CBPF-NF-080/98

DO-TH 98/21
ESI 98

On the Free-Energy of Three-Dimensional CFTs and Polylogarithms

Anastasios C. Petkou *†
Institut für Physik, Universität Dortmund, D-44221, Germany
and
Marcello B. Silva Neto‡
Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 150 - Urca
CEP:22290-180, Rio de Janeiro, Brazil

Abstract

We discuss the $O(N)$ vector model and the $U(N)$ Gross-Neveu model with fixed total fermion number, for large-N in three dimensions. Using non-trivial polylogarithmic identities, we calculate the renormalized free-energy density of these models at their conformal points in a "slab" geometry with one finite dimension of length L. We briefly comment on the possible implications of our results.

Key-words: Conformal field theory; Polylogarithms.

*email: petkou@catbert.physik.uni-dortmund.de
†Address after 1st December 1998: Department of Theoretical Physics, Aristotle University of Thessaloniki, Thessaloniki 54006, Greece.
‡e-mail: sneto@lafex.cbpf.br

Conformal field theories (CFTs) in dimensions $d > 2$ have recently attracted much interest [1]. Generic results based solely on conformal invariance are not very restrictive in $d > 2$, (see however [2, 3, 4]), and most of the information is extracted by studying explicit models. Although many CFTs have been recently discovered in $d = 4$ [5], a large amount of work has also been devoted to the study of CFT models in $2 < d < 4$, such as the $O(N)$ vector model [6, 7, 8] and the Gross-Neveu model [9] at their conformal points for large-N.

In this letter, we discuss the free-energy density of the $O(N)$ vector model and the $U(N)$ Gross-Neveu model with fixed total fermion number, to leading order in the $1/N$ expansion in $d = 3$. Using non-trivial polylogarithmic identities, we calculate the free-energy density of these models, at their conformal points in a "slab" geometry with one finite dimension of length L. The free-energy density of the three-dimensional $O(N)$ vector model in a "slab" geometry was first calculated in [10, 11]. Recently, polylogarithms have also appeared in calculations of the free-energy density of (super)conformal field theories in $d = 4$ [12].

We begin by reviewing the results of [11]. Consider the partition function of the $O(N)$ vector model in $d = 3$, obtained after integrating out the fundamental scalar fields $\phi^\alpha(x)$, $\alpha = 1, 2, .., N$,

$$Z_B = \int (\mathcal{D}\sigma) \exp\left[-N\, S_{eff}(\sigma, g)\right] , \tag{1}$$

$$S_{eff}(\sigma, g) = \frac{1}{2}\mathrm{Tr}[\ln(-\partial^2 + \sigma)] - \frac{1}{2g}\int \mathrm{d}^3x\, \sigma(x) , \tag{2}$$

where $\sigma(x)$ is the auxiliary scalar field and g the coupling. Setting $\sigma(x) = M^2 + (\mathrm{i}/\sqrt{N})\,\sigma_1(x)$, (1) can be calculated in a renormalisable $1/N$ expansion [13], provided the gap equation

$$\frac{1}{g} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2 + M^2} , \tag{3}$$

is satisfied. A non-trivial CFT, to any fixed order in $1/N$, is obtained by tuning the coupling to the critical value $1/g \equiv 1/g_* = (2\pi)^{-3}\int \mathrm{d}^3p/p^2$ [13]. Then, the renormalised mass (or inverse correlation length) $M(= 1/\xi) = 0$.

When the model is placed in a "slab" geometry with one finite dimension of length L and periodic boundary conditions, the gap equation reads

$$\frac{1}{g} = \frac{1}{L}\sum_{n=-\infty}^{\infty}\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{1}{p^2 + \omega_n^2 + M_L^2} , \tag{4}$$

