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ON PURE UNIFORM HOLOMORPHY IN SPACES OF HOLOMORPHIC GERMS

by

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To the memory of Bruno Kramm

ABSTRACT

If E is a complex (DFC)-space (see §2), we show that E leads to pure uniform holomorphy (see §2) if and only if its Fréchet dual space E' is separable (see Theorem 1, where these two conditions have other eight equivalent ones). By using a theorem of Mujica (see §4), we consider the (DFC)-space $\mathcal{H}(K)$ of germs around K of holomorphic $\mathbb{C}$-valued functions, where K is a nonvoid compact subset of a complex metrizable locally convex space E, and $\mathcal{H}(K)$ is endowed with the topology $\mathcal{T}_0$ obtained as an inductive limit of compact-open topologies (see §4). Not only Theorem 1 applies to $\mathcal{H}(K)$, with E replaced by $\mathcal{H}(K)$ in its statement, but also $\mathcal{H}(K)$ leads to pure uniform holomorphy if and only if E is separable (see Theorem 2).

1980 Mathematics Subject Classification: 46G20 Infinite dimensional holomorphy

1. INTRODUCTION

We dealt with the concept of uniform holomorphy [10], [11] of a holomorphic mapping $f: U \to F$ of a nonvoid open subset U of E to F, where E and F are complex locally convex spaces. When U is uniformly open (see §2), the definition of this concept simplifies. We say here that E leads to pure uniform holomorphy if every open subset U of E is uniformly open, and every holomorphic mapping $f: U \to F$ is uniformly holomorphic, for any U and F (see §2). Assume that E is metrizable, K is a nonvoid compact subset of E, and $H(K)$ is the vector space of germs around K of holomorphic $\mathbb{C}$-valued functions, $H(K)$ being endowed with the topology T_o obtained by an inductive limit of compact-open topologies (see §4, as well as Nicodemi [13] who seems to be the first to consider T_o on $H(K)$, and Mujica [8], [9] who has extensively used T_o, and given interesting applications of it). The purpose of this paper is to show that $H(K)$ leads to pure uniform holomorphy if and only if E is separable, or if and only if its Frêchet dual space $H'(K)$ is separable (see Theorem 2). This is done by using Mujica's theorem that $H(K)$ is (DFC)-space (see this concept in §2), and by proving a result about (DFC)-spaces on the equivalence of ten holomorphic, locally convex and topological conditions (see Theorem 1).

2. CONVENTIONS

E denotes a complex Hausdorff locally convex space (Theorem 1 remains true in the real case if we omit the first condition). E is a (DFC)-space if there is an increasing sequence of compact, convex, balanced subsets $K_m (m \in \mathbb{N})$ of E, whose union is E, such that, if α is a seminorm on E whose restriction $\alpha | K_m$ is continuous at 0 for every $m \in \mathbb{N}$, then α is continuous on E. If E is a Fréchet space, its dual space E'_c with the compact-open topology is a (DFC)-space; this follows from the Banach-Dieudonné theorem (see Horvath [6]). Conversely, if E is a (DFC)-space, every bounded subset of E is relatively compact; the dual space E'_c with the compact-open topology and the dual space E'_b with the bounded-open (strong) topology coincide, $E' = E'_c = E'_b$ is a Fréchet space, and E is surjectively isomorphic and homeomorphic to $(E'_c)'_c$ via the natural mapping $E \to (E'_c)'_c$. Concerning (DFC)-spaces, see Hollstein [4], [5], Mujica [7], [8]. They are akin to (DF)-spaces introduced by Grothendieck [3].

We define a subset U of E to be uniformly open if there is a continuous seminorm α on E such that U is α-open. We say that E leads to pure uniform holomorphy if, for every holomorphic mapping $f: U \to F$, where U is a nonvoid open subset of E and F is a complex normed space, there is a continuous seminorm α on E such that U is α-open and f is holomorphic when E is seminormed by α (compare with uniform holomorphy in Nachbin [10], [11]). We refer to Nachbin [11], [12], Noverraz [14], Dineen [2], Colombeau [1] for holomorphy.

3. PURE UNIFORM HOLOMORPHY.

We have

THEOREM 1. If E is a (DFC)-space, the following conditions are equivalent: (1) E leads to pure uniform holomorphy; (2) the Fréchet dual space $E' = E'_c = E'_b$ is separable; (3) every open subset of E is uniformly open; (4) E has a continuous norm; (5) E has a countable separating set of continuous seminorms; (6) E has a countable separating set of continuous linear forms; (7) every open subset of E has a countable basis of its compact subsets; (8) every open subset of E is a countable union of compact subsets; (9) every open subset of E is a Lindelöf space, that is, from any open cover of that subset we can extract a countable subcover of it; (10) every open subset of E is an F_σ-subset of E, that is, that subset is a countable union of closed subsets of E.

