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A Note on the Symplectic Structure on the Dressing Group in the sinh–Gordon Model

by

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Abstract

We analyze the symplectic structure on the dressing group in the sinh–Gordon model by calculating explicitly the Poisson bracket of the dressing group elements which create a generic one soliton solution from the vacuum. Our result is that the bracket between the dressing group elements does not coincide with the Semenov–Tian–Shansky one. However, a factor related to the topological charge interpolates between these two brackets.

Key-words: Integrable models; Sine-Gordon field theory; Dressing simmetry.

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1 Introduction

The solitons are particle like solutions of the integrable non–linear equations [1]. They describe elastic collision of solitary waves. After the interaction, the outgoing waves propagate with the same rapidities as the ingoing ones but with changed phases. The sine–Gordon theory is an example of an integrable model both at the classical and at the quantum level. The quantum scattering matrix was constructed in [2] by using bootstrap methods. It is well known [3] that the quantum sine–Gordon model for certain values of the coupling constant is a massive integrable perturbation of the minimal conformal models in two dimensions.

In [4] the dressing group symmetry was proposed as an alternative approach to solve classical integrable models. It was also argued that the dressing group is a semiclassical limit of the quantum group symmetry of a quantum integrable model. The action of the dressing group is realized via gauge transformations which act on the components of the Lax connection. A non–trivial Poisson bracket known as Semenov–Tian–Shansky bracket was introduced [5] in order to ensure the covariance of the Poisson brackets on the phase space under the dressing group action. The dressing group together with the Semenov–Tian–Shansky bracket becomes a Lie–Poisson group.

In [6] the dressing group elements which generate N–solitons from the vacuum in the sinh–Gordon model and its conformally invariant extension [7] are constructed explicitly. It was also shown that there is a relation between the dressing group and the vertex operator construction of the soliton solutions [8], [6].

In the present note we calculate the Poisson bracket $\{g \overset{\otimes}{,} g\}$ of the dressing group element g which generates a generic one soliton solution in the sinh–Gordon model from the vacuum. We use the fact that the phase space of the one solitons is two dimensional [1], [9]. Surprisingly, the Poisson bracket found by us is *not* identical to the Semenov–Tian–Shansky expression:$[r, g \otimes g]$. Our main results (3.10) and (3.11) suggest that the topological charge interpolates between these two symplectic structures.

We outline the content of the paper. Sec. 2 is devoted to the one soliton solutions of the sinh–Gordon equation and to the construction of the dressing group element which produces the one solitons from the vacuum. In Sec. 3 we present our calculation of the Poisson bracket of the dressing group element and compare it with the Semenov–Tian–Shansky bracket. In Sec. 4 we comment our result and outline the conclusions.

2 The sinh–Gordon equation, the one soliton solutions and the dressin group.

In this chapter we briefly review some basic facts concerning the sinh–Gordon model, its one soliton solutions and the dressing group [1], [6], [4]. We start by recalling the sinh–Gordon equation in two dimensions

$$\begin{aligned} \partial_+\partial_-\varphi &= 2m^2 sh2\varphi \qquad \partial_\pm = \frac{\partial}{\partial x^\pm} \\ x^\pm &= x \pm t \end{aligned} \tag{2.1}$$

It is clear that it has a vacuum solution $\varphi = 0$. The eq. (2.1) is equivalent to the zero–curvature condition $F_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0$ of the connection

$$A_\pm = \pm\partial_\pm \Phi + m e^{ad\Phi}\mathcal{E}_\pm \tag{2.2a}$$

$$\mathcal{E}_\pm = \lambda^{\pm 1}\left(E^+ + E^-\right) \qquad \Phi = \frac{1}{2}\varphi H \tag{2.2b}$$

where $E^\pm$ and H are the generators of the $sl(2)$ Lie algebra

$$[H, E^\pm] = \pm 2E^\pm \qquad [E^+, E^-] = H$$

and λ is the spectral parameter. The components of the Lax connection (2.2a) belong to the loop algebra $\tilde{sl}(2)$. In the *principal gradation* the last is generated by the elements $E_n^{\pm} = \lambda^n E^{\pm}, \quad n \in 2\mathbb{Z} + 1$ and $H_n = \lambda^n H, \quad n \in 2\mathbb{Z}$. The commutation relations are the following

$$[H_k, E_l^{\pm}] = \pm 2 E_{k+l}^{\pm} \quad [E_k^+, E_l^-] = H_{k+l} \tag{2.3}$$

