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Non-Linear Supersymmetric σ-Models and their Gauging in the Atiyah-Ward Space-Time

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Abstract

We present a supersymmetric non-linear σ-model built up in the $N = 1$ superspace of Atiyah-Ward space-time. A manifold of the Kähler type comes out that is restricted by a particular decomposition of the Kähler potential. The gauging of the σ-model isometries is also accomplished in superspace.

Key-words: Non-linear σ-models; Supersymmetry.

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1 Introduction

In the past few years, there has been a great deal of attention drawn to the formulation of globally and locally supersymmetric models in Atiyah-Ward space-times. One expects that self-dual (super) Yang-Mills theories in $D = (2+2)$ might act as a potential source of new examples of integrable models [1, 2, 3]. Besides, it is well-known that Atiyah-Ward space-times are the critical target manifolds for string models with 2 supersymmetries in the world-sheet [4] and that they also provide actions for N=1 and N=2 supersymmetric non-Abelian Chern-Simons theory in $D = (2+1)$ by means of a suitable dimensional reduction of a self-dual super-Yang-Mills theory [5].

Supersymmetry in $D = (2+2)$ reveals a number of peculiarities, mainly due to the special properties of spinors in such a space: Majorana-Weyl spinors may be defined [6] and, contrary to the case of $D = (3+1)$, the chirality constraint in superspace is not affected by complex conjugation of superfields. This statement is crucial in the process of building up actions for the matter sector: propagation is achieved only if independent superfields with opposite chiralities mix together [7].

This property of mixing different chirality sectors that are not related to one another by means of a simple complex conjugation has a major influence on the coupling to Yang-Mills superfields, as well as on the formulation of supersymmetric non-linear σ-models. These models, in $D = (3+1)$ dimensions, have played an important role in the coupling of supersymmetric gauge theories to supergravity. This was due to the non-linear nature of the coupling in a supergravity model, that can be interpreted in terms of a supersymmetric non-linear σ-model [8].

In the present work, we aim at an analysis of the geometrical properties of manifolds that may underline the construction of supersymmetric non-linear σ-models in $D = (2+2)$, as much as possible very close to the study of the strong connection that exists between complex manifolds and supersymmetries defined on space-times with a single timelike coordinate [9, 10, 11, 12, 13]. However, working in the Atiyah-Ward space-time brings new features to those formulations. Especifically, in the $N = 1$ formulation of the supersymmetric σ-model in terms of a Kähler manifold, we will be led to assume it as a 4n-dimensional manifold, its Kähler potential being constrained by a certain decomposition. This naturally restricts our manifold to a subclass of the more general possible Kähler manifolds. Our work is organized as follows: in Section 2, we discuss the superspace formulation of the model and establish its connection to Kähler manifolds. In Section 3, we contemplate the description of isometries and geometrical conditions are set that allows us to conclude whether or not there will be obstructions to the gauging of the isometries. The latter is the subject of Section 4, where we also perform the coupling of the σ-model to the Yang-Mills sector of $N = 1$, $D = (2+2)$ supersymmetry. The procedure adopted in Sections 2 and 3 follows very closely the one of ref.[10]. Finally, our Concluding Remarks are cast in Section 5. An Appendix follows, where we set up some useful remarks about Killing vectors in our Kähler space.

2 The Model in Superspace

In our construction, we shall follow the method used by Zumino [9] for deriving a supersymmetric $\sigma-$model action in $D=(3+1)$ dimensions. Here, the scalar fields defining the $\sigma-$model are the lowest components of a set of chiral and antichiral superfields, $(\Phi^i,\Xi^i)(i=1...n)$, which in $D=(2+2)$ are conveniently written as (we adopt the notation and conventions of Ref. [7])

$$\Phi^i = A^i + i\theta\psi^i + i\theta^2 F^i + i\tilde{\theta}\tilde{\partial\!\!\!/}\theta A^i + \frac{1}{2}\theta^2\tilde{\theta}\tilde{\partial\!\!\!/}\psi^i - \frac{1}{4}\theta^2\tilde{\theta}^2\Box A^i \quad , \tag{1}$$

$$\Xi^i = B^i + i\tilde{\theta}\tilde{\chi}^i + i\tilde{\theta}^2 G^i + i\theta\partial\!\!\!/\tilde{\theta}B^i + \frac{1}{2}\tilde{\theta}^2\theta\partial\!\!\!/\tilde{\chi}^i - \frac{1}{4}\theta^2\tilde{\theta}^2\Box B^i \quad , \tag{2}$$

where $A,\ B$ are complex scalars, $\psi,\ \chi$ are Weyl spinors and $F,\ G$ are complex scalar auxiliary fields. It should be noted that, contrary to the $D=(3+1)$ case, complex conjugation does not change chirality, i.e.

$$\begin{aligned} \widetilde{D}_{\dot\alpha}\Phi^i = 0 \quad \text{and} \quad \widetilde{D}_{\dot\alpha}\Phi^{i*} = 0 \ , \\ D_\alpha\Xi^i = 0 \quad \text{and} \quad D_\alpha\Xi^{i*} = 0 \ , \end{aligned} \tag{3}$$

with

$$D_\alpha = \partial_\alpha - i\partial\!\!\!/_{\alpha\dot\alpha}\widetilde{\theta}^{\dot\alpha} \quad \text{and} \quad \widetilde{D}_{\dot\alpha} = \widetilde{\partial}_{\dot\alpha} - i\widetilde{\partial\!\!\!/}_{\dot\alpha\alpha}\theta^\alpha \quad , \tag{4}$$

$$\begin{aligned} \{D_\alpha,\widetilde{D}_{\dot\alpha}\} = -2i\ \sigma^\mu_{\alpha\dot\alpha}\ \partial_\mu\ ,\ \{D_\alpha,D_\beta\} = \{\widetilde{D}_{\dot\alpha},\widetilde{D}_{\dot\beta}\} = 0 \ , \\ [D_\alpha,\partial_\mu] = [\widetilde{D}_{\dot\alpha},\partial_\mu] = 0 \ , \end{aligned}$$

