis leads us to the conclusion that these UV divergences do not have perturbative origin [23].

3 Regularization

In this section we will regularize the theory using the point of view which is called *gauge non-invariant formalism* [5] where one does not introduce a Wess-Zumino field to restore gauge symmetry [24, 25]. However, at least for the case dealt with in this paper the results are coincident [23]. In the gauge non-invariant formalism, the vector field A_μ is decomposed in his longitudinal and transverse parts, so

$$eA_\mu = \partial_\mu\rho - \tilde{\partial}_\mu\phi \tag{20}$$

We make a change of fermionic field variables, with the purpose to decouple the longitudinal part of A_μ from the fermion field,

$$\psi = e^{i\rho}\psi' \quad , \quad \overline{\psi} = \overline{\psi}' e^{-i\rho}. \tag{21}$$

The fermionic measure is not invariant under (21), and changes as

$$d\psi d\overline{\psi} \;=\; d\psi' d\overline{\psi}' \, \exp\left[\frac{i\,(a-1)}{4\pi}\int dx\; \partial_\mu\rho\partial^\mu\rho\right]. \tag{22}$$

Puting into the generating functional (5), we obtain

$$\begin{aligned} Z[\eta,\overline{\eta},J] \;=\; & \int d\rho d\phi d\psi d\overline{\psi}\; \exp\left[i\int dx\left(\frac{1}{2e^2}\phi\Box^2\phi + \overline{\psi}(i\partial\!\!\!/ - \tilde{\partial\!\!\!/}\phi)\psi + \right.\right. \\ & \left.\left. + \frac{(a-1)}{4\pi}\partial_\mu\rho\partial^\mu\rho + \overline{\eta}e^{i\rho}\psi + \overline{\psi}e^{-i\rho}\eta + \frac{1}{e}J_\mu\partial^\mu\rho - \frac{1}{e}J_\mu\tilde{\partial}^\mu\phi\right)\right]. \end{aligned} \tag{23}$$

The divergence in the fermionic Green's functions arises when we integrate over the longitudinal part of A_μ (the field ρ), due to the bad behavior, in the UV limit, of his propagator (as k^{-2}). To regularize the theory, we notice that we could make everything finite if we had a better UV behavior of the ρ propagator. We can do this by means of Pauli-Villars regularization. We add to the generating functional a new field β which has a large mass Λ^2 [26] ($\Lambda^2 \to +\infty$). This defines the regularized generating functional $Z_\Lambda[\eta,\overline{\eta},J]$, which reduces to $Z[\eta,\overline{\eta},J]$ in the limit of infinite Λ. After manipulations into the regularized generating functional, we get

$$\begin{aligned} Z_\Lambda[\eta,\overline{\eta},J] \;=\; & \int d\rho d\phi d\psi d\overline{\psi}\, \exp\left[i\int dx\left(\frac{1}{2e^2}\phi\Box^2\phi + \overline{\psi}(i\partial\!\!\!/ - \tilde{\partial\!\!\!/}\phi)\psi + \right.\right. \\ & \left.\left. - \frac{(a-1)}{4\pi\Lambda^2}\rho\Box(\Box + \Lambda^2)\rho + \overline{\eta}e^{i\rho}\psi + \overline{\psi}e^{-i\rho}\eta + \frac{1}{e}J\cdot\partial\rho - \frac{1}{e}J\cdot\tilde{\partial}\phi\right)\right]. \end{aligned} \tag{24}$$

Now the propagator of the ρ field has a better UV behavior (k^{-4}, for finite Λ^2). We are interested in the way the original action changes after regularization. Then, we come back to the original fields of the theory,

$$\begin{aligned} \psi &= e^{-i\rho}\psi', \\ \overline{\psi} &= \overline{\psi}' e^{i\rho}, \\ d\psi d\overline{\psi} \;&=\; d\psi' d\overline{\psi}' \, \exp\left[-\frac{i\,(a-1)}{4\pi}\int dx\; \partial_\mu\rho\partial^\mu\rho\right], \end{aligned} \tag{25}$$

and perform the inverse transformations $\rho = \frac{\partial_\mu}{\Box} A^\mu$, $\phi = \frac{\tilde{\partial}_\mu}{\Box} A^\mu$, to get the regularized generating functional in terms of the original fields of the theory (1),

$$Z_\Lambda[\eta,\overline{\eta},J] = \int dA_\mu d\psi d\overline{\psi}\ \exp\left(i \int dx\ \mathcal{L}_\Lambda[\psi,\overline{\psi},A_\mu] + J{\cdot}A + \overline{\eta}\psi + \overline{\psi}\eta \right)\ , \tag{26}$$

where $\mathcal{L}_\Lambda[\psi,\overline{\psi},A_\mu]$ is the Lagrangian density for the regularized theory

$$\mathcal{L}_\Lambda[\psi,\overline{\psi},A_\mu] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/)\psi - \frac{e^2(a-1)}{4\pi\Lambda^2}(\partial{\cdot}A)^2. \tag{27}$$