with the momentum along the finite dimension taking the values $\omega_n = 2\pi n/L$, $n = 0, \pm 1, \pm 2, \ldots$ We may then study the existence of a finite-temperature phase transition in the dimensionally continued version of the model, (i.e. in $d-1$ infinite dimensions) [13, 14, 15]. Since UV renormalisation is insensitive to finite geometry, the coupling constant on the l.h.s. of (4) can be put to its renormalised value in the bulk, which explicitly depends on the mass M_0 of the fundamental scalar fields $\phi^\alpha(x)$. This means that the system is in its $O(N)$-ordered phase for zero "temperature" $T \sim 1/L$. Then, we can obtain an equation which gives the dependence of M_L on M_0 and T. The finite temperature phase transition, corresponding to $O(N)$ symmetry restoration, occurs when $M_L = 0$ for some critical temperature T_*, which in turn is related to M_0 through the above equation. It can be shown that the finite-temperature phase transition cannot take place for $2 < d \leq 3$ in accordance to the Mermin-Wagner-Coleman theorem, but can only occur for $3 < d < 4$. The critical temperature T_* obtained this way agrees [16] with the well-known results, see for example [17]. The theory at T_* is a $d-1$-dimensional CFT, however the OPE structure of its correlation functions is less understood [16].

On the other hand, when the coupling is fixed to its bulk critical value $1/g_*$, (4) has the solution

$$M_L \equiv M_* = \frac{2}{L} \ln\left(\frac{1+\sqrt{5}}{2}\right), \tag{5}$$

which corresponds to the physical situation of finite-size scaling [18] of the correlation length. Then, the subtracted [1] free-energy density for this configuration reads

$$\begin{aligned} \frac{f_\infty - f_L}{N} &= \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \ln p^2 - \frac{1}{2L}\sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \ln(p^2 + \omega_n^2 + M_*^2) + \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{M_*^2}{p^2} \\ &= \frac{M_*^3}{12\pi} - \frac{1}{2\pi L^3}\left[\ln(\mathrm{e}^{-LM_*}) Li_2(\mathrm{e}^{-LM_*}) - Li_3(\mathrm{e}^{-LM_*})\right] \\ &= \frac{4}{5}\frac{\zeta(3)}{2\pi L^3}, \end{aligned} \tag{6}$$

where $Li_n(x)$ are the usual polylogarithms [19]. The third line in (6) follows from the second, by virtue of non-trivial polylogarithmic identities [19] and fits into the general formula [20]

$$f_\infty - f_L = \tilde{c}\,\frac{2\zeta(d)}{S_d L^d}, \tag{7}$$

[1]Since the UV singularities are the same in the bulk and in the finite geometry, subtraction of the bulk free-energy density ensures a UV-finite result.

for the finite-size scaling of the free-energy density in conformal field theories, with $\tilde{c}/N = 4/5$. It is quite remarkable that a rational number appears as the outcome of the calculations in (6). This is reminiscent of free-energy calculations in two-dimensional CFTs, (see [22] for a recent reference). There is strong evidence [21] that correlation functions at the above finite-size critical point can by described by operator product expansions (OPEs) of the bulk $O(N)$ vector model, in accordance with earlier ideas [20]. Note that for free bosons in $d = 3$, $\tilde{c}/N = 1$.

Following the considerations above, it is possible to study the free-energy density of other CFT models. Consider, for example, the $U(N)$ invariant Gross-Neveu model in $d = 3$ whose partition function, after integrating out the fundamental Dirac fermionic fields $\psi^\alpha(x)$, $\bar{\psi}^\alpha(x)$, $\alpha = 1, 2, .., N$, reads

$$Z_F = \int (\mathcal{D}\lambda) \exp\left[-N\, I_{eff}(\lambda, G)\right] , \tag{8}$$

$$I_{eff}(\lambda, G) = \frac{1}{2G}\int \mathrm{d}^3x\, \lambda^2(x) - \mathrm{Tr}[\ln(\not\partial + \lambda)] , \tag{9}$$

where $\lambda(x)$ is the auxiliary scalar field and G is the coupling. We use the notation $\not\partial = \gamma_\mu\partial_\mu$ and the following two-dimensional Hermitian representation for the Euclidean gamma matrices in $d = 3$ [23, 24]

$$\gamma_1 = \sigma^1 , \qquad \gamma_2 = \sigma^2 , \qquad \gamma_3 \equiv \gamma_0 = \sigma^3 , \tag{10}$$

where σ^i, $i = 1, 2, 3$ are the usual Pauli matrices. This model describes fermion mass generation through the breaking of space parity [24]. The partition function (8) can be evaluated in a renormalisable $1/N$ expansion [23] when one sets $\lambda(x) = m + (1/\sqrt{N})\, \lambda_1(x)$, provided the following gap equation is satisfied

$$\frac{1}{G} = 2\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2 + m^2} . \tag{11}$$

At the critical coupling $1/G = 1/G_* = 2(2\pi)^{-d}\int \mathrm{d}^dp/p^2$ the theory is conformally invariant and $m = 0$.