$\Phi^{i*}(\Xi^{i*})$ being the complex conjugates of $\Phi^i(\Xi^i)$. Following Zumino, we take for the supersymmetric action [1]

$$S \ = \ \frac{1}{8}\int d^4x d^2\theta d^2\widetilde{\theta}\ K(\Phi^i,\Xi^i;\Phi^{i*},\Xi^{i*}) \quad , \tag{5}$$

where the potential K is a real function. It is obvious from (5) that we need to take the manifold spanned by the scalar fields as a 4n- dimensional manifold. Terms involving only one chirality, e.g., functions of Φ^i and Φ^{i*} or Ξ^i and Ξ^{i*}, would not provide the kinetic term for the $\sigma-$model. Then, from the component expansion of (5), we get

$$\begin{aligned} S \ = \ 2\int d^4x \Big(& \frac{\partial^2 K}{\partial A^i\partial B^j}\partial_\mu A^i\partial^\mu B^j + \frac{\partial^2 K}{\partial A^i\partial B^{*j}}\partial_\mu A^i\partial^\mu B^{*j} \\ & + \frac{\partial^2 K}{\partial A^{*i}\partial B^j}\partial_\mu A^{*i}\partial^\mu B^j + \frac{\partial^2 K}{\partial A^{*i}\partial B^{*j}}\partial_\mu A^{*i}\partial^\mu B^{*j} + interaction\ \ terms \Big)\ . \end{aligned} \tag{6}$$

[1] $\int d^4x d^2\theta d^2\widetilde{\theta} \equiv \int d^4x D^\alpha \widetilde{D}^{\dot\alpha}\widetilde{D}_{\dot\alpha}D_\alpha$

In the latter expression, we have written only the piece associated to the kinetic term of the complete action, which gives us the metric of the manifold as

$$g_{\mathcal{I}\mathcal{J}} = \begin{pmatrix} \mathbf{0} & \frac{\partial^2 K}{\partial A^i \partial B^j} & \mathbf{0} & \frac{\partial^2 K}{\partial A^i \partial B^{*j}} \\ \frac{\partial^2 K}{\partial B^i \partial A^j} & \mathbf{0} & \frac{\partial^2 K}{\partial B^i \partial A^{*j}} & \mathbf{0} \\ \mathbf{0} & \frac{\partial^2 K}{\partial A^{*i} \partial B^j} & \mathbf{0} & \frac{\partial^2 K}{\partial A^{*i} \partial B^{*j}} \\ \frac{\partial^2 K}{\partial B^{*i} \partial A^j} & \mathbf{0} & \frac{\partial^2 K}{\partial B^{*i} \partial A^{*j}} & \mathbf{0} \end{pmatrix} \quad , \tag{7}$$

where

$$\mathcal{I}, \mathcal{J} = 1, ...4n \;\; and \;\; i, j = 1, ...n \;\; .$$

Equation (7) shows that in a four-dimensional space-time with signature 2+2, it is not necessary that a supersymmetric $\sigma-$model be associated with a Kähler manifold, contrary to what happens in $D = (3+1)$. In fact, a condition for having a Kähler metric is that $g_{\mathcal{I}\mathcal{J}}$ should be hybrid [14] and here this can only be achieved if K admits a decomposition as below:

$$K(\Phi^i, \Xi^i; \Phi^{i*}, \Xi^{i*}) \quad = \quad H(\Phi^i, \Xi^{i*}) \;\; + \;\; H^*(\Phi^{i*}, \Xi^i) \quad . \tag{8}$$

Consequently, if this is the case, the metric turns out to be

$$g_{\mathcal{I}\mathcal{J}} = \begin{pmatrix} \mathbf{0} & g_{I\bar{J}} \\ g_{\bar{I}J} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial^2 H}{\partial A^i \partial B^{j*}} \\ \mathbf{0} & \mathbf{0} & \frac{\partial^2 H^*}{\partial B^i \partial A^{j*}} & \mathbf{0} \\ \mathbf{0} & \frac{\partial^2 H^*}{\partial A^{i*} \partial B^j} & \mathbf{0} & \mathbf{0} \\ \frac{\partial^2 H}{\partial B^{i*} \partial A^j} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad . \tag{9}$$

In the above expression for $g_{\mathcal{I}\mathcal{J}}$, we suceeded in explicitly writing down the off-diagonal structure that characterizes the metric for Kähler manifolds [14]. But, since the holomorphic structure has been partitionned in two disjoint pieces according to (8), we can conclude that the manifold we arrived at is in fact more constrained than a general Kähler manifold. This will become clearer in the next section, when we shall discuss the isometries of this manifold.

It is also interesting to notice that the four-block Kählerian structure in (9) resembles that of a Hyper-Kähler space, although here we do not have the other complex structures (or equivalently, the second supersymmetry) which characterizes this latter space. The analysis of such $N = 2$ models shall be presented elsewhere [15].

With this choice for the potential K, and using the equations of motion to eliminate the auxiliary fields, we get from (5) the full action as

$$\begin{aligned} S \;\; = \;\; \int d^4x \Big(\; & 2\, h_{i\bar{\hat{j}}} \partial_\mu A^i \partial^\mu B^{*j} + 2\, h^*_{\bar{i}\hat{j}} \partial_\mu A^{*i} \partial^\mu B^j - \frac{1}{2} i \; h_{i\bar{\hat{j}}} \tilde{\chi}^{cj} \tilde{\sigma}^\mu \mathcal{D}_\mu \psi^i \\ & - \frac{1}{2} i \; h_{i\bar{\hat{j}}} \psi^i \sigma^\mu \mathcal{D}_\mu \tilde{\chi}^{cj} - \frac{1}{2} i \; h^*_{\bar{i}\hat{j}} \tilde{\chi}^j \tilde{\sigma}^\mu \mathcal{D}_\mu \psi^{ci} - \frac{1}{2} i \; h^*_{\bar{i}\hat{j}} \psi^{ci} \sigma^\mu \mathcal{D}_\mu \tilde{\chi}^j \\ & - \frac{1}{8} (\; h^{k\bar{\hat{l}}} \partial_{\bar{\hat{i}}} h_{k\bar{\hat{j}}} \partial_m h_{n\bar{\hat{l}}} - \partial_m \partial_{\bar{\hat{i}}} h_{n\bar{\hat{j}}}) \tilde{\chi}^{ci} \tilde{\chi}^{cj} \psi^m \psi^n \\ & - \frac{1}{8} (\; h^{*\bar{k}\hat{l}} \partial_{\hat{i}} h^*_{\bar{k}\hat{j}} \partial_{\bar{m}} h^*_{\bar{n}\hat{l}} - \partial_{\bar{m}} \partial_{\hat{i}} h^*_{\bar{n}\hat{j}}) \tilde{\chi}^i \tilde{\chi}^j \psi^{cm} \psi^{cn} \Big) \quad , \end{aligned} \tag{10}$$

where we have denoted

$$\begin{cases} \hat{i} = i + n, \ \ \bar{i} = i + 2n, \ \ and \ \bar{\hat{i}} = i + 3n \\ X_{\hat{i}} = B_i, \ \ X_{\bar{i}} = A_i{}^*, \ \ X_{\bar{\hat{i}}} = B_i{}^* \ \ . \end{cases}$$