The flatness of the connection (2.2a) implies that there exists a solution of the linear system

$$(\partial_{\pm} + A_{\pm}) T(x^+, x^-, \lambda) = 0 \tag{2.4}$$

which together with the initial condition $T(0,0,\lambda) = 1$ is known in the literature as a normalized transport matrix.

The canonical symplectic structure $\{\partial_t \varphi(x,t), \varphi(y,t)\} = 2\delta(x-y)$ can be equivalently written in the form

$$\begin{aligned} \{A(x,t) \overset{\otimes}{,} A(y,t)\} &= [r, A(x,t) \otimes 1 + 1 \otimes A(y,t)]\delta(x-y) && (2.5a) \\ r &= -\frac{\lambda^2 + \zeta^2}{\lambda^2 - \zeta^2} H \otimes H \\ &- 4 \frac{\lambda\zeta}{\lambda^2 - \zeta^2} \left(E^+ \otimes E^- + E^- \otimes E^+\right) && (2.5b) \end{aligned}$$

where $A = A_+ + A_-$ is the spatial component of the Lax connection; λ and ζ are the spectral parameters corresponding the left and the right tensor factors respectively.

To introduce the one soliton solution we consider the variable $\epsilon^+(x^+, x^-) = \epsilon^+(x)$ whose dependence on the light cone variables is dictaded by the relation

$$\begin{aligned} \frac{\epsilon^+(x) + \mu}{\epsilon^+(x) - \mu} &= a \exp\{2m(\mu x^+ + \frac{x^-}{\mu})\} \\ a &= \frac{\epsilon^+ + \mu}{\epsilon^+ - \mu}, \quad \epsilon^+ = \epsilon^+(0,0) \end{aligned} \tag{2.6}$$

The sinh–Gordon field is expressed in terms of ϵ^+ and the soliton rapidity μ as follows

$$e^{-\varphi(x)} = -\frac{\epsilon^+(x)}{\mu} \tag{2.7}$$

We shall also need the variable $\epsilon^-(x)$ related to $\epsilon^+(x)$ and μ by $\epsilon^+(x)\epsilon^-(x) = \mu^2$ or equivalently

$$\frac{\epsilon^+(x) + \mu}{\epsilon^+(x) - \mu} = -\frac{\epsilon^-(x) + \mu}{\epsilon^-(x) - \mu} \tag{2.8}$$

In order to construct a solution of the linear problem (2.4) one observes that (2.6) is equivalent to

$$\psi_{\pm}(x, \mu) = \pm a \psi_{\pm}(x, -\mu) \tag{2.9}$$

where

$$\psi_{\pm}(x, \lambda) = \left(\epsilon^{\pm}(x) + \lambda\right) e^{-m(\lambda x^+ + \frac{x^-}{\lambda})} \tag{2.10}$$

It was shown in [10] that the matrix

$$\mathcal{T} = e^{\Phi(x)} \begin{pmatrix} \psi_+(x,\lambda) & \psi_+(x,-\lambda) \\ \psi_-(x,\lambda) & -\psi_-(x,-\lambda) \end{pmatrix} \tag{2.11}$$

satisfies the linear system (2.4) with φ being the one soliton solution (2.7). Due to (2.9), the matrix $\mathcal{T}$ as a function on the spectral parameter λ is degenerated at the points $\lambda = \pm\mu$. A direct calculation shows that

$$\det \mathcal{T} = 2(\lambda^2 - \mu^2) \tag{2.12}$$

The normalized transport matrix is $T(x,\lambda) = \mathcal{T}(x,\lambda)\cdot\mathcal{T}^{-1}(0,\lambda)$.

