More About Gyroscope Precession in Cylindrically Symmetric Spacetimes

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Using gyroscopes we generalize results obtained for the gravitomagnetic clock effect in the particular case when the exterior spacetime is produced by a rotating dust cylinder to the case when the vacuum spacetime is described by the general cylindrically symmetric Lewis spacetime. Results are contrasted with those obtained for the Kerr space-time.

Key-words: Theory of gravitation; Cylindrical systems; Gyroscope.

1 Introduction

Recently Bonnor and Steadman [1] calculated and analysed the gravitomagnetic clock effect, which is the difference in periods of a test particle moving in prograde and retrogade circular geodesic orbits around the axis of a rotating body. They applied their results to a cylindrically symmetric system produced by van Stockum metric [2] describing a rotating dust cylinder. The exterior spacetime, containing two parameters, is a particular case of the general vacuum stationary cylindrically symmetric Lewis metric [2, 3, 4] containing four parameters. We extend some of their results to the general Lewis spacetime by using the results obtained by us [5] for the gyroscope precession in cylindrically symmetric spacetimes. The clock effect and the gyroscope precession amount to similar physical processes. However, as it will be seen below, using gyroscopes allows for wider class of possible "gedanken" experiments. Indeed, we have to face with two different effects, one is the influence of the rotation of the source on the gravitational field where the gyroscope is placed (the gravitomagnetic effect), which of course is absent in Newtonian theory. The other is related with the fact that the frame of the gyroscope may be rotating, producing a precession in the gyroscope (Thomas-like precession).

2 Precession of a gyroscope moving in a circle around the axis of symmetry

The Lewis metric can be written as

$$ds^2 = -fdt^2 + 2kdtd\phi + e^{\mu}(dr^2 + dz^2) + ld\phi^2, \quad (1)$$

where

$$f = ar^{1-n} - \frac{c^2 r^{1+n}}{n^2 a}, \quad (2)$$

$$k = -Af, \quad (3)$$

$$l = \frac{r^2}{f} - A^2 f, \quad (4)$$

$$e^{\mu} = r^{(n^2-1)/2}, \quad (5)$$

with

$$A = \frac{cr^{1+n}}{naf} + b. \quad (6)$$

We restrict the metric to the Weyl class by considering the parameters n, a, b and c real. These parameters can be either real or complex, and the corresponding

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solutions belong to the Weyl or Lewis classes respectively. Here we restrict our study to the Weyl class (not to confound with Weyl metrics representing static and axially symmetric spacetimes). The parameters n and a being proportional to the Newtonian energy per unit length and the topological defect respectively; while b and c describe the stationarity of the source being proportional to the angular momentum of the source producing a topological defect and the vorticity of the source respectively. Now it is important to stress that the transformations [6]

$$d\tau = \sqrt{a}(dt + bd\phi), \tag{7}$$

$$d\bar{\phi} = \frac{1}{n}[-cdt + (n - bc)d\phi], \tag{8}$$

casts the Weyl class of the Lewis metric into the Levi-Civita metric (static). However the transformations above are not valid globally, and therefore both metrics are equivalent only locally, a fact that can be verified by calculating the corresponding Cartan scalars [7]. In order to globally transform the Weyl class of the Lewis metric into the static Levi-Civita metric, we have to make $b = 0$. Indeed, if $b = 0$ and c is different from zero, (7) gives an admissible transformation for the time coordinate and (8) represents the transformation to a rotating frame (implying thereby that the frame of (1) is itself rotating). In other words, if $b = 0$, (1) is just the exterior line element of a static cylinder, as seen by a rotating observer. However, since rotating frames (as in special relativity) are not expected to cover the whole spacetime and furthermore since the new angle coordinate ranges from $-\infty$ to ∞, it has been argued in the past [7] that both b and c, have to vanish for (7) and (8) to be globally valid. This point of view is also reinforced by the fact that, assuming that only b has to vanish in order to globally cast (1) into Levi-Civita, we are lead to the intriguing result that there is no dragging outside rotating cylinders. We shall recall this question later.