The components of the metric were written as

$$h_{i\bar{j}} = \frac{\partial^2 H}{\partial A^i \partial B^{*j}} \ \ , \quad h^*_{\hat{i}\bar{j}} = \frac{\partial^2 H^*}{\partial B^i \partial A^{*j}} \ \ , \quad h^*_{\bar{i}\hat{j}} = \frac{\partial^2 H^*}{\partial A^{*i} \partial B^j} \ \ , \quad h_{\bar{\hat{i}}j} = \frac{\partial^2 H}{\partial B^{*i} \partial A^j} \ \ , \tag{11}$$

and the covariant derivatives for the fermions are directly read off:

$$\begin{cases} \mathcal{D}_\mu \psi^i = \partial_\mu \psi^i + h^{i\bar{\hat{l}}} \partial_k h_{j\bar{\hat{l}}} \psi^k \partial_\mu A^j \ \ , \\ \mathcal{D}_\mu \tilde{\chi}^{ci} = \partial_\mu \tilde{\chi}^{ci} + h^{l\bar{\hat{i}}} \partial_{\bar{\hat{k}}} h_{l\bar{\hat{j}}} \tilde{\chi}^{ck} \partial_\mu B^{*j} \ \ , \\ \mathcal{D}_\mu \psi^{ci} = \partial_\mu \psi^{ci} + h^{*\bar{i}\hat{l}} \partial_{\bar{k}} h^*_{\bar{j}\hat{l}} \psi^{ck} \partial_\mu A^{*j} \ \ , \\ \mathcal{D}_\mu \tilde{\chi}^i = \partial_\mu \tilde{\chi}^i + h^{*\bar{l}\hat{i}} \partial_{\hat{k}} h^*_{\bar{l}\hat{j}} \tilde{\chi}^k \partial_\mu B^j \ \ . \end{cases}$$

In the above expressions $\psi^c \equiv i\sigma_z \psi^*$ and $\tilde{\chi}^c \equiv i\sigma_z \tilde{\chi}^*$ [7]. Using the, we get:

$$\begin{aligned} -\frac{1}{2} i \ h_{i\bar{\hat{j}}} \tilde{\chi}^{cj} \tilde{\sigma}^\mu \mathcal{D}_\mu \psi^i - \frac{1}{2} i \ h^*_{\bar{i}\hat{j}} \tilde{\chi}^j \tilde{\sigma}^\mu \mathcal{D}_\mu \psi^{ci} \ &= \ 2Re\Big\{ -\frac{1}{2} i \ h_{i\bar{\hat{j}}} \tilde{\chi}^{cj} \tilde{\sigma}^\mu \mathcal{D}_\mu \psi^i \Big\} \\ \Big(\tilde{\chi}^{ci} \tilde{\chi}^{cj} \psi^m \psi^n \Big)^* \ &= \ \chi^i \chi^j \psi^{cm} \psi^{cn} \end{aligned}$$

from which we can easily conclude for the reality of the action.

We can get a very simplified expression if we introduce [2]

$$\Psi^I_A = \begin{pmatrix} \psi^i_\alpha \\ \tilde{\chi}^i_{\dot{\alpha}} \end{pmatrix}, \quad \Psi^{\bar{I}}_A = \begin{pmatrix} \psi^{ci}_\alpha \\ \tilde{\chi}^{ci}_{\dot{\alpha}} \end{pmatrix}, \quad Z^I = \begin{pmatrix} A^i \\ B^i \end{pmatrix}, \quad Z^{\bar{I}} = \begin{pmatrix} A^{*i} \\ B^{*i} \end{pmatrix} \ \ , \tag{12}$$

and the matrix

$$\gamma^\mu_{AB} = \begin{pmatrix} \mathbf{0} & \sigma^\mu_{\alpha\dot{\beta}} \\ \tilde{\sigma}^\mu_{\dot{\alpha}\beta} & \mathbf{0} \end{pmatrix} \ \ (A = \{\alpha, \dot{\alpha}\}, \ \ B = \{\beta, \dot{\beta}\}) \tag{13}$$

Then, the action (10) becomes

$$\begin{aligned} S \ &= \ \int d^4x \ \Big(2g_{I\bar{J}} \ \partial_\mu Z^I \partial^\mu Z^{\bar{J}} - \frac{i}{2} g_{\bar{I}J} \ \Psi^{\bar{I}} \gamma^\mu \mathcal{D}_\mu \Psi^J - \frac{i}{2} g_{I\bar{J}} \ \Psi^I \gamma^\mu \mathcal{D}_\mu \Psi^{\bar{J}} \\ &\quad + \frac{1}{8} R_{I\bar{M}J\bar{N}} \ \Psi^{\bar{M}} \Psi^{\bar{N}} \Psi^I \Psi^J \Big) \ \ , \end{aligned} \tag{14}$$

where

$$\begin{aligned} \mathcal{D}_\mu \Psi^{IA} \ &= \ \partial_\mu \Psi^{IA} + g^{I\bar{L}} \ \partial_K \ g_{J\bar{L}} \ \Psi^{KA} \partial_\mu Z^J \ \ , \\ R_{I\bar{M}J\bar{N}} \ &= \ \partial_I \partial_{\bar{M}} \ g_{J\bar{N}} - g^{\bar{K}L} \ \partial_I \ g_{\bar{K}J} \ \partial_{\bar{M}} \ g_{\bar{N}L} \ \ , \end{aligned} \tag{15}$$

[2] We will also use Z^I as denoting $Z^I = (\Phi^i, \Xi^i)$.

this expression being similar in form to the action appearing in [9].

It is worthwhile to notice that in $D = (2+2)$, we can also formulate a non-linear supersymmetric $\sigma-$model using chiral and anti-chiral superfields both subject to a reality condition ($\Phi^i = \Phi^{i*}$, $\Xi^i = \Xi^{i*}$). Here, we take our potential K as a function of (Φ^i, Ξ^i) and the action as in the usual form

$$S = \frac{1}{8}\int d^4x d^2\theta d^2\tilde{\theta}\; K(\Phi^i, \Xi^i) \ . \tag{16}$$

We obtain

$$S = \int d^4x \left(2g_{i\hat{j}}\partial_\mu Z^i \partial^\mu Z^{\hat{j}} - \frac{i}{2} g_{\hat{i}j}\Psi^{\hat{i}}\tilde{\sigma}^\mu \mathcal{D}_\mu \Psi^j - \frac{i}{2} g_{i\hat{j}}\Psi^i \sigma^\mu \mathcal{D}_\mu \Psi^{\hat{j}} + \frac{1}{8} R_{i\hat{m}j\hat{n}}\Psi^{\hat{m}}\Psi^{\hat{n}}\Psi^i\Psi^j \right) \ , \tag{17}$$

where we have used a notation similar to (12), but with the hatted components denoting the chiral conjugates, *i.e.* $(Z^i, Z^{\hat{i}}) \equiv (A^i, B^i)$ and $(\Psi^i, \Psi^{\hat{i}}) \equiv (\psi^i, \tilde{\chi}^i)$. Naturally, the metric is

$$g_{i\hat{j}} = \frac{\partial^2 K}{\partial A^i \partial B^j} \ ,$$

and the covariant derivatives and Riemann curvature are totally analogous to (15).