Starting from (2.11) one can construct algebraically solutions of the dressing problem. First we recall that the dressing group element which relates the vacuum to an one soliton solution is introduced by the equation

$$T(x,\lambda) \;=\; g(x,\lambda)\cdot T_0(x,\lambda)\cdot g^{-1}(0,\lambda) \tag{2.13}$$

where $T_0(x,\lambda) = \exp\{-m(x^+\mathcal{E}_+ + x^-\mathcal{E}_-)\}$ is the vacuum solution transport matrix. The last is expressed as $T_0(x,\lambda) = \mathcal{T}_0(x,\lambda)\mathcal{T}_0^{-1}(0,\lambda)$; $\mathcal{T}_0(x,\lambda)$ has the same form as (2.11) but with $\Phi = 0$ and $\psi_+(x,\lambda) = \psi_-(x,\lambda) = e^{-m(\lambda x^+ + \frac{x^-}{\lambda})}$ The element $g(x,\lambda)$ can be easily expressed in terms of the non–normalized transport matrices $\mathcal{T}$ and $\mathcal{T}_0$ as follows

$$g(x,\lambda) \;=\; \mathcal{T}(x,\lambda)\cdot\mathcal{T}_0^{-1}(x,\lambda)\cdot S(\lambda) \tag{2.14a}$$

$$S(\lambda) \;=\; \begin{pmatrix} a(\lambda) & b(\lambda) \\ b(\lambda) & a(\lambda) \end{pmatrix} \tag{2.14b}$$

$$detS(\lambda) \;=\; \frac{1}{\mu^2-\lambda^2} \tag{2.14c}$$

$$a(\lambda) \;=\; a(-\lambda) \quad b(\lambda) = -b(-\lambda) \tag{2.14d}$$

The form of the $x^\pm$–independent matrix $S(\lambda)$ (2.14b) is fixed by the requirement that it should commute with the matrix $T_0(x,\lambda)$. The equation (2.14c) guarantees that the dressing group element $g(x,\lambda)$ has a unit determinat; the condition (2.14d) reflects the fact that it is represented in the principal gradation. Setting $a(\lambda)+b(\lambda) = \lambda-\mu$ and $a(\lambda)-b(\lambda) = -\lambda-\mu$ one recovers the solution constructed in [6]

$$\begin{aligned} g(x,\lambda) \;&=\; \frac{e^{\Phi(x)}}{2(\lambda-\mu)}\begin{pmatrix} \lambda+\epsilon^+(x) & \lambda+\epsilon^+(x) \\ \lambda+\epsilon^-(x) & \lambda+\epsilon^-(x) \end{pmatrix} + \\ &+\; \frac{e^{\Phi(x)}}{2(\lambda+\mu)}\begin{pmatrix} \lambda-\epsilon^+(x) & -\lambda+\epsilon^+(x)) \\ -\lambda+\epsilon^-(x) & \lambda-\epsilon^-(x) \end{pmatrix} \end{aligned} \tag{2.15}$$

We note that the above expression is not the unique which satisfies the following requirement: the matrix elements of $g(x,\lambda)$ are meromorphic functions on the Riemann sphere with only simple poles at the points $\lambda = \pm\mu$. There exist four solutions: $a(\lambda)+b(\lambda) = (-)^p(\lambda-(-)^k\mu)$, $a(\lambda)-b(\lambda) = (-)^{p-1}(\lambda+(-)^k\mu)$ where $p,\ k = 0,\ 1$. These solutions have the following assymptotic behaviour

$$\begin{aligned} g(x,\lambda) \;&=\; (-)^p e^{\Phi(x)} + O(\lambda^{-1}) \;\; \lambda\to\infty \\ g(x,\lambda) \;&=\; (-)^{p+k} e^{-\Phi(x)} + O(\lambda) \;\; \lambda\to 0 \end{aligned} \tag{2.16}$$

The solution (2.15) is good since it turns to e^Φ for $\lambda\to\infty$ and to $e^{-\Phi}$ when $\lambda\to 0$ [4]. More than that it permits to make a relation with the vertex operators.