We see a new term into the Lagrangian density, equivalent to a Lorentz gauge fixing condition with a gauge parameter which is a dependent and proportional to Λ^{-2}. This new term allows us to regularize the full theory.

From (26), we calculate the regularized Green's functions. The photon propagator is

$$\tilde{G}_{\mu\nu}(k;\Lambda) = -\frac{i}{k^2 - m^2(a)}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \frac{i\,2\pi\Lambda^2}{e^2(a-1)}\frac{k_\mu k_\nu}{k^2(k^2-\Lambda^2)}\ , \tag{28}$$

and the 1PI two-point bosonic function is

$$\tilde{\Gamma}^{\mu\nu}(k;\Lambda) = -g^{\mu\nu}(k^2 - m^2(a)) + \left(1 - \frac{e^2(a-1)}{2\pi\Lambda^2}\right)k^\mu k^\nu - \frac{e^2}{\pi}\frac{k^\mu k^\nu}{k^2}\ . \tag{29}$$

Both functions are finite when $\Lambda^2 \to +\infty$, as in the non-regularized theory.

The regularized fermion propagator is given by

$$\begin{aligned} G(x-y;\Lambda) \ &= \ i\ \exp\left\{ \frac{i2\pi\Lambda^2}{a-1}\int \frac{dk}{(2\pi)^2}\frac{1-e^{-ik\cdot(x-y)}}{k^2\,(k^2-\Lambda^2)} \right\} \\ &\times \exp\left\{ -i\,e^2 \int \frac{dk}{(2\pi)^2}\ \frac{1-e^{-ik\cdot(x-y)}}{k^2(k^2-m^2(a))} \right\} S_F(x-y). \end{aligned} \tag{30}$$

In momentum space the regularized fermion propagator $\tilde{G}(p;\Lambda)$ also satisfies the equation

$$\tilde{G}(p;\Lambda) = \frac{i}{p\!\!\!/} - ie^2 \int \frac{dk}{(2\pi)^2}\ f_\Lambda(k)\,\frac{1}{p\!\!\!/}\,k\!\!\!/\,\tilde{G}(p-k;\Lambda), \tag{31}$$

where $f_\Lambda(k)$ is given by

$$f_\Lambda(k) \ = \ \frac{2\pi}{e^2(a-1)}\frac{\Lambda^2}{k^2(k^2-\Lambda^2)} - \frac{1}{k^2(k^2-m^2(a))}. \tag{32}$$

If we expand the above equation for $\tilde{G}(p;\Lambda)$ we get

$$\begin{aligned} \frac{1}{\hbar}\tilde{G}(p;\Lambda) = \frac{i}{p\!\!\!/} \ + \ \hbar\, e^2 \int \frac{dk}{(2\pi)^2}\ f_\Lambda(k)\,\frac{1}{p\!\!\!/}\,k\!\!\!/\,\frac{1}{p\!\!\!/ - k\!\!\!/} \ + \\ -\ i\hbar^2\, e^4 \int \frac{dk}{(2\pi)^2}\frac{ds}{(2\pi)^2}\ f_\Lambda(k) f_\Lambda(s)\frac{1}{p\!\!\!/}k\!\!\!/\frac{1}{p\!\!\!/ - k\!\!\!/}s\!\!\!/\frac{1}{p\!\!\!/ - k\!\!\!/ - s\!\!\!/} \ + \ \mathcal{O}(\hbar^3). \end{aligned} \tag{33}$$