PROOF. It will proceed as follows

$$\begin{array}{ccccccc}
 & & & & (2) & \Longrightarrow & (6) \\
 & & & & \Downarrow & \nwarrow & \Downarrow \\
(1) & \Longrightarrow & (3) & \Longrightarrow & (4) & \Longrightarrow & (5) \\
\Uparrow & & & & \Downarrow & & \\
(9) & \Longleftarrow & (8) & \Longleftarrow & (7) & & \\
 & & \Downarrow\Uparrow & & & & \\
 & & (10) & & & &
\end{array}$$

of which implications (1)$\Longrightarrow$(3), (2)$\Longrightarrow$(6), (4)$\Longrightarrow$(5), (6)$\Longrightarrow$(5), (7)$\Longrightarrow$(8), (8)$\Longrightarrow$(9), (8)$\Longrightarrow$(10) are clear. Let us next prove (9)$\Longrightarrow$(1). We claim that, if α_n $(n \in \mathbb{N})$ are continuous seminorms on E, there are a continuous seminorm α on E and real numbers $\lambda_n \geq 0$ $(n \in \mathbb{N})$ such that $\alpha_n \leq \lambda_n \alpha$ $(n \in \mathbb{N})$. In fact, set $c_{mn} = \sup\{\alpha_n(x);\ x \in K_m\} < +\infty$ for m, $n \in \mathbb{N}$, where K_m ($m \in \mathbb{N}$) are as in the definition of a (DFC)-space (see §2). Choose real numbers $\varepsilon_n > 0$ ($n \in \mathbb{N}$) so that, for every $m \in \mathbb{N}$, we have $\varepsilon_n c_{mn} \to 0$ as $n \to \infty$. Define $\alpha: E \to \mathbb{R}$ by $\alpha(x) = \sup\{\varepsilon_n \alpha_n(x); n \in \mathbb{N}\}$ for $x \in E$; since $x \in K_m$ for some $m \in \mathbb{N}$, hence $\varepsilon_n \alpha_n(x) \to 0$ as $n \to \infty$, we have $0 \le \alpha(x) < +\infty$, from which we get that α is a seminorm on E. To prove continuity of α on E, we use that E is a (DFC)-space, and show that $\alpha|K_m$ is continuous at 0 for every $m \in \mathbb{N}$; this follows from $\varepsilon_n \alpha_n \to 0$ as $n \to \infty$ uniformly on K_m for every $m \in \mathbb{N}$, and because each $\varepsilon_n \alpha_n$ ($n \in \mathbb{N}$) is continuous on E and vanishes at 0. Then $\alpha_n \le \lambda_n \alpha$ if we set $\lambda_n = 1/\varepsilon_n$ ($n \in \mathbb{N}$), proving our claim. Let $f: U \to F$ be holomorphic where U is a nonvoid open subset of E and F is a complex normed space. For every $x \in U$, there is a continuous seminorm α_x on E such that the open α_x-ball B_x of center x and radius 1 is contained in U, and the restriction $f|B_x : B_x \to F$ is holomorphic when E is seminormed by α_x. By (9), choose $x_n \in U$ ($n \in \mathbb{N}$) so that U is the union of the B_{x_n} ($n \in \mathbb{N}$). Then find a continuous seminorm α on E and real numbers $\lambda_n \ge 0$ ($n \in \mathbb{N}$) such that $\alpha_{x_n} \le \lambda_n \alpha$ ($n \in \mathbb{N}$). Then U is α-open and f is holomorphic when E is seminormed by α. This proves (1). Let us next prove (5)$\Longrightarrow$(2). By the homeomorphic bijective linear mapping $E \to (E'_c)'_c$, the topology on E is identified to the compact open topology on $(E'_c)'_c$. Thus (5) means that there are compact subsets L_m ($m \in \mathbb{N}$) of E'_c whose union generates a dense vector subspace of E'_c. Since $\mathbb{C}$ and every compact metrizable space L_m ($m \in \mathbb{N}$) are separable, the Fréchet space E'_c is separable. This proves (2). Let us next prove (4)$\Longrightarrow$(7). Fix a continuous norm α on E, by (4). If U is open in E, let F be its complement in E. For $n \in \mathbb{N}$, call K_{mn} the set of all x in K_m such that $\alpha(x-t) \geq 1/n$ for all t in $F \cap K_m$, where K_m $(m \in \mathbb{N})$ are as in the definition of a (DFC)-space (see §2). Given any compact subset K of U, there is $m \in \mathbb{N}$ such that $K \subset K_m$. Since K, $F \cap K_m$ are α-compact and disjoint, there is $n \in \mathbb{N}$ such that $K \subset K_{mn}$. This proves (7). Let us next prove (2)⟹(4). If $\phi_m \in E'_c (m \in \mathbb{N})$, we can find real numbers $\varepsilon_m > 0$ $(m \in \mathbb{N})$ such that $\varepsilon_m \phi_m \to 0$ as $m \to \infty$, and so the set L formed by $\varepsilon_m \phi_m$ $(m \in \mathbb{N})$ and 0 is compact in the Frēchet space E'_c. The continuous seminorm α on E defined by $\alpha(x) = \sup \{|\phi(x)|;\ \phi \in L\}$ for $x \in E$ is a norm, if ϕ_m $(m \in \mathbb{N})$ are chosen to be dense in E'_c, by (2). This proves (4). Let us next prove (3)⟹(4). By (3), the complement U of 0 in E is α-open for some continuous seminorm α on E. It results that α is a norm, proving (4). Let us finally prove (10)⟹(8). This follows from the fact that E is a countable union of the compact subsets K_m $(m \in \mathbb{N})$ used in defining (DFC)-spaces (see §2). All such implications prove the theorem. QED