3 Derivation of the Poisson brackets of the dressing group elements and the Semenov–Tian–Shansky bracket

In this paper we restrict ourselves to calculate the bracket $\{g \overset{\otimes}{,} g\}$ for (2.15) only. The other choices for g mentioned above will be analyzed in [11]. We shall use the coordinates ϵ^+ and μ to parametrize the phase space of the one solitons. It is clear that

$$\{g(\lambda) \overset{\otimes}{,} g(\zeta)\} \;=\; \left(\frac{\partial g(\lambda)}{\partial\epsilon^+}\otimes\frac{\partial g(\zeta)}{\partial\mu} - \frac{\partial g(\lambda)}{\partial\mu}\otimes\frac{\partial g(\zeta)}{\partial\epsilon^+}\right)\{\epsilon^+\,,\,\mu\} \tag{3.1}$$

We recall that (2.15) contains dependence on the variable ϵ^- but the last is a function on ϵ^+ and μ according to (2.8). To get an explicit expression for (3.1) we first obtain

$$\begin{aligned}
\frac{\partial g(\lambda)}{\partial \epsilon^+} g^{-1}(\lambda) &= -\frac{1}{2\epsilon^+(\lambda^2-\mu^2)}\left((\lambda^2+\mu^2)H + 2\lambda\mu(E^+ - E^-)\right) \\
\frac{\partial g(\lambda)}{\partial \mu} g^{-1}(\lambda) &= \left(\frac{\lambda^2+\mu^2}{2\mu(\lambda^2-\mu^2)} + \frac{((\epsilon^+)^2-\mu^2)\lambda^2}{\epsilon^+(\lambda^2-\mu^2)^2}\right) H - \\
-\frac{\lambda\mu(\lambda^2-(\epsilon^+)^2)}{\epsilon^+(\lambda^2-\mu^2)^2}E^+ &- \frac{\epsilon^+}{\mu}\left(\frac{\lambda}{\lambda^2-\mu^2} + 2\frac{\lambda\epsilon^-}{\mu(\lambda^2-(\epsilon^-)^2)}\right)\frac{\lambda^2-(\epsilon^-)^2}{\lambda^2-\mu^2}E^-
\end{aligned} \tag{3.2}$$

In what follows it will be convenient to introduce the following elements of the loop algebra $\tilde{sl}(2)$

$$\begin{aligned}
X_0(\lambda) &= H + 2\frac{\lambda\mu}{\lambda^2+\mu^2}(E^+ - E^-) \\
X_\pm(\lambda) &= H + \left(\frac{\lambda}{\mu}\right)^{\pm 1} E^+ - \left(\frac{\mu}{\lambda}\right)^{\pm 1} E^-
\end{aligned} \tag{3.3}$$

which are eigenvectors of the adjoint action of the element (2.15)

$$\begin{aligned}
g(\lambda)X_0(\lambda)g^{-1}(\lambda) &= X_0(\lambda) \\
g(\lambda)X_\pm(\lambda)g^{-1}(\lambda) &= e^{\pm\varphi}X_\pm(\lambda)
\end{aligned} \tag{3.4}$$

In terms of (3.3) the derivatives (3.2) are expressed as follows

$$\begin{aligned}
\frac{\partial g(\lambda)}{\partial \epsilon^+} g^{-1}(\lambda) &= -\frac{\lambda^2+\mu^2}{2\epsilon^+(\lambda^2-\mu^2)}X_0(\lambda) \\
\frac{\partial g(\lambda)}{\partial \mu} g^{-1}(\lambda) &= \frac{\lambda^2+\mu^2}{2\mu(\lambda^2-\mu^2)}X_0(\lambda) - \\
&- \frac{\lambda^2}{(\lambda^2-\mu^2)^2}\left((\mu+\epsilon^-)X_+(\lambda) - (\mu+\epsilon^+)X_-(\lambda)\right)
\end{aligned} \tag{3.5}$$