Obviously, this space is not Kählerian, as it is not a complex space. But it is curious to notice that it possesses some properties of a Kähler space if we just replace the notion of complex conjugation by that of chiral conjugation, that would take Z^i into $Z^{\hat{i}}$ and vice-versa. This class of σ-models is a feature of the $2+2$ signature of the space-time on which we build our supersymmetry. In $D = (3+1)$, a $N = 1$-supersymmetric σ-model requires a complex Kähler manifold as its target space [9, 11]. We then see another example, together with the one appearing in (7), of a non-Kähler manifold associated to $N = 1$-supersymmetric σ-models in $D = (2+2)$; however they exhibit the nice feature of being included in the class of theories generated by a scalar potential K.

3 Isometries

In the previous section, we have imposed the decomposition (8) in order to render manifest the Kählerian structure of the target space. From (9), we observe that the transformations for the potential K, allowed by the condition of metric invariance, are of the form

$$K \longrightarrow K' = K + F(Z_I) + G(Z_{\overline{I}}) \ . \tag{18}$$

These are the holomorphic transformations of a general Kähler manifold. Nevertheless, the $D = (2+2)$ spacetime structure forbids such a transformation, since terms out of the blocks $g_{I\overline{J}}$ would be generated. This happens because the invariance of the action (5) is ensured by chiral transformations

$$K \longrightarrow K' = K + F(\Phi, \Phi^*) + G(\Xi, \Xi^*) \ . \tag{19}$$

The way to make (18) and (19) compatible is to admit that the most general transformation of the potential K is

$$K \;\longrightarrow\; K' \;=\; K \;+\; \eta(\Phi) \;+\; \eta^*(\Phi^*) \;+\; \theta(\Xi) \;+\; \theta^*(\Xi^*) \;\;. \tag{20}$$

This has an immediate consequence on the possible coordinate transformations allowed for the manifold. The holomorphic transformations of a general Kähler manifold are decomposed into a more restricted subgroup, in which coordinates associated to different chiralities do not mix [3]

$$A^i \;\longrightarrow\; A'^i = f(A^i) \quad , \quad B^i \;\longrightarrow\; B'^i = f(B^i) \quad \textit{and c.c.} \;\; . \tag{21}$$

If we permitted that a coordinate A^i could have been taken into a B^i, terms out of the anti-diagonal in the metric of (9) would have been generated. In this way, we see that we are dealing with a subset of manifolds among those that have the most general Kähler form. Also, from these facts, we can conclude that the Killing vectors will be parametrized in terms of different chiral components:

$$\mathcal{K}_a^I \;=\; \begin{pmatrix} \kappa_a^i(A) \\ \tau_a^i(B) \end{pmatrix} \quad , \quad \mathcal{K}_b^{\overline{I}} \;=\; \begin{pmatrix} \kappa_b^{*i}(A^*) \\ \tau_b^{*i}(B^*) \end{pmatrix} \;\; . \tag{22}$$

The possibility of working with the above Killing vectors is due to the fact that the metric does not contain the components $g_{i\bar{j}}$, $g_{\hat{i}\bar{\hat{j}}}$, $g_{\bar{i}j}$, $g_{\bar{\hat{i}}\hat{j}}$ (see Appendix). Under a global isometry, the coordinates of the Kähler manifold will transform as

$$Z'^I \;=\; \exp\left(L_{\lambda\cdot\mathcal{K}}\right) Z^I \rightarrow \begin{cases} A'^i \;=\; \exp\left(L_{\lambda\cdot\kappa}\right) A^i \\ B'^i \;=\; \exp\left(L_{\lambda\cdot\tau}\right) B^i \end{cases} \quad \textit{and c.c.} \;\; , \tag{23}$$

where L_X is the Lie derivative along X and λ is a global parameter. The Killing vectors generate the algebra of the isometry group of the Kähler manifold, i.e. $[\mathcal{K}_a, \mathcal{K}_b] = f_{ab}^c\, \mathcal{K}_c$. The isometries induce transformations in the potential K, which are described in their general form by

$$\delta K \;=\; \lambda^a \left(\frac{\partial K}{\partial Z_I} \mathcal{K}_a^I \;+\; \frac{\partial K}{\partial Z_{\overline{I}}} \mathcal{K}_a^{\overline{I}} \right) \;=\; \lambda^a \frac{\partial H}{\partial A^i} \kappa_a^i + \lambda^a \frac{\partial H^*}{\partial B^i} \tau_a^i + \lambda^a \frac{\partial H^*}{\partial A^{*i}} \kappa_a^{*i} + \lambda^a \frac{\partial H}{\partial B^{*i}} \tau_a^{*i} \;\; . \tag{24}$$

Comparing eq.(24) with eq.(20), which is also an invariance of the metric, we can write

$$\begin{aligned}
\eta_a(A) \;&=\; \frac{\partial H(A,B^*)}{\partial A^i}\, \kappa_a^i(A) \;+\; Y_a(A,B^*) \;\; , \\
\theta_a(A) \;&=\; \frac{\partial H^*(B,A^*)}{\partial B^i}\, \tau_a^i(B) \;-\; Y_a^*(B,A^*) \;\; , \\
\eta_a^*(A^*) \;&=\; \frac{\partial H^*(B,A^*)}{\partial A^{*i}}\, \kappa_a^{*i}(A^*) \;+\; Y_a^*(B,A^*) \;\; , \\
\theta_a^*(A^*) \;&=\; \frac{\partial H(A,B^*)}{\partial B^{*i}}\, \tau_a^{*i}(B^*) \;-\; Y_a(A,B^*) \;\; .
\end{aligned} \tag{25}$$

[3]From now on, the term holomorphic will mean not only a splitting in terms of fields and their conjugated, but also a splitting in different chiralities.

