A NEW FAMILY OF FUZZY OPERATORS AND THE ASSOCIATED BREAK-COLLAPSE METHOD

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ABSTRACT

We introduce a new family of fuzzy operators which satisfies all the standard requirements and contains, as particular cases, both Hamacher and Sugeno algorithms. In addition to this, we show that the Break-collapse Method (used in graph theory for some magnetic models) is applicable for this operators. The procedure is illustrated on the Wheatstone Bridge topology.

Key-words: Fuzzy logic; Hamacher and Sugeno algorithms; Break-collapse method.

1 BASIC DEFINITIONS

For general reviews of Fuzzy Logic see [1, 2, 3].

An operator $n : [0,1] \to [0,1]$ is called a negation (or involutive complement) when it satisfies the following properties:

Boundary conditions: $n(0) = 1$ and $n(1) = 0$.

Nonincreasing monotonicity: $\forall a, b \in [0,1], n(a) \leq n(b)$ if $a > b$.

Involution: $\forall a \in [0,1], n(n(a)) = a$.

An important parameterized family of negations is due to Sugeno (page 39 of [2]):

$$n_\alpha(a) = \frac{1-a}{1+\alpha a} \quad (\alpha > -1). \tag{1}$$

The commutativity, associativity, and monotonicity properties for an operator $\nabla : [0,1] \to [0,1]$ are described as:

Commutativity: $\forall a, b \in [0,1], \nabla(a,b) = \nabla(b,a)$.

Associativity: $\forall a, b, c \in [0,1], \nabla(a, \nabla(b,c)) = \nabla(\nabla(a,b), c)$.

Monotonicity: $\forall a, b, c, d \in [0,1], \nabla(a,b) \leq \nabla(c,d)$ if $a \leq c$ and $b \leq d$.

A operator $\top : [0,1] \to [0,1]$ is called a t-norm when it is commutative, associative, monotonic and has 1 as neutral element:

Neutral element: $\forall a \in [0,1], \top(a,1) = a$.

A operator $\perp : [0,1] \to [0,1]$ is called a t-conorm (or s-norm) when it is commutative, associative, monotonic and has 0 as neutral element:

Neutral element: $\forall a \in [0,1], \perp(a,0) = a$.

From the above properties we derive

Absorbing element: $\forall a \in [0,1], \top(a,0) = 0$.

Absorbing element: $\forall a \in [0,1], \perp(a,1) = 1$.

A t-norm $\top$ and a t-conorm $\perp$ are said to be dual in relation to a negation operation n when they satisfy the De Morgan relations, given by $n(\top(a,b)) = \perp(n(a), n(b))$ and $n(\perp(a,b)) = \top(n(a), n(b))$.

An important parameterized family of t-norms and t-conorms is described by Hamacher[4] operators:

$$\top_\gamma(a,b) = \frac{ab}{\gamma + (1-\gamma)(a+b-ab)} \quad (\gamma > 0) \tag{2}$$

$$\perp_\gamma(a,b) = \frac{a+b+(\gamma-2)ab}{1+(\gamma-1)ab} \tag{3}$$

These operators are dual in relation to the negation operator $n(a) = 1 - a$.

Let $\mu_A, \mu_B : X \to [0,1]$ respectively denote the membership function of fuzzy sets A and B in X. Then the intersection, union and complement of fuzzy sets are given by:

Intersection: $\mu_{A\cap B} = \top(\mu_A(x), \mu_B(x))$

Union: $\mu_{A\cup B} = \perp(\mu_A(x), \mu_B(x))$

Complement: $\mu_{A^c} = n(\mu_A(x))$

2 PROPOSED OPERATORS

Let $a, b \in [0,1]$. The following parameterized families of operators are proposed:

$$\top_{\gamma,\lambda}(a,b) = \frac{ab}{\gamma + (1-\gamma)(a+b-ab)} \quad (\gamma > 0) \tag{4}$$

$$\perp_{\gamma,\lambda}(a,b) = \frac{a+b+(\lambda-2)ab}{1+(\lambda-1)ab} \quad (\lambda > 0) \tag{5}$$

$$n_{\gamma,\lambda}(a) = \frac{\gamma(1-a)}{\gamma + (\lambda-\gamma)a} \quad (\gamma > 0, \lambda > 0) \tag{6}$$

$\top_{\gamma,\lambda}$ is the dual of $\perp_{\gamma,\lambda}$ in relation to $n_{\gamma,\lambda}$.

The proposed families of operators have some interesting characteristics:

. When $\lambda = \gamma$, $\top_{\gamma,\lambda}$ and $\perp_{\gamma,\lambda}$ reproduce[3] Hamacher t-norm and t-conorm operators (i.e. $\perp_{\gamma,\gamma} \equiv \perp_\gamma$ and $\top_{\gamma,\gamma} \equiv \top_\gamma$).

. When $\gamma = 1$ and $\lambda = \alpha+1$, $n_{\gamma,\lambda}$ reproduces Sugeno negation operators (i.e. $n_{1,\alpha+1} \equiv n_\alpha$). In fact, this case precisely corresponds to the composition algorithms[5] associated with the Potts ferromagnet, which in turn contains as particular cases the Ising ferromagnet ($\lambda = 2$), bond percolation ($\lambda = 1$) and resistors($\lambda \to 0$)).

. Equation (4) is the most general ratio of bilinear forms [1] which is commutative, with neutral element 1 and absorbing element 0.

