CBPF-NF-078/98

Feynman Path Integral as a Quantum-Mechanical Wild-Rosen Turbulent Functional Integral

Luiz C.L. Botelho

Departamento de Física
Universidade Federal Rural do Rio de Janeiro
23851–180 – Itaguaí, RJ, Brasil

We deduce the Feynman Path integral expression for the propagator of a (non relativistic) quantum particle by considering the quantum trajectory satisfying a turbulent equation soluble by means of the Wild-Rosen turbulent functional integral.

PACS numbers: 11.25 Hf; 05.50 + q, 11.10. Kk

An important progress in physics of turbulence was the (formal) solubility of the turbulent motions equations by means of functional integrals called now Wild-Rosen path integrals ([1]).

In this report we propose to deduce the Feynman path integral expression in non relativistic quantum mechanics by considering the Heisenberg indetermination principle as a kind of turbulent relation among the generalized momenta and the usual Newtonian velocities.

Let us start our analysis by considering a classical particle of mass M moving under the presence of a potential $V(\vec{x})$ in R^D with a trajectory $\vec{x}(t)$. Its classical action $S(\vec{x},(t),t) = W(\vec{x}(t)) - Et$, with E denoting the particle energy is such that it satisfies the Hamilton-Jacobi equation on the trajectory

$$\frac{1}{2M}|\vec{\nabla}W|^2(\vec{x}(t)) + V(\vec{x}(t)) = E \tag{1}$$

The generalized momenta associated to the contact transformation generated by $S(\vec{x},t)$ is given by

$$\vec{\rho} = \vec{\nabla}W(\vec{x}(t)) \tag{2}$$

At this point we propose to define that the ensemble of quantum trajectories $\vec{x}(t)$ is defined by a kind of a quantum stochastic equation for the Newtonian velocity (our "Heisenberg" like indetermination quantum principal

$$\frac{d\vec{x}(t)}{dt} - \frac{\vec{\rho}(t)}{M} = \vec{\eta}(t) \tag{3}$$

where $\vec{\eta}(t)$ is an intrinsic white noise Gaussian process (a kind of universal quantum non-relativistic "aether" vaccum) living in the quantum mechanical world whose strength depends on the interaction with the particle by means of the only classical parameter which must be preserved in its integrity after quantization: the classical inertial particle mass

$$\langle \eta_i(t) n_j(t') \rangle = \frac{\hbar}{M}\delta(t-t')\delta_{ij}. \tag{4}$$

Here $\hbar$ is the Planck constant and M the Newtonian particle mass. It is worth to remark that the classical mechanics may be obtained alternatively in our approach by considering the vacuum turn off large mass $M \to \infty$ in relation to the energy quantum mechanical process instead of the usual W.K.B. limit of $\hbar \to 0$.

Let us show that our proposed quantum motion equation (3) leads to the Feynman Quantum Mechanics and consequently to the usual Schrödinger Wave mechanics.

Let us, thus, consider the (formal) realization of the *quantum mechanical noise* above considered by the following Gaussian functional integral generating functional

$$Z[\vec{J}(t)] = \frac{1}{Z(0)} \int D^F[\vec{\eta}(\sigma)] exp\left\{ \frac{i}{\hbar} M \int_0^t d\sigma \frac{1}{2} \left(\frac{d\vec{\eta}(\sigma)}{d\sigma} \right)^2 \right\} exp\left\{ \frac{i}{\hbar} \int_0^t d\sigma \vec{J}(\sigma)\vec{\eta}(\sigma) \right\} \quad (5)$$

At this point it is worth to remark that eq. (5) is *not* the usual mathematical well defined white-noise probabilistic path integral since the objects inside in eq. (5) are not measures on the distributional spaces sampling the stochastic process realization of eq. (4) in space of distributions (Minlos theorem - [4]) that is the reason that we call eq. (4)-eq. (5) by the name of *quantum mechanical noise*!. Anyway, the two-point correlation function associated to the *Feynman* Path Integral eq. (5) still to be given by eq. (4).