The rotation Ω of the compass of inertia, or the gyroscope, with respect to a rotating frame with angular velocity ω moving around the axis of symmetry given by metric (1) can be easily calculated by using the Rindler-Perlick method [8]. This consists in transforming the angular coordinate ϕ by

$$\phi = \phi' + \omega t, \tag{9}$$

where ω is a constant (observe that (8), with $b = 0$, defines a rotation in the sense opposite to that in (9)). Then the transformed metric is written in a canonical form,

$$ds^2 = -e^{2\Psi}(dt - \omega_i dx^i)^2 + h_{ij}dx^i dx^j, \tag{10}$$

with latin indexes running from 1 to 3 and Ψ, ω_i and h_{ij} depend on the spatial coordinate x^i only (we are omitting primes). Then, it may be shown that the four acceleration A_μ and the rotation three vector Ω^i of the congruence of world lines $x^i =$ constant are given by,

$$A_\mu = (0, \Psi_{,i}), \tag{11}$$

$$\Omega^i = \frac{1}{2}e^{\Psi}(\det h_{mn})^{-1/2}\epsilon^{ijk}\omega_{k,j}, \tag{12}$$

where the comma denotes partial derivative. It is clear from the above that if $\Psi_{,i} = 0$, then particles at rest in the rotating frame follow a circular geodesics. On the other hand, since Ω^i describes the rate of rotation with respect to the proper time at any point at rest in the rotating frame, relative to the local compass of inertia, then $-\Omega^i$ describes the rotation of the compass of inertia (the *gyroscope*) with respect to the rotating frame. Applying (9) to the original frame of (1), with $t = t', r = r'$ and $z = z'$, we cast (1) into the canonical form (10), and obtain (see (45) in [5])

$$\Omega = MNr^{(1-n^2)/4}\left(M^2ar^{1-n} - \frac{N^2r^{1+n}}{n^2a}\right)^{-1}, \tag{13}$$

where

$$M = 1 + b\omega, \quad N = n\omega - c(1 + b\omega). \tag{14}$$

From (13) we can ask if there are ω's for which the gyroscope precession is null. We see from (13) that the gyroscope does not precess if $M = 0$ or $N = 0$ producing $\Omega = 0$ and implying respectively for the angular velocity of the frame

$$\omega_M = -\frac{1}{b}, \quad \omega_N = \frac{c}{n - bc}. \tag{15}$$

The physical meaning of this result will be discussed below. A similar result has been obtained in [1] but in the particular context of van Stockum solution, while our result is general and independent of the source.

The tangential velocity W of the gyroscope moving around the axis of symmetry for metric (1) is given by (see (53) in [9])

$$W = \frac{\omega(fl + k^2)^{1/2}}{f - \omega k}. \tag{16}$$

Substituting (2-5) into (16), we obtain

$$W = \frac{n\omega\chi}{c(1 + b\omega)(1 - c^2\chi^2) + nc\omega\chi^2}, \tag{17}$$

where

$$\chi = \frac{r^n}{na}. \tag{18}$$

The angular velocities (15) give, respectively, from (17) the tangential velocities,

$$W_M = \frac{1}{c\chi}, \quad W_N = c\chi, \tag{19}$$

and we observe that these velocities do not depend upon b in spite of the corresponding angular velocities depend upon b. The Newtonian energy per unit length σ is given, in terms of n, by

$$\sigma = \frac{1}{4}(1-n), \tag{20}$$

and we consider the range $1 > n > -1$ or $0 < \sigma < 1/2$. This range produces physically reasonable cylindrically symmetric sources [7]. However there exist no circular timelike geodesics for $n < 0$, and furthermore it is not clear that $n < 0$ represent cylinders [10]. From (19) we see that when $r \to 0$ produces, for $1 > n > 0$, $W_{M+} \to \infty$ and $W_{N+} \to 0$, while for $0 > n > -1$, $W_{M-} \to 0$ and $W_{N-} \to -\infty$. We discard W_{M+} and W_{N-} as being unphysical. Now, let us suppose that $1 > n > 0$, then Ω vanishes for $\omega = \omega_N$. If furthermore $b = 0$, then it follows at once from (8), that transformation (9) brings the system back to the non-rotating frame (the frame in which the line element is static), thereby explaining the vanishing of the precession. The remarkable fact, however, is that Ω vanishes for ω_N, even if b is different from zero. As for ω_M, we have not a reasonable interpretation, unless we accept that (1) describes a cylinder only if $1 > n > 0$.

Now we study the case of infinite precession, $\Omega \to \infty$, for the gyroscope moving around the axis of symmetry. From (13) we have then

$$r^n = \frac{Mna}{N}, \tag{21}$$

and considering (14), we can rewrite (21) for the angular velocity of the rotating frame,

$$\omega = \frac{1 + c\chi)}{n\chi - b(1 + c\chi)}. \tag{22}$$

The corresponding tangential speed of the gyroscope becomes, using (16), (17) and (22)

$$W = 1, \tag{23}$$

which means that the gyroscope attains infinite precession when its tangential velocity around the axis becomes the light velocity.