The introduction of the complex functions Y_a is necessary, so that no further restriction is imposed on the potentials $H's$. The $Y_a's$ are naturally related to the structure of Killing vectors in Kähler space. To show this, we may start by the derivation of the first equation in (25) with respect to B^*:

$$\frac{\partial^2 H}{\partial A^i \partial B^{*j}}\, \kappa_a^i(A) \;=\; -\,\frac{\partial Y_a}{\partial B^{*j}} \;\;, \tag{26}$$

or deriving θ_a with respect to A^*:

$$\frac{\partial^2 H^*}{\partial B^i \partial A^{*j}}\, \tau_a^i(B) \;=\; \frac{\partial Y_a^*}{\partial A^{*j}} \;\;. \tag{27}$$

These equations and their conjugates can be written in a compact form in terms of a real potential $\mathcal{Y}_a = iY_a^*(B, A^*) - iY_a(A, B^*)$

$$g_{I\overline{J}}\, \mathcal{K}_a^I \;=\; -i\, \frac{\partial \mathcal{Y}_a}{\partial Z^{\overline{J}}} \quad \textit{and c.c.} \;\;. \tag{28}$$

This equation is just the restriction imposed by the Killing equation with mixed indices,

$$\nabla_I\, \mathcal{K}_{\overline{J}} \;+\; \nabla_{\overline{J}}\, \mathcal{K}_I \;=\; 0 \;\;, \tag{29}$$

on the form of the Killing vectors, which become described by the potential $\mathcal{Y}_a$.

The determination of this potential is crucial for the process of gauging, as we shall see in what follows. In order to accomplish this goal, we will use the method established in [10]. Contracting eq.(28) with $\mathcal{K}_b^{\overline{J}}$ and its conjugate with $\mathcal{K}_b^I$, and then comparing them both, we get the identity

$$\mathcal{K}_b^I\, \frac{\partial \mathcal{Y}_a}{\partial Z^I} \;+\; \mathcal{K}_a^{\overline{I}}\, \frac{\partial \mathcal{Y}_b}{\partial Z^{\overline{I}}} \;=\; 0 \;\;. \tag{30}$$

Now, under an isometry transformation, $\mathcal{Y}_a$ transforms as

$$\delta \mathcal{Y}_a \;=\; \lambda^b \left(\frac{\partial \mathcal{Y}_a}{\partial Z^I}\, \mathcal{K}_b^I \;+\; \frac{\partial \mathcal{Y}_a}{\partial Z^{\overline{I}}}\, \mathcal{K}_b^{\overline{I}} \right) \;\;, \tag{31}$$

which, by virtue of (30), may be written as

$$\delta \mathcal{Y}_a \;=\; \frac{\lambda^b}{2} \left(\frac{\partial \mathcal{Y}_{[a}}{\partial Z^I}\, \mathcal{K}_{b]}^I \;+\; \frac{\partial \mathcal{Y}_{[a}}{\partial Z^{\overline{I}}}\, \mathcal{K}_{b]}^{\overline{I}} \right) \;\;. \tag{32}$$

With the help of eqs.(24 - 28), we get the fundamental relation

$$\mathcal{K}_{[a}^I\, \frac{\partial \mathcal{Y}_{b]}}{\partial Z^I} \;+\; \mathcal{K}_{[a}^{\overline{I}}\, \frac{\partial \mathcal{Y}_{b]}}{\partial Z^{\overline{I}}} \;=\; f_{ab}^c\, (\xi_c \;+\; \xi_c^*) \;\;, \tag{33}$$

where $\xi_a = \eta_a + \theta_a$ and f_{ab}^c are the structure constants of the isometry group. In components, this last equation means

$$\begin{aligned} \kappa_{[a}^i\, \frac{\partial \eta_{b]}}{\partial A^i} \;&=\; f_{ab}^c\, \eta_c \;+\; c_{ab} \\ \tau_{[a}^i\, \frac{\partial \theta_{b]}}{\partial B^i} \;&=\; f_{ab}^c\, \theta_c \;-\; c_{ab}^* \qquad \textit{and c.c.} \;\;, \end{aligned} \tag{34}$$

and $c^*_{ab} \ = \ - \ c^*_{ba}$ is a complex constant. Finally, from (32 - 34) we get

$$\delta Y_a \ = \ \lambda^b \left(\frac{\partial Y_a}{\partial A^i} \, \kappa^i_b \ + \ \frac{\partial Y_a}{\partial B^{*i}} \, \tau^{*i}_b \right) \ = \ - \, \lambda^b \, f^c_{ab} \, Y_c \ - \ \lambda^b \, c_{ab} \quad and \; c.c. \ . \qquad (35)$$

At this point we see that, in order to make explicit the potentials Y_a as functions of the Killing vectors, we have to restrict the isometry groups to be semi-simple. This becomes clearer if we combine (28) in (35):

$$Y_a \ = \ 2 \, f^c_{ab} \, \kappa^i_d \, \tau^{*j}_c \frac{\partial^2 H}{\partial A^i \partial B^{*j}} \, g^{bd} \ + \ f^c_{ab} \, c_{dc} \, g^{bd} \quad and \; c.c. \ . \qquad (36)$$

To define Y_a we needed to introduce the inverse Killing metric, and this means that Abelian factors would spoil the definition of Y_a, so that only semi-simple groups are allowed [16]. The constants c_{ab} express an arbitrariness in the definition of Y_a, as they can be reabsorbed by the shift $Y_a \ \rightarrow \ Y^{'}_a \ = \ Y_a \ - \ f^c_{ab} c_{dc} g^{bd}$, whenever g^{ab} is defined. This property will be of fundamental importance in the procedure of gauging the model.

In the particular case of a non-semi-simple group, G, of isometries, for which f^c_{ab} is non-vanishing only when all its indices are associated to generators in the semi-simple factor S, *i.e.* G has the form

$$G \ = \ S \ \otimes \ A_N \ , \qquad (37)$$

where A_N represents the direct product of N Abelian factors, and if all the constants c_{ab} (determined by (34)) with indices associated to the latter vanish, then from (35) we can conclude that the potential Y_a will be allways determined up to N arbitrary complex constants associated to each Abelian factor.

In the general case of a non-semi-simple group, with Abelian factors generating non-zero constants c_{ab}, eq.(35) may not admit any solution and this will be an obstruction to the gauging, as we shall see in the following.