4. SPACES OF HOLOMORPHIC GERMS

Let $H(U)$ be the vector space of holomorphic $\mathbb{C}$-valued functions on the nonvoid open subset U of E, and $H(K) = \lim\limits_{\to U \supset K} H(U)$ be the vector space of germs around K of elements of $H(U)$ for U containing K. We shall endow $H(K)$ with the inductive limit topology T_o of the compact-open topology T_o on $H(U)$, instead of the topology T_ω on $H(U)$ gotten likewise from the topology T_ω on $H(U)$. A fundamental result due to Mujica [8], used by him in [9], says that, if E is metrizable, then $H(K)$ endowed with

$\mathcal{T}_o$ is a (DFC)-space, an important example of an abstract (DFC)-space, not a priori presented in the concrete form as the dual space of a Fréchet space with the compact-open topology. We have the Fréchet space $\mathcal{H}'(K)$, the dual space of $\mathcal{H}(K)$ with the compact-open topology, identical to the bounded-open (strong) topology. That topology $\mathcal{T}_o$ on $\mathcal{H}(K)$ is defined by the family of all seminorms of the following two types:

1) $p: f \in \mathcal{H}(K) \longmapsto \sup_{n \in \mathbb{N},\ x \in K,\ s \in \varepsilon_n L} \left| \frac{1}{n!} d^n f(x)(s^n) \right| \in \mathbb{R}$

where L is a compact subset of E, $\varepsilon_n \geq 0$ $(m \in \mathbb{N})$ are real numbers with $\varepsilon_n \to 0$ as $n \to \infty$.

2) $q: f \in \mathcal{H}(K) \longmapsto \sup_{k \in \mathbb{N}} 2^{n_k} \left| \sum_{n=0}^{n_k} \frac{1}{n!} \{d^n f(x_k)(s_k{}^n) - d^n f(y_k)(t_k{}^n)\} \right| \in \mathbb{R}$

where $x_k, y_k \in K$, $s_k, t_k \in E$, $s_k \to 0$, $t_k \to 0$ as $k \to 0$, $x_k + s_k = y_k + t_k$, $n_k \in \mathbb{N}$ $(k \in \mathbb{N})$.

THEOREM 2. If E is metrizable and $\mathcal{H}(K)$ is endowed with the topology $\mathcal{T}_o$, so that $\mathcal{H}(K)$ is a (DFC)-space whose topological dual space $\mathcal{H}'(K)$ is a Fréchet space, not only Theorem 1 applies to $\mathcal{H}(K)$ with E replaced by $\mathcal{H}(K)$ in its statement, but also $\mathcal{H}(K)$ leads to pure uniform holomorphy if and only if E is separable.

PROOF. Let E be separable. There is a compact subset L of E which generates a dense vector subspace of E. Fix real numbers $\varepsilon_n > 0$ $(n \in \mathbb{N})$ with $\varepsilon_n \to 0$ as $n \to \infty$. Then the corresponding continuous seminorm p is a norm on $\mathcal{H}(K)$. Hence (4) of Theorem 1 is satisfied by $\mathcal{H}(K)$ in place of E. Let now E be nonseparable. Consider any continuous seminorms p_m, q_m $(m \in \mathbb{N})$ on $\mathcal{H}(K)$ of the preceeding types p, q. We claim that the countable family formed by them is not separating on $\mathcal{H}(K)$. Each p involves a compact subset L of E, so that we have a compact subset L_m of E involved in p_m ($m \in \mathbb{N}$). Each q involves s_k, t_k ($k \in \mathbb{N}$) in E, so that we have s_{km}, t_{km} ($k \in \mathbb{N}$) involved q_m ($m \in \mathbb{N}$). Since E is not separable, and every compact metrizable space is separable, we see that K, L_m ($m \in \mathbb{N}$), and s_{km}, t_{km} ($k, m \in \mathbb{N}$) generate a vector subspace of E which is not dense in it; hence there is a continuous linear form $f \neq 0$ on E vanishing on this vector subspace. We have $df = f$ and $d^n f = 0$ ($n \in \mathbb{N}, n \geq 2$) on E. Consider the germ $f_K \in \mathcal{H}(K)$ of f around K. Then $f_K \neq 0$ as the natural linear mapping $E' \to \mathcal{H}(K)$ is an isomorphism. However $p_m(f_K) = 0$, $q_m(f_k) = 0$ ($m \in \mathbb{N}$). Hence (5) of Theorem 1 is not satisfied by $\mathcal{H}(K)$ in place of E. QED

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