3 Precession of a gyroscope at rest

If the gyroscope is at rest in the original lattice, then we have (see (32) in [5])

$$\Omega = \frac{cr^{(1-n)(n-3)/4}}{a(1 - c^2\chi^2)}. \tag{24}$$

(Observe that it is the absolute value of Ω what appears in (31),(32),(33) and (34) in [5].) We see that the precession is infinite if $c\chi = 1$. It is remarkable that for $c\chi = 1$, if the gyroscope is moving around the axis of symmetry, produces a tangential speed of light (19), $W_{N+} = 1$, with null precession, but in this case, while at rest its precession becomes infinite. On the other hand, when $b = 0$ and $c = 0$, i.e., when the Weyl class of Lewis becomes the static Levi-Civita spacetime, the precession of a gyroscope moving around the axis of symmetry results in

$$\Omega = \frac{n\omega r^{(1-n)(n-3)/4}}{a(1 - n^2\omega^2\chi^2)}, \tag{25}$$

with a tangential velocity obtained from (17)

$$W = n\omega\chi. \tag{26}$$

We observe that the gyroscope precession is the same in both cases, (24) and (25), if the angular velocity of the gyroscope, in the Levi-Civita spacetime, is related to the vorticity, of Lewis spacetime, by

$$\omega = \frac{c}{n}. \tag{27}$$

These two equal precessions, (24) and (25), suggest that (if $b = 0$) it is equivalent to measure the precession of a gyroscope at rest with respect to the rotating Lewis source or moving around the corresponding static source. This situation in turn, is a reminiscense of the non-Machian behaviour of Newtonian gravity, where gravitomagnetic effects are absent.

4 Precession of a gyroscope in a locally non rotating frame

Using the transformation

$$d\phi = d\bar{\phi} + \omega dt, \tag{28}$$

where ω is

$$\omega = -\frac{k}{l}, \tag{29}$$

the Lewis metric (1) transforms into a diagonal form near $r = r_0$. This frame is called locally non rotating [11, 12]. ¿From (28) for the Lewis metric (1) we have

$$\omega = \frac{n^3a^2c - 2n^2a^2bc^2 + (bc-n)c^3r_0^{2n} + n^4a^4br_0^{-2n}}{n^4a^2 - 2n^3a^2bc + 2n^2a^2b^2c^2 - (n-bc)c^2r_0^{2n} - n^4a^4b^2r_0^{-2n}}, \tag{30}$$

which can be rewritten with (18),

$$\omega = \frac{(n-bc)c\chi_0^2 + b}{(n-bc)^2\chi_0^2 - b^2}, \tag{31}$$

where $\chi_0 = \chi(r_0)$. The tangential velocity (17) with (31) becomes,

$$W = \frac{(n-bc)c\chi_0^2 + b}{n\chi_0}. \tag{32}$$

and the precession (13) with (31) becomes,

$$\Omega = \frac{b(n-bc)r_0^{(1-n)(n-3)/4}}{a[(n-bc)^2\chi_0^2 - b^2]}. \tag{33}$$

From (31) there are two cases where ω does not depend upon a particular radius r_0 and produce no precession according to (33). These cases are, for $b = 0$

$$\omega = \frac{c}{n}, \;\; W = c\chi_0; \tag{34}$$

and for $bc = n$,

$$\omega = -\frac{1}{b}, \;\; W = \frac{1}{c\chi_0}; \tag{35}$$

where we have included, from (32), the corresponding tangential velocities. We see from (34) that the result corresponds to what we obtained for ω_N in (15) and agreeing with the analysis of the gyroscope at rest (24) compared to the precession in Levi-Civita's spacetime (25). However the case (35), while producing a similar result compared to ω_M in (15), imposes the relation $b = n/c$. When $b \neq 0$ and $b \neq n/c$ the locally non rotating frame produces non null precession.