Substituing back the above expression into (3.1) we arrive at the expression

$$\begin{aligned}
&\{g(\lambda) \overset{\otimes}{,} g(\zeta)\} \cdot g^{-1}(\lambda) \otimes g^{-1}(\zeta) \quad = \\
&-\{\epsilon^+,\mu\}\frac{\lambda^2(\zeta^2+\mu^2)}{2\epsilon^+(\lambda^2-\mu^2)^2(\zeta^2-\mu^2)}\left((\mu+\epsilon^-)X_+(\lambda) - (\mu+\epsilon^+)X_-(\lambda)\right)\otimes X_0(\zeta) + \\
&+\{\epsilon^+,\mu\}\frac{\zeta^2(\lambda^2+\mu^2)}{2\epsilon^+(\lambda^2-\mu^2)(\zeta^2-\mu^2)^2}X_0(\lambda)\otimes\left((\mu+\epsilon^-)X_+(\zeta) - (\mu+\epsilon^+)X_-(\zeta)\right)
\end{aligned} \tag{3.6}$$

In the basis (3.3) the r–matrix (2.5b) reads

$$\begin{aligned}
r &= -2\frac{\mu^2\lambda^2(\zeta^2+\mu^2)}{(\lambda^2-\mu^2)^2(\zeta^2-\mu^2)}\left(X_+(\lambda)+X_-(\lambda)\right)\otimes X_0(\zeta) + \\
&+ 2\frac{\mu^2\zeta^2(\lambda^2+\mu^2)}{(\lambda^2-\mu^2)(\zeta^2-\mu^2)^2}X_0(\lambda)\otimes\left(X_+(\zeta)+X_-(\zeta)\right) + \ldots
\end{aligned} \tag{3.7}$$

where we have omited terms $X_0 \otimes X_0$ and $X_\pm \otimes X_\mp$ since they do not contribute to the commutator $[r, g\otimes g]$.

Let $h(\lambda)$ be the element (2.15) with $\epsilon^\pm$ replaced by the variables $\eta^\pm$. We shall keep the rapidity unchanged: $\eta^+\eta^- = \mu^2$. Denote by $\tilde{\varphi}$ the corresponding sinh–Gordon field (2.7). Using (3.3), (3.4) and (3.7) we immediately obtain

$$r - h(\lambda)\otimes h(\zeta)\cdot r \cdot h^{-1}\otimes h^{-1}(\zeta) \quad = \tag{3.8}$$

$$\begin{aligned}
&-2\frac{\mu\lambda^2(\zeta^2+\mu^2)}{(\lambda^2-\mu^2)^2}\left((\mu+\eta^-)X_-(\lambda) + (\mu+\eta^+)X_-(\lambda)\right)\otimes X_0(\zeta) + \\
&+2\frac{\mu\zeta^2(\lambda^2+\mu^2)}{(\lambda^2-\mu^2)(\zeta^2-\mu^2)^2}X_0(\lambda)\otimes\left((\mu+\eta^-)X_+(\zeta) + (\mu+\eta^+)X_-(\zeta)\right)
\end{aligned} \tag{3.9}$$

Comparing (3.6) with the above expression we conclude that

$$\{g(\lambda) \stackrel{\otimes}{,} g(\zeta)\} \cdot g^{-1}(\lambda) \otimes g^{-1}(\zeta) \;=\; r - h(\lambda) \otimes h(\zeta) \cdot r \cdot h^{-1}(\lambda) \otimes h^{-1}(\zeta) \tag{3.10}$$

provided that

$$\begin{aligned} \epsilon^+ &= -\eta^+ \\ \{\epsilon^+, \mu\} &= 4\mu\epsilon^+ \frac{\epsilon^+ - \mu}{\epsilon^+ + \mu} \end{aligned} \tag{3.11}$$

The generalization of this result to the case of dressing group elements wich produce solutions with an arbitrary number of solitons will be done in [12].