A power counting analysis of the terms above shows that the UV logarithmic divergence is controlled. The regularized 1PI two-point fermionic function is

$$\tilde{\Gamma}(p;\Lambda) \;=\; \not{p}\left[1 + i\hbar\, e^2 \int \frac{dk}{(2\pi)^2}\, f_\Lambda(k)\, \not{k}\, \frac{1}{\not{p}-\not{k}} \;+\; \mathcal{O}(\hbar^2)\right] \tag{34}$$

and the regularized three-point function

$$G_\mu(x,y,z;\Lambda) = i\, e \int \frac{dk}{(2\pi)^2}\, g_\mu^\Lambda(k) \left[\, e^{-ik\cdot(z-x)} - e^{-ik\cdot(z-y)}\,\right] G(x-y;\Lambda), \tag{35}$$

with $g_\mu^\Lambda(k)$ is given by

$$g_\mu^\Lambda(k) = -\frac{2\pi\, \Lambda^2\, k_\mu}{e^2(a-1)\, k^2(k^2-\Lambda^2)} - \frac{\gamma_5 \tilde{k}_\mu}{k^2(k^2-m^2(a))}\ . \tag{36}$$

In momentum space $\tilde{G}_\mu(p,-p-q,q;\Lambda) \equiv \tilde{G}_\mu(p,q;\Lambda)$ is such that

$$\tilde{G}_\mu(p,q;\Lambda) = i\, e\, g_\mu^\Lambda(q) \left[\tilde{G}(p+q;\Lambda) - \tilde{G}(p;\Lambda)\right]\ , \tag{37}$$

and the regularized 1PI three-point function $\tilde{\Gamma}_\mu(p,q)$ is

$$\tilde{\Gamma}_\mu(p,q;\Lambda) = e\, \frac{\gamma_\mu \not{q}}{q^2} \left[\tilde{\Gamma}(p+q;\Lambda) - \tilde{\Gamma}(p;\Lambda)\right]. \tag{38}$$

So we showed that the 1PI three-point function is regularized if the 1PI two-point fermionic function is regularized. We could show, in a similar way, that all the fermionic Green's functions are regularized too. Thus, we only need to renormalize the fermion two-point function, as we will do in the next sections.

4 Ward Identities

We start from the regularized generating functional (26), $Z_\Lambda[\overline{\eta},\eta,J]$. We make the following gauge transformation into this regularized generating functional,

$$\begin{aligned} A_\mu &\;\rightarrow\; A_\mu + \frac{1}{e}\partial_\mu \lambda(x), \\ \psi &\;\rightarrow\; \psi + i\lambda(x)\psi, \\ \overline{\psi} &\;\rightarrow\; \overline{\psi} - i\lambda(x)\overline{\psi}, \end{aligned} \tag{39}$$

with infinitesimal $\lambda(x)$. In our framework, the fermionic measure is gauge non-invariant under a gauge transformation, then it changes as

$$d\psi d\overline{\psi} \;\rightarrow\; d\psi d\overline{\psi}\, \exp\left[-\frac{i\,(a-1)}{2\pi} \int dx \left(\frac{1}{2}\lambda \Box \lambda + e\lambda\, \partial\cdot A\right)\right] \tag{40}$$

With this, we get the Ward identity satisfied by the generating functional of the 1PI functions $\Gamma_\Lambda[\psi, \overline{\psi}, A_\mu]$,

$$i\frac{\delta\Gamma_\Lambda}{\delta\psi(x)}\psi(x) - i\frac{\delta\Gamma_\Lambda}{\delta\overline{\psi}(x)}\overline{\psi}(x) + \frac{1}{e}\partial^\mu_x\frac{\delta\Gamma_\Lambda}{\delta A^\mu(x)} = \frac{e}{2\pi}(a-1)\left(1+\frac{\Box}{\Lambda^2}\right)\partial^\mu_x A_\mu(x). \tag{41}$$

Now we can calculate the Ward identity satisfied by the 1PI two-point bosonic function (in momentum space),

$$k_\mu\tilde{\Gamma}^{\mu\nu}(k;\Lambda) = \frac{e^2}{2\pi}(a-1)\left(1-\frac{k^2}{\Lambda^2}\right)k^\nu. \tag{42}$$

We see the non-transversality of the photon propagator, which is the sign of a gauge anomaly. Then, we obtain another important Ward identity, which we have seen to be satisfied by the 1PI two-point fermionic function and the 1PI three-point function

$$\frac{1}{e}\, q^\mu\,\tilde{\Gamma}_\mu(p,q;\Lambda) = \tilde{\Gamma}(p+q;\Lambda) - \tilde{\Gamma}(p;\Lambda)\ . \tag{43}$$

This identity can be obtained by direct manipulation from equation (38). It shows that we only need to renormalize the 1PI two-point fermionic function. This Ward identity will be important for analysis of the renormalizability of the theory.