When the system is placed in a "slab" geometry with one finite dimension of length L, the fermions acquire antiperiodic boundary conditions and the gap equation reads

$$\frac{1}{G} = \frac{2}{L}\sum_{n=-\infty}^{\infty}\int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\frac{1}{\mathbf{p}^2 + \omega_n^2 + m_L^2} , \tag{12}$$

with $\omega_n = (2n+1)\pi/L$, $n = 0, \pm 1, \pm 2, ...$ Again, we may study the finite-temperature phase transition in the dimensionally continued version of the model, (i.e. in $d-1$ infinite dimensions) [23, 14], by putting $1/G$ to its bulk renormalised value which explicitly depends on the mass m_0 of the fundamental fermionic fields. This means that the system is in its "broken phase" for zero "temperature" $T \sim 1/L$. Then, we can obtain an equation which gives the dependence of m_L on m_0 and T. The second order finite-temperature phase transition, corresponding to space parity restoration, occurs when $m_L = 0$ for some critical temperature T_*, which is in turn related to the bulk fermion mass m_0. The finite-temperature phase transition is now possible for all $2 < d < 4$ [23, 14], due to the existence of zero modes for fermions and antiperiodic boundary conditions.

On the other hand, when the coupling stays at its bulk critical value $1/G_*$, (12) is satisfied for

$$m_L \equiv m_* = 0. \tag{13}$$

This essentially means that, to leading-N, the free-energy density of the system is given by the free-field theory result [2]. Indeed we easily find

$$\frac{f_\infty - f_L}{N} = \frac{3}{4}\frac{\zeta(3)}{2\pi\, L^3}\,, \tag{14}$$

which implies $\tilde{c}/N = 3/4$, in agreement with the results in [25].

The Gross-Neveu model can also be studied [26, 27] for fixed total fermion number B. To this effect, we introduce a delta-function constraint $\delta(\hat{N} - B)$ into the functional integral (8) at finite temperature, where

$$\hat{N} = \int \mathrm{d}^2\mathbf{x}\, \psi^\dagger(\mathbf{x})\psi(\mathbf{x})\,, \qquad\qquad \mathbf{x} = (x_1, x_2) \tag{15}$$

is the fermion number operator. Using an auxiliary scalar field $\theta(x_3)$, the above delta-function constraint is exponentiated and after integrating out the fermions we obtain the partition function

$$Z_f \;=\; \int (\mathcal{D}\lambda)(\mathcal{D}\theta)\, \exp\left[-N\,\mathcal{I}_{eff}(\lambda, \mathcal{G}; \theta, \tilde{B})\right]\,, \tag{16}$$

$$\mathcal{I}_{eff}(\lambda, \mathcal{G}; \theta, \tilde{B}) \;=\; \mathrm{i}\tilde{B}\int_L \theta(x_3)\,\mathrm{d}x_3 + \frac{1}{2\mathcal{G}}\int_L \mathrm{d}^3x\, \lambda^2(x) - \mathrm{Tr}[\ln(\partial\!\!\!/ + \mathrm{i}\gamma_3\theta + \lambda)]_L\,, \tag{17}$$

[2] There exist solutions of (12) with imaginary m_L, which will be discussed elsewhere.

where $\tilde{B} = B/N$ is assumed to be finite for large-N and the subscript L denotes x_3-integration up to L, the latter plays here the rôle of inverse temperature $1/T$.