5 The Kerr spacetime

It is instructive to compare the situation described above with that in the Kerr spacetime. In Boyer-Lindquist coordinates with $\theta = \pi/2$ the Kerr metric has the form (the Kerr parameter a, describing the specific angular momentum, no to be confounded with the parameter a of the Lewis metric),

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 - \frac{4am}{r}dt d\phi + \frac{1}{\Pi}dr^2 + \left(r^2 + a^2 + \frac{2a^2m}{r}\right)d\phi^2. \tag{36}$$

where

$$\Pi = 1 - \frac{2m}{r} + \frac{a^2}{r^2}. \tag{37}$$

Then, applying the Rindler-Perlick method, one obtains after some lengthy calculations

$$e^{2\Psi} = \Lambda, \tag{38}$$
$$\omega_i = (0, 0, \omega_\phi), \tag{39}$$
$$\omega_\phi = \frac{1}{\Lambda}\left[\omega(r^2 + a^2) - \frac{2am}{r}(1 - a\omega)\right], \tag{40}$$
$$h_{rr} = \frac{1}{\Pi}, \tag{41}$$
$$h_{\phi\phi} = \frac{\Pi}{\Lambda}r^2. \tag{42}$$

with

$$\Lambda = 1 - \omega^2(r^2 + a^2) - \frac{2m}{r}(1 - a\omega)^2. \tag{43}$$

Substituting (38-42) into (12) we obtain

$$\Omega = \frac{2}{\Lambda}\left[\omega - \frac{3m}{r}\omega(1 - a\omega) + \frac{am}{r^3}(1 - a\omega)^2\right]. \tag{44}$$

The value of the angular velocity ω for which there is no precession ($\Omega = 0$), is easily obtained from (44) to be

$$\omega = -\frac{r^2(r - 3m) - 2ma^2 - \sqrt{r^4(r - 3m)^2 - 4ma^2r^3}}{2ma(3r^2 + a^2)} \tag{45}$$

which is the same value for which prograde and retrograde circular geodesics have the same period [13], and which leads to the condition of no clock effect in [1], after replacing ω by its expression for a circular geodesic. This result was obtained before [14] and (together with other properties) led some authors to suggest that natural non-rotating observers are those moving with angular velocity (45) (see [13] and references therein). This identification however, is not necessarily correct. In fact observe that a gyroscope at rest in the frame of (36) ($\omega = 0$) will precess unless $a = 0$, reflecting the well known fact that the original frame of (36) is itself rotating with respect to a compass of inertia [8]. Therefore the vanishing of Ω for observers rotating with angular velocity (45) only shows that the gravitational dragging effect of the source exactly cancels the Thomas-like precession due to the rotation of the frame where the gyroscope is placed. A frame, which as shown in [1] rotates relative to distant stars. Under these circumstances it becomes difficult to accept that those observers represent "the most natural standard of non-rotation"

6 Conclusions

We have seen that a gyroscope at rest in the frame of (1) will precess independently of b and in a similar way as a gyroscope moving around a static source with angular velocity given by (27). This result together with the fact that transformations (7) and (8) cast (1) into a static Levi-Civita's line element if $b = 0$, would indicate that the rotation of the source does not affect the gyroscope. However for the gyroscope moving around the source, there exist two possible angular velocities for which there is no precession. The physical meaning of one of them (ω_M) is not understood by the authors, unless the range of n is restricted to $1 > n > 0$, in which case it is discarded. The situation with ω_N is clear if $b = 0$, in which case (9) is just a transformation to the non rotating frame if $\omega = \omega_N$. However if b is not vanishing, then the reasons for the vanishing of Ω are obscure. Finally if we define a locally non rotating frame acording to (28) and (29), then we see that a gyroscope at rest in such a frame will precess according to (33). The origin of this precession is rather surprising if we note that it appears even if $n = a = 1$ (Minkowski) and $c = 0$. But under these conditions, (1) is not the Minkowskian line element corresponding to a rotating frame. So the question here is, what is the nature of b, that makes the gyroscope precess? In the Kerr case we have seen that the frame in which $\Omega = 0$ can hardly be called non-rotating. The difference with Lewis case (with $b = 0$) becomes intelligible, if we note that the frame of (36) with $m = 0$ does not represent a rotating Minkowskian observer, a conclusion confirmed by the fact that (44) with $m = \omega = 0$ yields $\Omega = 0$. However, as mentioned before, the frame of (36) is rotating with respect to a compass of inertia if $m \neq 0$ (yielding $\Omega \neq 0$). This is in contrast with the Lewis case, where Ω is not vanishing for the gyroscope at rest in (1) in the case $n = 1$ (Minkowski).This conspicuous difference in the relation between the source of the field and the rotation, in both cases, seems to suggest, loosely speaking, that the behaviour of the Kerr metric is more "Machian" than that of Lewis.

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