Now we re-write eq. (5) in terms of the $\vec{x}(t)$ quantum trajectories space. It is a simple functional variable change to see that ([4])

$$Z[\vec{K}(t)] = \frac{1}{Z(0)} \int D^F[(\vec{x}(\sigma)].det \left[\frac{\delta[\vec{\eta}(\sigma) = \frac{d\vec{x}(\sigma)}{d\sigma} - \frac{\vec{\nabla} W(\vec{x}(\sigma))}{M}]}{\delta \vec{x}(\sigma)} \right]$$
$$exp\left\{ \frac{iM}{2\hbar} \int_0^t d\sigma \left[\frac{d\vec{x}(\sigma)}{d\sigma} - \frac{\vec{\nabla} W(\vec{x}(\sigma))}{M} \right]^2 \right\} exp\left\{ i \int_0^t d\sigma \vec{k}(\sigma).\vec{x}(\sigma) \right\} \quad (6)$$

Where we have used eq. (1), eq. (2) and eq. (3). It is, thus, straightforward to get the following result for eq. (6).

$$Z[\vec{k}(t)] = \frac{1}{Z(0)} \int D^F[\vec{x}(\sigma)] exp\left\{ \frac{i}{\hbar} \int_0^t d\sigma \left(\frac{1}{2} M \frac{d\vec{x}(\sigma)}{d\sigma} \right)^2 \right\}$$
$$exp\left\{ -\frac{i}{\hbar} \int_0^t d\sigma V(\vec{x}(\sigma)) \right\} exp\left\{ \frac{i}{\hbar} \int_0^t \vec{k}(\sigma) \cdot \vec{x}(\sigma) \right\}$$

Note that the functional determinant arising from the variable change eq. (3) is unity (see refs. [1]).

It is worth considering the associated "intercept point probability distribution" ([4] -

pag. 53)

$$\int d\vec{k}_2 \int d\vec{k}_1 e^{-i\vec{k}_1\cdot\vec{x}_1} \cdot e^{+i\vec{k}_2\cdot\vec{x}_2} Z[\vec{K}(\sigma) = [\vec{k}_1 \cdot \delta(\sigma - t_1) + \vec{k}_2\delta(\sigma - t_2)] \stackrel{def}{=} P[(x_1, t_1); (x_2, t_2)] \tag{7}$$

in order to see that eq. (8) is the celebrated the quantum mechanical transitions amplitudes in the Feynman operational framework for Quantum mechanics

$$P[(\vec{x}_1, t_1); (\vec{x}_2, t_2)] = \int_{\vec{x}(t_1)=\vec{x}_1;\vec{x}(t_2)=\vec{x}_2} D^F[\vec{x}(\sigma)] e^{\frac{i}{\hbar}\int_0^t d\sigma\left(\frac{1}{2}M\frac{d\vec{x}(\sigma)}{d\sigma}\right)^2} - e^{-\frac{i}{\hbar}\int_0^t d\sigma V(\vec{x}(\sigma))} \tag{8}$$

It is important remark that all above exposed formal Feynman path integrals procedures can be mathematically well defined if one considers firstly the "Euclidean" Hamilton-Jacobi equation

$$\frac{1}{2M}|\vec{\nabla}W|^2(\vec{x}_E(t)) - V(\vec{x}_E(t)) = E \tag{9}$$

instead of eq. (1) and by secondly, the mathematically well defined functional integral over the Schwartz space topological dual of $C_0^\infty([0,T]$ ([4]).

$$Z[\vec{J}_E(t)] = \frac{1}{Z(0)} \int D^F[\vec{\eta}_E(\sigma)] exp\left\{-\frac{M}{2\hbar}\int_0^t d\sigma \left(\frac{d\vec{\eta}_E(\sigma)}{d\sigma}\right)^2\right\} exp\left\{\frac{i}{\hbar}\int_0^t d\sigma\ \vec{J}_E(\sigma)\vec{\eta}_E(\sigma)\right\} \tag{10}$$

One could proceed exactly as in eqs. (6)-eq. (9) to arrive to the Wick-rotated (difusion) propagator

$$P_{euclidean}[(x_1, t_1); (x_2, t_2)] = \int_{\vec{x}(t_1) = \vec{x}_1; \vec{x}(t) = \vec{x}_2} d\mu^{Wiener}[\vec{x}(\sigma)] exp\left(-\frac{1}{\hbar}\int_0^t\ V(\vec{x}(\sigma))\right) \tag{11}$$

where the Wiener *measure* $d\mu^{Wiener}[\vec{x}(\sigma)]$ is rigorously defined in ref. [4].

These are our proposed results, their generalizations to Quantum field path integrals are straightforward as we show next.

Let us exemplify the theory above proposed in a $\lambda\sigma^4$ scalar (two-dimensional) massless bosonic quantum field in a box $-L \leq x \leq L$ with Dirichlet conditions ([4]).