For large-N, the functional integral (16) can be calculated by the steepest descent method, since it is dominated by the uniform stationary points $\langle\lambda\rangle$ and $\langle\theta\rangle$ of $\mathcal{I}_{eff}$. These stationary points are obtained as the solutions of the following set of saddle-point equations

$$\frac{1}{\mathcal{G}} = \frac{2}{L}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\frac{1}{\mathbf{p}^2+(\omega_n+\langle\theta\rangle)^2+\langle\lambda\rangle^2}\,, \tag{18}$$

$$i\tilde{b} = \lim_{\tau\to 0}\frac{2}{L}\int\sum_{n=-\infty}^{\infty}\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\frac{e^{\mathrm{i}\omega_n\tau}(\omega_n+\langle\theta\rangle)}{\mathbf{p}^2+(\omega_n+\langle\theta\rangle)^2+\langle\lambda\rangle^2}\,, \tag{19}$$

where $\tilde{b} = (\tilde{B}L/\Omega)$, with Ω the total volume. The regulating term $e^{i\omega_n\tau}$ on the r.h.s of (19) has been discussed in [27] and ensures a finite result in the limit $\tau \to 0$, *after* the Matsubara sum has been performed. For $\langle\theta\rangle$ purely imaginary, which corresponds to having a real chemical potential $\mu = -\mathrm{i}\langle\theta\rangle$ [26, 27], (18) coincides with a similar saddle-point equation obtained in [14, 23]. One can then renormalise (18) by substituting for $1/\mathcal{G}$ the bulk renormalised coupling $1/G$, i.e. from (11), since the presence of $\langle\theta\rangle$ does not alter its UV behavior. In this way, one studies the finite-temperature phase transition of the model, in terms of the renormalised mass of the bulk fermionic fields and the chemical potential. For example, (19) would now give the critical fermion number $\tilde{B}_{cr}$ at which space parity is restored.

Here however, we are interested in possible real values of $\langle\theta\rangle$ which satisfy (18) and (19). The reason is that, if $\langle\theta\rangle$ is a real number, we can put in (18) the coupling $1/\mathcal{G}$ to its bulk critical value $1/G_*$ and obtain the following equation for $\langle\lambda\rangle$

$$L\,\langle\lambda\rangle + \ln\left(1+e^{-L\langle\lambda\rangle-\mathrm{i}L\langle\theta\rangle}\right) + \ln\left(1+e^{-L\langle\lambda\rangle+\mathrm{i}L\langle\theta\rangle}\right) = 0\,. \tag{20}$$

This has a real solution for $\langle\lambda\rangle$ in terms of the "temperature" $T \equiv 1/L$ and $\langle\theta\rangle$, whenever we have $-1 \le \cos(L\langle\theta\rangle) \le -1/2$, or simply $\pi \ge L\langle\theta\rangle \ge 2\pi/3$. Note now that $\langle\lambda\rangle$, which is the renormalised inverse correlation length, is non-zero for $L\langle\theta\rangle \ne 2\pi/3$, which corresponds to a finite-size scaling regime for our fermionic model. The dimensionless quantity $L\langle\lambda\rangle$ is plotted in Fig. 1 for the allowed values of $\langle\theta\rangle$.

The second saddle-point equation (19) ensures a fixed mean fermion number $\langle\hat{N}\rangle = B$, as imposed by the constraint. It turns out that its r.h.s. is real and has a quadratic divergence which should be subtracted. Again, if $\langle\theta\rangle$ is purely imaginary (19) assumes its usual form [28]

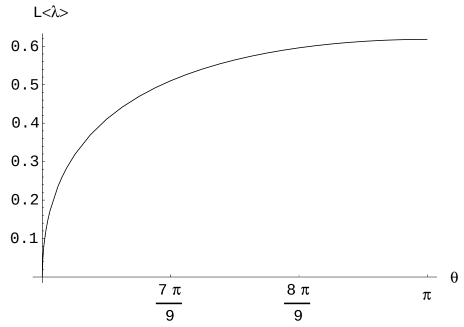

Figure 1: $L\langle\lambda\rangle$ for the allowed region of θ.

as the expression for the conserved charge in a system exchanging particles with a reservoir. It also vanishes for $\mu \equiv -\mathrm{i}\langle\theta\rangle = 0$, as it should, since this would correspond to having no conserved charges.