5 Renormalization

In the usual way, we express the regularized Lagrangian density (27) in terms of the renormalized quantities and its respective renormalization constants

$$\mathcal{L}_\Lambda = -\frac{1}{4}Z_A\,F_{\mu\nu}F^{\mu\nu} - \frac{Z_e^2}{Z_\psi^2}\frac{e^2(a-1)}{4\pi\Lambda^2}(\partial_\mu A^\mu)^2 + Z_\psi\,\overline{\psi}i\partial\!\!\!/\psi + Z_e\,eA_\mu\overline{\psi}\gamma^\mu\psi\ . \tag{44}$$

We take the a parameter to be finite, Z_A is the renormalization constant of the A_μ field, Z_ψ is the renormalization constant of the ψ field and Z_e is the charge renormalization constant. A_μ and ψ are the renormalized fields, e is the renormalized coupling constant. We define the bare fields A_o^μ and ψ_o, and the bare coupling constant e_o as

$$A_o^\mu = \sqrt{Z_A}A^\mu \quad , \quad \psi_o = \sqrt{Z_\psi}\psi \quad , \quad e_o = \frac{Z_e}{Z_\psi\,Z_A^{1/2}}\,e\ . \tag{45}$$

The object of the renormalization procedure is to determinate Z_ψ, Z_A and Z_e that make all the Green's function of the theory finite. Possible ambiguities in the choice of these constants, will be parameterized through the imposition of renormalization conditions.

The pure bosonic Green's functions do not have UV divergences. Then, we do not need counterterms to renormalize them, which means

$$Z_A = 1. \tag{46}$$

We remember the Ward identity (43) satisfied for 1PI bare functions

$$\frac{1}{e_o}\, q^\mu\, \tilde{\Gamma}_\mu(p,q;\Lambda) \;=\; \tilde{\Gamma}(p+q;\Lambda) - \tilde{\Gamma}(p;\Lambda)\ . \tag{47}$$

Substituting in this equation the relation between the bare and renormalized 1PI functions,

$$\tilde{\Gamma}_R(p) \;=\; Z_\psi\, \tilde{\Gamma}(p;\Lambda) \tag{48}$$

$$\tilde{\Gamma}^\mu_R(p,q) \;=\; Z_\psi\, \tilde{\Gamma}^\mu(p,q;\Lambda)\ , \tag{49}$$

we obtain

$$\frac{1}{e}\, q_\mu\, \tilde{\Gamma}^\mu_R(p,q) \;=\; Z_e\, Z_\psi^{-1} \left[\tilde{\Gamma}_R(p+q) - \tilde{\Gamma}_R(p)\right]. \tag{50}$$

On the other side, we can *verify* that the renormalized functions also satisfy the Ward identity

$$\frac{1}{e}\, q_\mu\, \tilde{\Gamma}^\mu_R(p,q) \;=\; \tilde{\Gamma}_R(p+q) - \tilde{\Gamma}_R(p)\ . \tag{51}$$

If we compare equations (50) and (51), we obtain

$$Z_e \;=\; Z_\psi\ . \tag{52}$$

Coming back to equation (45), and remembering that $Z_A = 1$, we see that

$$e_o = e\ . \tag{53}$$

We see that the coupling constant of the theory is not renormalized, even when the theory is gauge non-invariantly quantized. This shows that the universality of the electromagnetic interaction, usually expressed by $eA^\mu = e_o A^\mu_o$, can be preserved into a gauge non-invariant renormalization scheme. In particular, we see that the coupling constant will not depend on the energy scale μ selected to impose the renormalization conditions. Hence, we have a null Callan-Symanzik beta function

$$\beta = \mu \frac{\partial}{\partial \mu} e(\mu) \quad \Rightarrow \quad \beta = 0\ . \tag{54}$$

5.1 Semi-perturbative analysis

We start again from the Lagrangian density $\mathcal{L}_\Lambda[\psi,\overline{\psi},A_\mu]$ (27),

$$\mathcal{L}_\Lambda[\psi,\overline{\psi},A_\mu] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{e^2(a-1)}{4\pi\Lambda^2}(\partial\!\cdot\! A)^2 + \overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/)\psi\ . \tag{55}$$

With a perturbative calculation in mind, we get the free photon propagator

$$-\,i\tilde{G}^o_{\mu\nu}(k) \;=\; -\frac{i\, g_{\mu\nu}}{k^2} - i\frac{k_\mu k_\nu}{k^4}\left(\frac{2\pi\Lambda^2}{e^2(a-1)} - 1\right)\ . \tag{56}$$