The infinite variable analogous of the Hamilton-Jacobi equation, after considering the

Fourier expansion $\phi(x,t) = \sum_{n=-\infty}^{+\infty} c_n(t) e^{\frac{2\pi i n x}{L}}$, reads

$$\sum_{n=-\infty}^{+\infty} \left| \frac{\partial W}{\partial C_n}(C_0(t), C_1(t), \cdots, C_n, \cdots) \right|^2$$
$$+ \left(\sum_{n=-\infty}^{\infty} \left(\frac{2\pi n}{L}\right)^2 C_n^2 \right) + \lambda \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} \sum_{n_3=-\infty}^{+\infty} \sum_{n_4=-\infty}^{+\infty}$$
$$C_{n_1}(t) C_{n_2}(t) C_{n_3}(t) C_{n_4}(t)\ \delta(n_1+n_2+n_3+n_4) = E \tag{12}$$

The proposed first order in time and (infinite variable) evolution equation for each normal mode quantum field oscilator is given by

$$\frac{d}{dt} C_n(t) - \left(\frac{\partial}{\partial C_n} \cdot W \right) [C_0(t), \cdots, C_n(t), \cdots] = \eta_n(t) \tag{13}$$

where the normal mode "aether" quantum field oscilator is the white noise process $\left(\eta(x,t) = \sum_{n=-\infty}^{+\infty} \eta_n(t) e^{\frac{2\pi i n x}{L}}\right)$

$$\langle \eta_n(t) \eta_m(t') \rangle = \delta_{nm} \delta(t-t') \tag{14}$$

By following the same analysis exposed in the paper, we arrive at the well-known Feynman path integral representation for the quantum field theory under study in the space-time field representation:

$$Z[J(x,t)] = \frac{1}{Z(0)} \int D^F[\varphi(x,t)]\ exp\left\{ \frac{i}{\hbar} \int_{-\infty}^{+\infty} dt\ \int_{-\infty}^{+\infty} \left(\left(\frac{\partial \varphi}{\partial t}\right)^2 - \left(\frac{\partial \varphi}{\partial x}\right)^2 + \lambda \varphi^4 \right)(x,t) \right\}$$
$$exp\left\{ \frac{i}{\hbar} \int_{-\infty}^{+\infty} dt dx\ J(x,t)\varphi(x,t) \right\} \tag{15}$$

Let us finally stress that new numerical procedures originated from eq. (3) and eq. (4) for quantum path integrals perhaps would be easier to implement than the usual Fast Fourier Transforms for the Feynman path integral eq. (9) or Monte Carlo sampling for the Euclidean path integral eq. (12). Numerical studies are in progress and will appear elsewhere (see the Appendix A for a outline of these computer-oriented proposals).

Acknowledgement

Luiz C.L. Botelho was supported by CNPq, the Brazilian Science Agency.

References

[1] Luiz C.L. Botelho, Mod. Phys. Lett. B5, 391 (1991); ibid B6, 203 (1993).
Brazilian Journal of Physics, **20**, 290 (1991).
Modern Physics Letters B12, 1153 n. 27 (1998).
Luiz C.L. Botelho and Edson P. Silva, International Journal of Modern Physics B12, 2857 (1998).
Modern Phys. Lett. B12, 569 (1998); ibid. B12, 301 (1998); ibid. B12, 1191 (1998).

[2] H. Goldstein, "Classical Mechanics", Addison-Wesley Publishing Company, (1980).

[3] A.S. Monin and A.M. Yaglon, "Statistical Fluid Mechanics" (MIT, Cambridge, MA) – 1971.

[4] B. Simon, "Functional Integration and Quantum Physics", Academic Press INC, 1979.

Appendix A

New Numerical Procedures and Paraxial Wave Propagation

In this appendix, we would like to point out that eq. (3) gives a simple stochastic-numerical procedure to evaluate the Feynman quantum mechanical propagator eq. (6). In order to display this approximate procedure, let us rewrite eq. (6) as the vertex average defined by eq. (3), namely:

$$G[(\vec{x}_1,t_1);(\vec{x}_2,t_2)] = \int d\vec{k}_1\ d\vec{k}_2\ e^{-i\vec{k}\cdot\vec{x}_1} e^{i\vec{k}_2\cdot\vec{x}_2} \left\langle e^{i\vec{k}_1\cdot\vec{x}(t_1)} e^{-i\vec{k}_2\cdot\vec{x}(t_2)} \right\rangle_{\vec{x}} \tag{A.1}$$

Here the quantum (turbulent) trajectory average $\langle\ \rangle_{\vec{x}}$ is defined by the (non-linear) stochastic eq. (3) with the boundary conditions $\vec{x}(t_1) = x_1$ and $\vec{x}(t_2) = x_2$.