On the other hand, for real $\langle\theta\rangle$ we obtain, after subtraction of the divergence, a purely imaginary $\tilde{b}$ for the allowed values of $\langle\lambda\rangle$ and $\langle\theta\rangle$, namely

$$\tilde{b} = \frac{\mathrm{i}}{2\pi L^2}\left[Cl_2(2\phi) - Cl_2(2\phi - 2L\langle\theta\rangle) - Cl_2(2L\langle\theta\rangle)\right] , \tag{21}$$

$$\phi = \arctan\left[\frac{e^{-L\langle\theta\rangle}\sin\left(L\langle\lambda\rangle\right)}{1 + e^{-L\langle\lambda\rangle}\cos\left(L\langle\theta\rangle\right)}\right] , \tag{22}$$

where $Cl_2(\omega) = \mathrm{Im}\left[Li_2(e^{\mathrm{i}\omega})\right]$ is Clausen's function [19]. Now, $\tilde{b}$ is related to the total fermion number of the system and, in principle, should be real and positive. Therefore, the only allowed real value for $\langle\theta\rangle$ which satisfies both saddle-point equations (18) and (19) is $\langle\theta\rangle = \pi/L$, for which the r.h.s. of (21) vanishes. In this case, since $\tilde{b} = 0$, it seems that there are no fermions left in the system. This is consistent with the apparent bosonisation of the theory for $\langle\theta\rangle = \pi/L$, which will be discussed shortly. It is also conceivable that the imaginary solutions for $\tilde{b}$ may also have physical meaning, as they give rise to a real value for the free-energy density of the theory. The latter result is rather surprising and is obtained by virtue of non-trivial polylogarithmic identities.

In order to demonstrate the above points, we calculate the free-energy density of the model and we obtain

$$\begin{aligned}\frac{f_\infty - f_L}{N} &= \frac{1}{L}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\ln(\mathbf{p}^2 + (\omega_n + \langle\theta\rangle)^2 + \langle\lambda\rangle^2) - \int\frac{\mathrm{d}^3p}{(2\pi)^3}\ln p^2 - \frac{\langle\lambda\rangle^2}{2G_*} \\ &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\left[\ln\left(\frac{p^2 + \langle\lambda\rangle^2}{p^2}\right) - \frac{\langle\lambda\rangle^2}{p^2}\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{L}\int\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\left\{\ln\left(1+e^{-L\sqrt{\mathbf{p}^2+\langle\lambda\rangle^2}-\mathrm{i}L\langle\theta\rangle}\right)+\ln\left(1+e^{-L\sqrt{\mathbf{p}^2+\langle\lambda\rangle^2}+\mathrm{i}L\langle\theta\rangle}\right)\right\}\\
&+i\Omega\langle\theta\rangle\int\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\left\{\frac{1}{1+e^{L\sqrt{\mathbf{p}^2+\langle\lambda\rangle^2}-iL\langle\theta\rangle}}-\frac{1}{1+e^{L\sqrt{\mathbf{p}^2+\langle\lambda\rangle^2}+iL\langle\theta\rangle}}\right\}\\
=\;&-\frac{\langle\lambda\rangle^3}{6\pi}+\frac{1}{\pi\,L^3}\left[\ln\left(e^{-L\langle\lambda\rangle}\right)Li_2\left(-e^{-L\langle\lambda\rangle},L\langle\theta\rangle\right)-Li_3\left(-e^{-L\langle\lambda\rangle},L\langle\theta\rangle\right)\right]\\
&+\frac{L\langle\theta\rangle}{2\pi L^3}\left[Cl_2(2\phi)-Cl_2(2\phi-2L\langle\theta\rangle)-Cl_2(2L\langle\theta\rangle)\right]\,. \qquad (23)
\end{aligned}
$$

$Li_n(r,\theta)$ is the real part of the polylogarithm $Li_n\left(re^{\mathrm{i}\theta}\right)$ in Lewin's notation [19]. As mentioned before, the free-energy density (23) is real for all values of $\langle\lambda\rangle$ and $\langle\theta\rangle$ obtained from (20) and only for them.