4 The Gauging

The isometry transformations of the coordinates on a Kähler manifold are given in eq.(23). Now we can make this symmetry local by taking the constant parameter λ as superfields of definite chirality. Those transformations are then written in superfields as

$$\begin{aligned} \Phi \longrightarrow \quad \Phi^{'} \quad &= \quad \exp{(L_{\Lambda \cdot \kappa})} \Phi \\ \Xi \longrightarrow \quad \Xi^{'} \quad &= \quad \exp{(L_{\Gamma \cdot \tau})} \Xi \quad and \;\; c.c. \ , \end{aligned} \qquad (38)$$

The superfields Λ and Γ are chiral and anti-chiral respectively. But as we have already seen, in $D = (2+2)$ this does not make any restriction on their reality. In $D = (3+1)$ they would be necessarily complex conjugates of each other.

Let us then take $\Lambda \ = \ \Lambda^*, \ \ \Gamma \ = \ \Gamma^*$. Here, the local infinitesimal isometries read as

$$\begin{aligned} \delta \Phi^i \ &= \ \Lambda^a k^i_a \\ \delta \Xi^i \ &= \ \Gamma^a \tau^i_a \quad and \;\; c.c. \ , \end{aligned} \qquad (39)$$

and the Kähler potential transforms like

$$\delta K \;=\; \Lambda^a\left(\frac{\partial H}{\partial A^i}\kappa_a^i + \frac{\partial H^*}{\partial A^{*i}}\kappa_a^{*i}\right) + \Gamma^a\left(\frac{\partial H^*}{\partial B^i}\tau_a^i + \frac{\partial H}{\partial B^{*i}}\tau_a^{*i}\right) \;. \tag{40}$$

In order to have a transformation which could be compared with (20), all superfields should transform with the same parameter. This can be obtained if we introduce a real vector superfield V, which in $D=(2+2)$ assumes the form,

$$\begin{aligned} V(x,\theta,\widetilde{\theta}) = {} & C(x) + i\theta\zeta(x) + i\widetilde{\theta}\widetilde{\eta}(x) + \frac{1}{2}i\theta^2 M(x) + \frac{1}{2}i\widetilde{\theta}^2 N(x) + \\ & + \frac{1}{2}i\theta\sigma^\mu\widetilde{\theta}A_\mu(x) - \frac{1}{2}\widetilde{\theta}^2\theta\lambda(x) - \frac{1}{2}\theta^2\widetilde{\theta}\widetilde{\rho}(x) - \frac{1}{4}\theta^2\widetilde{\theta}^2 D(x) \;\;, \end{aligned} \tag{41}$$

where C, M, N and D are real scalars, ζ, $\widetilde{\eta}$, λ and $\widetilde{\rho}$ are Majorana-Weyl spinors and A_μ is a vector field. Now we replace the superfields Ξ^i [17] by

$$\tilde{\Xi}^i \;\equiv\; \exp\left(L_{V\cdot\tau}\right)\Xi^i \quad and \quad c.c. \;, \tag{42}$$

so that $\tilde{\Xi}^i$ can transform as

$$\tilde{\Xi}'^i \;=\; \exp\left(L_{\Lambda\cdot\tau}\right)\tilde{\Xi}^i \;\;. \tag{43}$$

This is only possible if the vector superfield transforms as

$$\exp\left(L_{V'\cdot\tau}\right) \;=\; \exp\left(L_{\Lambda\cdot\tau}\right)\exp\left(L_{V\cdot\tau}\right)\exp\left(-L_{\Gamma\cdot\tau}\right) \;\;. \tag{44}$$

Since the parameters Λ and Γ are real, we have from (44) that V transforms indeed as a real vector superfield. The infinitesimal isometries have the form

$$\begin{aligned} \delta\Phi^i &= \Lambda^a k_a^i \\ \delta\tilde{\Xi}^i &= \Lambda^a\tau_a^i \qquad and \;\; c.c. \;, \end{aligned} \tag{45}$$

and the transformation (40) takes a form comparable to (20), with the replacements $\{\Xi,\ \Xi^*\} \longrightarrow \{\tilde{\Xi},\ \tilde{\Xi}^*\}$. But now, since the parameter Λ is a chiral superfield, we do not have the action invariant under local isometries, for

$$S \quad\longrightarrow\quad S' \;=\; \frac{1}{8}\int d^4x d^2\theta d^2\widetilde{\theta}\;\; \Lambda^a\left(\theta_a(\tilde{\Xi}) \;+\; \theta_a^*(\tilde{\Xi}^*)\right) \;\neq\; 0 \;\;. \tag{46}$$

However, the invariance of the action can be recovered if we introduce an antichiral superfield and its complex conjugate, υ and υ^*, such that they transform like

$$\begin{aligned} \delta\upsilon &= \lambda^a\theta_a\;\;(\Xi), \\ \delta\upsilon^* &= \lambda^a\theta_a^*(\Xi^*) \;\;. \end{aligned} \tag{47}$$

Then, we take our action as

$$S_\upsilon \;=\; \frac{1}{8}\int d^4x d^2\theta d^2\widetilde{\theta}\left(H(\Phi,\Xi^*) \;+\; H^*(\Phi^*,\Xi) \;-\; \upsilon \;-\; \upsilon^*\right) \;\;. \tag{48}$$

This action is globally invariant under the infinitesimal form of the transformations (23) and (47), and since υ and υ^* are anti-chiral, we also have $S_\upsilon = S$. Those superfields should be thought of as extra coordinates extending our manifold [10]. In this way, we write two new Killing vectors

$$\tau_a(\Xi) \longrightarrow \tau_a^{'}(\Xi) = \tau_a^i(\Xi)\frac{\partial}{\partial \Xi^i} + \theta_a(\Xi)\frac{\partial}{\partial \upsilon} \quad and \quad c.c. \ , \tag{49}$$

and the new Kähler potential $K^{'} = K - \upsilon - \upsilon^*$ is invariant under their action. Finally, the gauging of the isometry is simply performed by replacing $\Xi \longrightarrow \tilde{\Xi}$, $\upsilon \longrightarrow \tilde{\upsilon}$ and c.c. in (48). Now using the result

$$\begin{aligned} K(\Phi,\tilde{\Xi},\Phi^*,\tilde{\Xi}^*) &= K(\Phi,\Xi,\Phi^*,\Xi^*) + 2Re\left\{\frac{\exp{(L^{'})}-1}{L^{'}}\, V^a\Big(\theta_a(\Xi) + Y_a^*(\Phi^*,\Xi)\Big)\right\} \ , \\ \tilde{\upsilon} &= \upsilon + \frac{\exp{(L^{'})}-1}{L^{'}}\, V^a\theta_a(\Xi) \ , \\ L^{'} &\equiv L_{V\cdot\tau^{'}} \ , \end{aligned} \tag{50}$$

we are left with the form for the action that couples the σ-model to Yang-Mills fields through the gauging of the isometries:

$$S = \frac{1}{8}\int d^4x d^2\theta d^2\tilde{\theta}\left(H(\Phi,\Xi^*) + H^*(\Phi^*,\Xi) + 2Re\left\{\frac{\exp{(L)}-1}{L}\, V^a\, Y_a^*(\Phi^*,\Xi)\right\}\right) \ . \tag{51}$$