4 Remarks and Conclusions

This final chapter is devoted to the analysis of our result. First, we recall that the sinh–Gordon model has a conformally invariant extension [7]. To get the equations of motion of it one imposes the zero–curvature condition on the connection (2.2a) in the affine Lie algebra $\hat{sl}(2)$ with $\Phi = \frac{1}{2}\varphi + \eta\hat{d} + \frac{1}{4}\omega\hat{c}$; $\hat{c}$ is the central element and $\hat{d} = \lambda\frac{\partial}{\partial\lambda}$ is the derivation which counts the grades of the elements of the algebra. The fields η and ω are auxiliary and are introduced in order to restore the conformal invariance. The bracket (2.5a) remains valid provided that the r–matrix in (2.5b) is changed as follows $r \rightarrow \hat{r} = r + \hat{c} \otimes \hat{d} + \hat{d} \otimes \hat{c}$. In [6] The N–solitons of this model were studied and it was shown that to get the dressing group elements which create solitons from the vacuum one has to multiply the corresponding dressing group elements in the sinh–Gordon model by a factor which is in the center of the affine Lie group $\hat{SL}(2)$. A generic solution can be expressed as

$$\begin{aligned} e^{-2\Lambda(\Phi(x))} &= \xi_\Lambda(x) \cdot \bar{\xi}_\Lambda(x) \\ \xi_\Lambda(x) &= <\Lambda|e^{-\Phi(x)}T(x) \\ \bar{\xi}_\Lambda(x) &= T^{-1}(x)e^{-\Phi(x)}|\Lambda> \end{aligned} \tag{4.1a}$$

where $|\Lambda>$ is a highest weight vector of the affine Lie algebra $\hat{sl}(2)$ and $<\Lambda|$ is its dual. The action of the dressing group on (4.1a) is the following [4]

$$\begin{aligned} \xi_\Lambda(x) &\rightarrow \xi_\Lambda(x) \cdot g_-^{-1}(0) \\ \bar{\xi}_\Lambda(x) &\rightarrow g_+(0) \cdot \bar{\xi}_\Lambda(x) \end{aligned} \tag{4.2}$$

where g_+ (g_-) is the expansion of the dressing group element around the point $\lambda = 0$ ($\lambda = \infty$).

In [5], [4] the following brackets on the dressing group

$$\begin{aligned} \{g_\pm \stackrel{\otimes}{,} g_\pm\} &= [r^\pm, g_\pm \otimes g_\pm] \\ \{g_+ \stackrel{\otimes}{,} g_-\} &= [r^+, g_+ \otimes g_-] \end{aligned} \tag{4.3}$$

are imposed in order to ensure the covariance of the Poisson brackets of (4.1a) under the dressing group action; in the r. h. s. of (4.3) $r^\pm$ is the expansion of the affine analogue of (2.5b) on powers of $\left(\frac{\lambda}{\zeta}\right)^{\pm 1}$.

Going back to our result we first observe that in order to extend it to the affine case we have only to add terms which contain the central element $\hat{c}$. Therefore, if (4.3) is valid for the affine algebra, it has to be true for the loop algebra calculation *also*. On the other hand, taking into account (2.15) and (3.11) we observe that

$$\begin{aligned} g(\lambda)h^{-1}(\lambda) &= -e^{\Phi-\tilde{\Phi}} + O(\lambda^{-1}) \quad \lambda \rightarrow \infty \\ g(\lambda)h^{-1}(\lambda) &= e^{\Phi-\tilde{\Phi}} + O(\lambda) \quad \lambda \rightarrow 0 \end{aligned}$$

Since the blocks (4.1a) corresponding to the vacuum are proportional to $< \Lambda|$ and $|\Lambda >$ respectively we see that the basic Poisson brackets (i. e. the brackets between the blocks (4.1a)) transform covariantly also under the dressing group action when one assumes (3.10) instead of (4.3).

We shall finish this section with the following remark: looking at (2.15) as a function of the sinh–Gordon field φ one sees that in order to get the r. h. s. of (3.10) one has only to shift φ by $i\pi$. This makes us to believe that in general the symplectic structure on the dressing group coincides with the Semenov–Tian–Shansky one (4.3) up to a shift of the sinh–Gordon field which is related to the topological charge.

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