For $\langle\theta\rangle=\pi/L$, (20) becomes simply equation (5) and has the solution $\langle\lambda\rangle=M_L$ (the "golden mean"), which corresponds to the physical situation of finite-size scaling of the correlation length in the $O(N)$ vector model in $d=3$. This apparent "bosonisation" of the Gross-Neveu model happens because the solution $\langle\theta\rangle=\pi/L$ induces a transmutation between Fermi and Bose statistics by removing the fermion zero mode. Also, for this particular value of $\langle\theta\rangle$ we have seen that $\tilde{b}$ is zero. This can be further interpreted as the nonexistence of conserved charges in the bosonized version of the system, in a much similar way as discussed earlier. The bosonisation of the model for $\langle\theta\rangle=\pi/L$ is also discussed in [27, 29] and a possible connection with anyonic physics is made in [30]. In this case, by the same polylogarithmic identities used in (6) we obtain

$$
\frac{f_\infty-f_L}{N}=-\frac{8}{5}\frac{\zeta(3)}{2\pi\,L^3}\,, \qquad (24)
$$

which is consistent with the expected CFT result (7) with $\tilde{c}/N=-8/5$. In fact, (23) is consistent with the scaling form of (7) for all the allowed values of $\langle\lambda\rangle$ and $\langle\theta\rangle$, however the corresponding expressions for $\tilde{c}/N$ involve polylogarithms and Clausen's functions at non-exceptional arguments and are not illuminating. In. Fig. 2 we plot $\tilde{c}/N$ for the allowed values of $\langle\theta\rangle$.

Nevertheless, for the other end-point of the allowed $\langle\theta\rangle$ region, $\langle\theta\rangle=2\pi/3L$, (20) has the solution $\langle\lambda\rangle=0$, which corresponds to infinite correlation length. Remarkably, (23) simplifies considerably for $\langle\theta\rangle=2\pi/3L$ giving

$$
\frac{f_\infty-f_L}{N}=-\frac{1}{2\pi L^3}\left[Cl_2\left(\frac{\pi}{3}\right)-\frac{2}{3}\zeta(3)\right]\,. \qquad (25)
$$

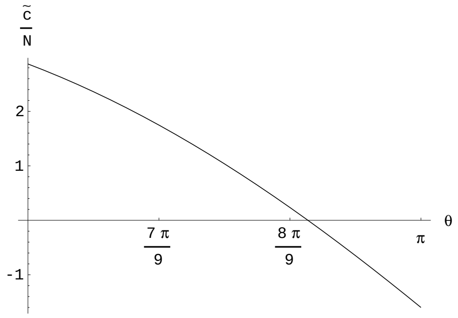

Figure 2: $\tilde{c}/N$ for the allowed region of θ.

It may be interesting to point out that $Cl_2(\pi/3)$ is the absolute maximum of Clausen's function [19].

Consistency of our main result (23) with the scaling form (7) expected from conformal invariance raises the possibility that, for $\pi \geq L\langle\theta\rangle \geq 2\pi/3$ the Gross-Neveu model with fixed total fermion number is related to some conformal field theory. This is most clearly seen for $\langle\theta\rangle = \pi/L$, when from (24) one concludes that the model is related to the $O(N)$ vector model at its finite-size scaling critical point. Note that the free-energy density given by (24) is negative. This shows that such a critical point for the Gross-Neveu model may be unstable by itself. However, it may be a viable critical point in the context of supersymmetric CFTs in $d = 3$. To this end, we note that in the $\mathcal{N} = 1$ supersymmetric σ-model in $d = 3$ [31], the supermultiplet contains Majorana fermions so that the large-N fermionic contribution to the free-energy of the system with fixed total fermion number, is half the r.h.s. of (23). Therefore, from (6) and Fig. 2, we see that the sum of the bosonic and fermionic contributions is always greater that zero and vanishes for $\langle\theta\rangle = \pi/L$.

The relevance of non-trivial polylogarithmic identities to the calculation of free-energy densities in two-dimensional CFTs is well-known (see [22] for a recent reference). Their appearance in studies of CFTs in higher dimensions requires further investigation. It would be interesting to study the OPE structure of correlation functions in the Gross-Neveu model at the above critical points for fixed total fermion number. Although our results simplify for $d = 3$, they can presumably be generalised for all $2 < d < 4$.

Acknowledgments

A. C. P. would like to acknowledge the hospitality of the "Erwin Schrödinger Institute" in Vienna, where this work started. He would also like to acknowledge enlightening conversations with Prof. I. Todorov. This work was partially supported by CAPES, a Brazilian agency for the development of science and by DAAD A/98/19026 Fellowship.

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