We can still implement a simpler expression for this action if we choose to work in the Wess-Zumino gauge (44) (see for instance [17]). We also make use of eqs.(22) and (28). In this way, the action (51) is rewritten in the following very simple final form

$$\begin{aligned} S \;=\; & \frac{1}{8}\int d^4x d^2\theta d^2\tilde{\theta}\Big(H(\Phi,\Xi^*) + H^*(\Phi^*,\Xi) + 2\,V^a\,Y_a^* + 2\,V^a\,Y_a \\ & - 2\,V^a\,V^b\,\mathcal{K}_a^I\,g_{I\overline{J}}\,\mathcal{K}_b^{\overline{J}}\Big) \ . \end{aligned} \tag{52}$$

Then, we see how the potential Y_a, determined in eq.(36) for semi-simple isometry groups, couples to the vector superfield V^a in the gauged action. As we discussed in the end of Section 3, Abelian factors in the isometry group may lead to the appearance of arbitrary constants in the potential Y_a. These will also couple to the vector superfield generating the so-called Fayet-Iliopoulos terms [10, 18]. In the general case of non-semi-simple isometry groups, as it happens in $D = (3 + 1)$ dimensions, the potential Y_a may not be determined, and this will represent an obstruction to the gauging of the non-linear σ-model.

It would be perhaps interesting to consider the possibility of working with superfield parameters, Λ and Γ, that are not real. This would lead to the introduction of a family of complex vector superfields to perform the gauging; however, the appearance of more than one Yang-Mills multiplet in the gauging of the isometry group is beyond the scope of the present work.

5 Concluding Remarks

We have here considered a few geometrical aspects concerning non-linear σ-models in the context of an $N = 1$ supersymmetry defined in $D = (2+2)$. We have shown that such models in general do not need to be of a Kähler type, even if they are generated by a potential K. As an explicit example, the construction of a real supersymmetric σ-model has been worked out. Then, restricting ourselves to a special sub-class of Kähler manifolds, we proceeded to an investigation of the main points involved in the process of gauging its isometries. In particular, we have choosen the gauge parameters as constant real superfields, which would not be possible in a $D = (3+1)$ space-time. We ended up with a superspace action, eq.(51), that is invariant under local isometry transformations. The kinetic terms of $D = (2+2)$ σ-models are off-diagonal (6) and this would signal the presence of ghosts (negative-norm states) in a space-time of the Minkowski type. However, the next step would be to carry out a dimensional reduction from $D = (2+2)$ to $D = (1+2)$ and $D = (1+1)$, where the propagation of fields is better controlled. Following the results of [5] and [7], one could go to lower dimensions in such a way that non-physical modes be eliminated and σ-models coupled to Yang-Mills fields may be of some relevance in connection with conformal theories and integrable models.

The relation of $N = 1$ models after dimensional reduction to chiral σ-models in 2 dimensions [19], and also the construction of an $N = 2$ σ-model in Atiyah-Ward space-time will be the subject of further investigation [15].

6 Appendix

The Kähler space treated in this work is of the type $\mathcal{C}^{2m} \times \overline{\mathcal{C}^{2m}}$ with metric (9), where each of the blocks is a (2n x 2n) matrix whose respective components $g_{i\bar{j}}$, $g_{\hat{i}\bar{\hat{j}}}$, *and* $g_{\bar{i}j}$, $g_{\bar{\hat{i}}\hat{j}}$ vanish. Since the more general Kähler space would allow those components, our Kähler space is a subclass of the more general one.

From (9), we obtain for the connections

$$
\begin{aligned}
\Gamma^{i}_{jk} &= g^{i\bar{\hat{r}}}\partial_{j} g_{k\bar{\hat{r}}} \ , \\
\Gamma^{\hat{i}}_{\hat{j}\hat{k}} &= g^{\hat{i}\bar{r}}\partial_{\hat{j}} g_{\hat{k}\bar{r}} \ , \\
\Gamma^{\bar{i}}_{\bar{j}\bar{k}} &= g^{\bar{i}\hat{r}}\partial_{\bar{j}} g_{\bar{k}\hat{r}} \ , \\
\Gamma^{\bar{\hat{i}}}_{\bar{\hat{j}}\bar{\hat{k}}} &= g^{\bar{\hat{i}}r}\partial_{\bar{\hat{j}}} g_{\bar{\hat{k}}r} \ ,
\end{aligned} \tag{53}
$$

and for the curvatures

$$
\begin{array}{llll}
\mathcal{R}^{i}_{jkL} = \partial_L \Gamma^{i}_{jk} & \text{with} \quad L = \{\hat{l}, \bar{l}, \bar{\hat{l}}\} \ , & \mathcal{R}^{i}_{jKl} = -\partial_K \Gamma^{i}_{jl} & \text{with} \quad K = \{\hat{k}, \bar{k}, \bar{\hat{k}}\} \ , \\
\mathcal{R}^{\hat{i}}_{\hat{j}\hat{k}L} = \partial_L \Gamma^{\hat{i}}_{\hat{j}\hat{k}} & \text{with} \quad L = \{l, \bar{l}, \bar{\hat{l}}\} \ , & \mathcal{R}^{\hat{i}}_{\hat{j}K\hat{l}} = -\partial_K \Gamma^{\hat{i}}_{\hat{j}\hat{l}} & \text{with} \quad K = \{k, \bar{k}, \bar{\hat{k}}\} \ , \\
\mathcal{R}^{\bar{i}}_{\bar{j}\bar{k}L} = \partial_L \Gamma^{\bar{i}}_{\bar{j}\bar{k}} & \text{with} \quad L = \{l, \hat{l}, \bar{\hat{l}}\} \ , & \mathcal{R}^{\bar{i}}_{\bar{j}K\bar{l}} = -\partial_K \Gamma^{\bar{i}}_{\bar{j}\bar{l}} & \text{with} \quad K = \{k, \hat{k}, \bar{\hat{k}}\} \ , \\
\mathcal{R}^{\bar{\hat{i}}}_{\bar{\hat{j}}\bar{\hat{k}}L} = \partial_L \Gamma^{\bar{\hat{i}}}_{\bar{\hat{j}}\bar{\hat{k}}} & \text{with} \quad L = \{l, \bar{l}, \hat{l}\} \ , & \mathcal{R}^{\bar{\hat{i}}}_{\bar{\hat{j}}K\bar{\hat{l}}} = -\partial_K \Gamma^{\bar{\hat{i}}}_{\bar{\hat{j}}\bar{\hat{l}}} & \text{with} \quad K = \{k, \bar{k}, \hat{k}\} \ .
\end{array}
$$

Now, we shall analyse the assumption on the structure of the Killing vectors shown in eq.(22). We intend to show just a sketch of a proof that is in complete analogy to the one given in [20], so that it will be just a slight modification of the Theorems 2.4 and 2.5 of that reference.