The free photon propagator (56) diverges quadratically in the limit $\Lambda \to +\infty$. If we insert the propagator calculated at *tree level* in the perturbative calculation of the correlation functions, we will make the ultraviolet behavior of the individual graphics worse in each consecutive perturbative order, implicating an apparent non-renormalizability. However, the exact photon propagator does not exhibit that divergence, which is cancelled when we add the terms of the geometric sum that defines it. This shows that, in this anomalous theory, the exact or complete photon propagator has to be considered, the *tree level* of the bosonic sector being indefinite.

However, as we saw [21], the expansions (33) and (34) are equivalent to a loopwise expansion using the exact photon propagator, and in this case a power e^{2n} corresponds to $\hbar^n$, or n loops. We can account for the $\hbar$ powers that appear in the expansions (33) or (34), if we consider that

$$\begin{aligned} \text{fermion propagator} &\to \hbar \\ \text{exact photon propagator} &\to \hbar^2 \\ \text{vertex} &\to \hbar^{-3/2} \end{aligned}$$

5.2 Renormalization to 1-loop order

Now, we can calculate the 1PI functions to 1-loop order (in this semi-perturbative sense) and impose renormalization conditions, to determinate the finite part of the renormalization constants.

The 1PI two point fermion function (34) $\tilde{\Gamma}(p;\Lambda)$ to 1-loop order is

$$\tilde{\Gamma}(p;\Lambda) = \not{p}\left[\,1 + \frac{\hbar}{2(a-1)}\ln\left|1-\frac{\Lambda^2}{p^2}\right| - \frac{\hbar}{2(a+1)}\ln\left|1-\frac{m^2(a)}{p^2}\right| + \mathcal{O}(\hbar^2)\,\right] \tag{57}$$

The renormalized $\tilde{\Gamma}_R(p)$ function is given by

$$\tilde{\Gamma}_R(p) \;=\; Z_\psi\,\tilde{\Gamma}(p;\Lambda)\;, \tag{58}$$

where Z_ψ is the renormalization constant of the fermion field. The next step, is the imposition of the renormalization conditions that can fix the finite part of the renormalization constants.

At this point we must ask the following question: how many renormalization conditions are necessary to parameterize the ambiguities of the theory? The ambiguities in the finite part of correlation functions manifest themselves in two circumstances into the Schwinger model. One of them is in the determination of the Z_ψ constant. Another is present in a massive counterterm $\delta m^2(a)A_\mu A^\mu$, generated dynamically and finite. The difference between the two situations is that the second does not represent the finite residue of any renormalization of mass, since it does not correspond to a multiplicative renormalization (classically, the photon mass is zero). This does not imply, however, that the coefficient of this counterterm does not have to be fixed with the help of renormalization conditions, as happens to any correlation function. We conclude that we need *two* renormalization conditions on the theory, one to parameterize Z_ψ and another to make the same to a (which is obviously related to the massive counterterm mentioned above). The renormalization condition, compatible with the tree level, that will fix Z_ψ (as a function of μ) is easily chosen

$$\tilde{\Gamma}_R(p)\Big|_{\not p=\not \mu} = \not \mu \ . \tag{59}$$

Using it, we get Z_ψ to 1-loop order,

$$Z_\psi \ = \ 1 - \frac{\hbar}{2(a-1)} \ln\left|1 - \frac{\Lambda^2}{\mu^2}\right| + \frac{\hbar}{2(a+1)} \ln\left|1 - \frac{m^2(a)}{\mu^2}\right| + \mathcal{O}(\hbar^2). \tag{60}$$

To this order, we get $\tilde{\Gamma}_R$ as

$$\tilde{\Gamma}_R(p) = \not p \left[1 + \frac{\hbar}{2(a-1)} \ln\left|\frac{1-\Lambda^2/p^2}{1-\Lambda^2/\mu^2}\right| - \frac{\hbar}{2(a+1)} \ln\left|\frac{1-m^2(a)/p^2}{1-m^2(a)/\mu^2}\right| + \mathcal{O}(\hbar^2)\right]. \tag{61}$$