Let the *regularized membership degree* of a be defined as

$$r(a) = \frac{a}{\gamma + (1-\gamma)a} \tag{7}$$

a monotonically increases from 0 to 1 if and only if the same happens for $r(a)$. The inverse transformation of a is given by

$$a = \frac{\gamma\ r(a)}{1 + (\gamma - 1)r(a)} \tag{8}$$

Let us employ the following notation in the remainder of the text: $\top_s \equiv \top_{\gamma,\lambda}(a,b)$, $\perp_p \equiv \perp_{\gamma,\lambda}(a,b)$,$a^c \equiv n_{\gamma,\lambda}(a)$ and $a' \equiv r(a)$ (s stands for *series* and p stands for *parallel*).
Equation (4) can be elegantly rewritten as follows:

$$\frac{\top_s}{\gamma + (1-\gamma)\top_s} = \frac{a}{\gamma + (1-\gamma)a}\ \frac{b}{\gamma + (1-\gamma)b} \tag{9}$$

or, in terms of regularized membership degrees, as

$$\top_s' = a'\ b' \tag{10}$$

which is in the form of the "probabilistic" t-norm!

In relation to the negation we have

$$(a^c)' = \frac{1 - a'}{1 + (\gamma\lambda - 1)a'} \qquad (\lambda > 0) \tag{11}$$

In relation to the proposed t-conorm we have

$$\perp_p' = \frac{\perp_p}{\gamma + (1-\gamma)\perp_p} \tag{12}$$

hence

$$\perp_p = \frac{\gamma\ \perp_p'}{\gamma + (1-\gamma)\perp_p'} \tag{13}$$

We also have that

$$(\perp_p')^c = (a')^c\ (b')^c \tag{14}$$

which is equivalent to

$$\perp_p' = \frac{a' + b' + (\gamma\lambda - 2)a'b'}{1 + (\gamma\lambda - 1)a'b'} \tag{15}$$

which is in turn equivalent to (5).

[1]A function $f(x_1, ..., x_n)$ is said to be a bilinear ratio when $f(x_1, ..., x_n) = \frac{g(x_1,...,x_n)}{h(x_1,...,x_n)}$ such that g and h are linear in relation to each x_i.

3 BREAK-COLLAPSE METHOD

Let us illustrate the Break-collapse Method (BCM[5]) with a particular two-terminal graph, namely the Wheatstone Bridge. This method is however valid for all (planar and nonplanar) two-terminal graphs.

$W \equiv$ *Wheatstone bridge* = two terminals, one "top" and one "bottom"; 5 bonds (a, b, c, d, e) between these two terminals; the "left" bonds are a and b; the "right"bonds are c and d; the central bond is e; bonds a and c join the "top" terminal; bonds b and d join the "bottom" terminal.

$$t_W = \frac{N_W}{D_W} \tag{16}$$

We wish to calculate N_W and D_W through simple topological operations on the graph. We choose anyone of the 5 bonds in order to break-collapse it. The result will NOT depend on our choice! In the present illustration we shall choose the bond e for break-collapsing.

BROKEN GRAPH: we take $e = 0$, hence the graph becomes two parallel branches (between terminals), the "left"one is a series array of the bonds a and b, the "right"one is a series array of the bonds c and d. It follows

$$t_b = \frac{N_b}{D_b} \tag{17}$$

(b stands for *broken*)

We calculate N_b and D_b by using Eqs. (4) and (5). We obtain

$$N_b = ab[\gamma + (1-\gamma)(c+d-cd)] + cd[\gamma + (1-\gamma)(a+b-ab)] + (\lambda-2)abcd \tag{18}$$

and

$$D_b = [\gamma + (1-\gamma)(a+b-ab)][\gamma + (1-\gamma)(c+d-cd)] + (\lambda-1)abcd \tag{19}$$

COLLAPSED GRAPH: we take $e = 1$, hence the graph becomes a series array of two branches (between terminals), the "upper"one is a parallel array of the bonds a and c, the "lower"one is a parallel array of the bonds b and d. It follows

$$t_c = \frac{N_c}{D_c} \tag{20}$$

(c stands for *collapsed*)

We calculate N_c and D_c by using Eqs. (4) and (5). We obtain

$$N_c = [a + c + (\lambda-2)ac][b + d + (\lambda-2)bd] \tag{21}$$

and

$$D_c = \gamma[1 + (\lambda-1)ac][1 + (\lambda-1)bd] + (1-\gamma)(R_1 - R_2) \tag{22}$$

with

$$R_1 \equiv [a + c + (\lambda-2)ac][1 + (\lambda-1)bd] + [b + d + (\lambda-2)bd][1 + (\lambda-1)ac] \tag{23}$$

and

$$R_2 \equiv [a + c + (\lambda-2)ac][b + d + (\lambda-2)bd] \tag{24}$$

Since both N_W and D_W are multilinear functions of the variables a, b, c, d, e, we have that

$$N_W = N_b + e(N_c - N_b) \tag{25}$$

and

$$D_W = D_b + e(D_c - D_b) \tag{26}$$

If we replace the results (18,19,21-24) into Eqs. (25,26) and these into Eq. (16), we have solved, for arbitrary $(a, b, c, d, e, \gamma, \lambda)$, the Wheatstone bridge, which is a nontrivial graph, since it is NOT reducible to only series/parallel operations. This was done by performing simple topological operations on the graph!

4 CONCLUSIONS

We summarize here our main conclusions, namely

(i)Eqs. (4), (5) and (6) implement a new fuzzy transformation which contains both the Hamacher and the Sugeno ones as particular cases.

(ii)Eqs. (4), (6), (16) and (25,26) basically enable the calculation, through topological operations, of ANY two-terminal graph with arbitrary membership degrees.

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