As it is well known, in a compact Kähler space a necessary and suficient condition for a contravariant vector $\mathcal{K}^I$ be a Killing vector is

$$\begin{aligned} g^{JK}\nabla_J\nabla_K\mathcal{K}^I + \mathcal{R}^I_J\mathcal{K}^J = 0 \ , \\ \nabla_I\mathcal{K}^I = 0 \ , \end{aligned} \tag{54}$$

where $\mathcal{R}^I_J$ is the Ricci tensor.

Let us impose that the Killing vector $\mathcal{K}^I = (k^i, k^{\hat{i}}, k^{\bar{i}}, k^{\bar{\hat{i}}})$ satisfies

$$\nabla_i k^i = \nabla_{\hat{i}} k^{\hat{i}} = 0 \ \ and \ \ c.c. \ . \tag{55}$$

Then, from (54), we also have $\zeta^I = (k^i, 0, 0, 0)$, $\tau^I = (0, k^{\hat{i}}, 0, 0)$, $\lambda^I = (0, 0, k^{\bar{i}}, 0)$ and $\eta^I = (0, 0, 0, k^{\bar{\hat{i}}})$ as Killing vectors. This allows us to write for each of them,

$$\nabla_I\zeta_J + \nabla_J\zeta_I = 0 \ \ etc. \ , \tag{56}$$

with $\zeta_I = (0, 0, 0, k_{\bar{\hat{i}}})$, $k_{\bar{\hat{i}}} = g_{\bar{\hat{i}}j} k^j$. Recalling that $\Gamma^{\bar{\hat{k}}}_{\bar{\hat{i}}\bar{\hat{j}}}$ is the only non-vanishing component of $\Gamma^{\bar{\hat{k}}}_{IJ}$, we have from (56) that $\zeta_{\bar{\hat{i}}} = \zeta_{\bar{\hat{i}}}(\Xi^*)$ or $k_{\bar{\hat{i}}} = k_{\bar{\hat{i}}}(\Xi^*)$, and in an analogous way $k_i = k_i(\Phi)$, $k_{\hat{i}} = k_{\hat{i}}(\Xi)$ and $k_{\bar{i}} = k_{\bar{i}}(\Phi^*)$. Those covariant components of the Killing vector $\mathcal{K}^I$ being holomorphic, we have from [20] that $\mathcal{K}^I$ is harmonic, i.e., it satisfies,

$$\nabla_I\mathcal{K}_J - \nabla_J\mathcal{K}_I = 0 \ . \tag{57}$$

Since $\mathcal{K}^I$ is a Killing vector we also have $\nabla_I\mathcal{K}_J + \nabla_J\mathcal{K}_I = 0$. This, together with eq.(57), gives $\nabla_I\mathcal{K}_J = 0$, and then $\nabla_I\mathcal{K}^J = 0$, which also implies

$$k^i = k^i(\Phi), \ k^{\hat{i}} = k^{\hat{i}}(\Xi), \ k^{\bar{i}} = k^{\bar{i}}(\Phi^*), \ k^{\bar{\hat{i}}} = k^{\bar{\hat{i}}}(\Xi^*) \ . \tag{58}$$

We have then proven that Killing vectors satisfying (55) are holomorphic in all their coordinates.

Conversely, let $\mathcal{K}^I$ be a vector satysfying (55) and holomorphic in all its coordinates (58). From the Ricci identities

$$\nabla_J\nabla_K\mathcal{K}^I - \nabla_K\nabla_J\mathcal{K}^I = \mathcal{R}^I_{LKJ}\mathcal{K}^L \ , \tag{59}$$

we get

$$\begin{aligned} \nabla_{\bar{\hat{j}}}\nabla_k k^i &= \mathcal{R}^i_{lk\bar{\hat{j}}} k^l \ , \\ \nabla_{\bar{j}}\nabla_{\hat{k}} k^{\hat{i}} &= \mathcal{R}^{\hat{i}}_{\hat{l}\hat{k}\bar{j}} k^{\hat{l}} \ , \\ \nabla_{\hat{j}}\nabla_{\bar{k}} k^{\bar{i}} &= \mathcal{R}^{\bar{i}}_{\bar{l}\bar{k}\hat{j}} k^{\bar{l}} \ , \\ \nabla_j\nabla_{\bar{\hat{k}}} k^{\bar{\hat{i}}} &= \mathcal{R}^{\bar{\hat{i}}}_{\bar{\hat{l}}\bar{\hat{k}}j} k^{\bar{\hat{l}}} \ . \end{aligned} \tag{60}$$

Contracting each of them respectively with $g^{\bar{\hat{j}}k}$, $g^{\bar{j}\hat{k}}$, $g^{\hat{j}\bar{k}}$, $g^{j\bar{\hat{k}}}$, and using (55), we can write

$$
\begin{aligned}
& g^{\bar{\hat{j}}k}\nabla_{\bar{\hat{j}}}\nabla_k k^i + \mathcal{R}^i_l\, k^l = 0 \quad , \quad \nabla_i k^i = 0 \quad and \quad c.c. \ , \\
& g^{\bar{j}\hat{k}}\nabla_{\bar{j}}\nabla_{\hat{k}} k^{\hat{i}} + \mathcal{R}^{\hat{i}}_{\hat{l}}\, k^{\hat{l}} = 0 \quad , \quad \nabla_{\hat{i}} k^{\hat{i}} = 0 \quad and \quad c.c. \ ,
\end{aligned} \tag{61}
$$

or in a compact way,

$$
g^{JK}\nabla_J\nabla_K\mathcal{K}^I + \mathcal{R}^I_L\mathcal{K}^L = 0 \quad \text{and} \quad \nabla_I\mathcal{K}^I = 0 \ .
$$

This is exactly the condition (54) for a Killing vector. We have proven then that a vector satisfying (55) is a Killing vector if and only if its components are holomorphic in all coordinates Φ, Ξ, Φ^*, Ξ^*.

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