The job is less obvious when we deal with a. We note that, due to equation (56), there is no *tree level* in the bosonic sector because the free photon propagator is divergent. We observe, however, in equation (60), that Z_ψ involves a. We can solve the problem by imposing another renormalization condition over $\tilde{\Gamma}_R(p)$, that may stablish a relationship between a and μ. The more natural condition, compatible with the fermionic *tree level* is

$$\frac{\partial}{\partial \not p}\, \tilde{\Gamma}_R(p)\Big|_{\not p=\not \mu} = 1 \ . \tag{62}$$

Doing this, we obtain the following equation at 1-loop order (for finite Λ^2),

$$\hbar \left(\frac{1}{a-1}\frac{\Lambda^2}{\mu^2-\Lambda^2} - \frac{e^2/2\pi}{\mu^2 - \frac{e^2}{2\pi}(a+1)} \right) + \mathcal{O}(\hbar^2) = 0. \tag{63}$$

To solve this equation, we must consider the following restrictions on the a and μ parameters

i) $a \neq 1$, due to the gauge non-invariant nature of the regularization scheme;

ii) $\mu^2 \neq \Lambda^2$, $\mu^2 \neq \frac{e^2}{2\pi}(a+1)$, points where the function (61) is singular.

So, we can isolate a in this equation,

$$a \ = \ 1 - 2\Lambda^2 \left(\frac{1}{\mu^2} - \frac{1}{m_{gi}^2} \right), \tag{64}$$

where m_{gi}^2 is

$$m_{gi}^2 \ = \ \frac{e^2}{\pi} \ . \tag{65}$$

m_{gi} is the mass that would be generated dynamically if we would have made a gauge invariant quantization, that is, with $a = 1$. The above equation can be solved in two ways:

i) For μ^2 fixed, (64) defines a as a divergent function of Λ^2. This would implicate the need of a mass renormalization for the A_μ field. It does not correspond to the explicit computation of the 1PI two point bosonic function and to the fact that a is finite. This eliminates this solution.

ii) The second option is to parameterize μ^2 as a function of Λ^2 such that equation (64) produces a finite limit when Λ^2 becomes infinite. The most general parameterization is given by

$$\frac{1}{\mu^2} = \frac{1}{m_{gi}^2}\left[1 + \frac{\delta_1}{2}\frac{m_{gi}^2}{\Lambda^2} + \mathcal{O}\left(\frac{1}{\Lambda^4}\right)\right] \tag{66}$$

where δ_1 is a finite coefficient, independent of Λ^2. So a becomes

$$a = 1 - \delta_1 + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) \tag{67}$$

remaing arbitrary, but related to μ^2 through δ_1. However, μ^2 becomes fixed, in the limit $\Lambda^2 \to +\infty$ (that has to be taken in the end of the calculus), as

$$\mu^2 = m_{gi}^2, \tag{68}$$

but, a remains arbitrary. The energy scale, where we have to impose the renormalization conditions, is fixed exactly in the value of the mass dynamically generated when the theory is quantized in a gauge invariant way.

6 Conclusions

We have seen, through the example of the Schwinger model, how to renormalize an anomalous gauge theory. The main feature is that the theory is renormalizable, in the usual sense, if the complete photon propagator can be computed. This could be a good starting point to attack the same question in four dimensions, if we could estimate or take into account the main characteristics of the exact photon propagator.

One of our most interesting results, concerns the dislocation of the renormalization group parameterization from μ to a. This gives us the hope that the S matrix of the theory is independent of a, which would show that one could quantize the theory preserving or not gauge invariance, to obtain the same results. One indication for this is the fact that, in the regularized version of the theory, the dependence on a is completely contained in the cut-off dependent "gauge fixing" parameter. However, one must be careful in analysing this, because of the non-gauge invariant nature of the quantum theory. It is necessary to remember here that we are not fixing the gauge (we do not have the right to do that, it is an anomalous theory), but regularizing divergences of non-perturbative nature.

There are two complementary directions to investigate. One is to look for the new version of the Callan-Symanzik equations, in terms of a. This seems to be a highly non-trivial task, as the bare quantities depend on a (one can remember that an usual way to arrive at the Callan-Symanzik equations is to consider the bare Green's functions as μ-independent, take their derivatives with respect to μ as zero and relate them with the renormalized ones). Perhaps one way out of this is to compute directly the a-dependence of all the quantities involved in the computation of the S matrix.

The other direction is the one that points to the chiral Schwinger model. In this context, as there is no privileged value for a, it would be very interesting to investigate if physical predictions depend or not on it. Of course, it is necessary to first discuss and clear the questions above, in the context of the Schwinger model. Progress in this direction will be reported elsewhere.

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