If one discretizes the propagation time interval steps of a fixed length, $t_1 - t_2/N = \varepsilon$, one can solve numerically eq. (3) which takes the following discetrized form

$$(\vec{x}_{\ell+1} - \vec{x}_\ell)/\varepsilon + (\vec{\nabla}\cdot W)(\vec{x}_\ell) = \vec{\zeta}(\varepsilon\ell) \overset{def}{\underset{a}{=}} \vec{\eta}(\varepsilon\ell) \tag{A.2}$$

with $1 \leq \ell \leq N$

and, thus, one is able to estimate eq. (A.1) by means of the vertex

$$G[(\vec{x}_1,t_1);(\vec{x}_2,t)] \sim \int d\vec{k}_1\ d\vec{k}_2 \cdot e^{-i\vec{k}_1\cdot\vec{x}_1} \cdot e^{i\vec{k}_2\cdot\vec{x}_2} \left\langle e^{+i\vec{k}_1\cdot\vec{x}_1} \cdot e^{-i\vec{k}_2\cdot\vec{x}_N} \right\rangle_\zeta \tag{A.3}$$

Here $\vec{x}_2 = \vec{x}_N = f(x_1,\zeta(0),\zeta(1),\cdots,\zeta(N))$ is obtained directly from the recurrence relationship eq. (A.2) and $\langle\ \rangle$ is the white-noise Gaussian (latticed) average:

$$\langle\theta(\zeta)\rangle_\zeta = \frac{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}(d\zeta(N))e^{+\frac{i}{\lambda}(\zeta(0))^2}\cdots e^{+\frac{i}{\lambda}(\zeta(N))^2}\theta(\zeta)}{\int_{-\infty}^{+\infty}(d\zeta(0))\cdots\int_{-\infty}^{+\infty}(d\zeta(N))e^{+\frac{i}{\lambda}(\zeta(0))^2}\cdots e^{+\frac{i}{\lambda}(\zeta(N))^2}} \tag{A.4}$$

Finally, we would like to display a application of the results of this note on quantum physics to a classical problem of determinate the electromagnetic strength $E[(x,y);z,t]$ originating from a monochromatic point source and propagating in a medium characterized by a deterministic refractive index $\eta(x,y,z)$ in the half-space $R^3_+ = \{(x,y,z); 0 \leq z < \infty, (x,y) \in R^2\}$.

In the famous Paraxial approximation, this pulse is supposed to have explicitly the following structural form

$$E[(x,y);z,t] = Real\{A[(x,y);z]e^{i(kz-wt)}\} \tag{A.5}$$

where the Paraxial amplitude satisfies the following *two dimensional Schrödinger initial value equation with the depth coordinate playing the role of the time variable*

$$\left(i\frac{\partial}{\partial z} + \frac{1}{2k}\Delta_{(x,y)} - k(1-\eta(x,y,z))\right) A[(x,y);z] = 0 \tag{A.6}$$

Here the initial date condition is supposed to be known, namelly:

$$A[(x,y);z \to 0^+] = A^{(N)}(x,y) \tag{A.7}$$

Now it is straightforward to apply the results of this note to the problem of time-dependent bounded potentials $V(\vec{x},t)$ ($V(\vec{x},t) \in C_c^\infty(R^4)$) by considering instead of eq. (1) in the text, the following time-dependent Hamilton-Jacobi Equation

$$|(\vec{\nabla}W)(\vec{x},t)|^2 + 2\cdot\frac{\partial W}{\partial t}(\vec{x},t) = E - V(x,t) \tag{A.8}$$

with $E = max|U(\vec{x},t)| < \infty$.

At this point, we remark that the numerical analogous of eq. (B.2) is explicitly given by

$$(\vec{x}_{\ell+1} - \vec{x}_\ell)/\varepsilon + (\vec{\nabla}\cdot W)(\vec{x}_\ell, \varepsilon\ell) = \zeta(\varepsilon\ell) \tag{A.9}$$

Finally, we remark that the same numerical-stochastic procedure works within in the context of the diffusion equation propagator eq. (12), even in the case of *randomnes of the potential* $V(\vec{x})$ (see refs. [1]), a potential advantage in comparison to other stochastic methods.