Levi-Civita Solutions Coupled with Electromagnetic Fields

A.Y. Miguelote[*], M.F.A. da Silva[†], Anzhong Wang[‡]

Departamento de Física Teórica, Instituto de Física, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, Maracanã, 20550-013 Rio de Janeiro – RJ, Brazil

N.O. Santos[§]

Centro Regional Sul de Pesquisas Espaciais - INPE/MCT, Cidade Universitária, CEP. 97105-900, Santa Maria, RS, Brazil

and

LAFEX, Centro Brasileiro de pesquisas Físicas - CBPF, Rua Dr. Xavier Sigaud 150, Urca, CEP: 22290-180, Rio de Janeiro – RJ, Brazil

(November 24, 2000)

The local and global properties of the Levi-Civita (LC) solutions coupled with an electromagnetic field are studied and some limits to the vacuum LC solutions are given. By doing such limits, the number of physically independent parameters and their physical and geometrical interpretations are made clear. Sources for both the LC vacuum solutions and the LC solutions coupled with an electromagnetic field are studied, and in particular it is found that all the LC vacuum solutions with $\sigma \geq 0$ can be produced by cylindrically symmetric thin shells that satisfy all the energy conditions, weak, dominant, and strong. When the electromagnetic field is present, the situation changes dramatically. In the case of a purely magnetic field, all the solutions with $\sigma \geq 1/\sqrt{8}$ or $\sigma \leq -1/\sqrt{8}$ can be produced by physically acceptable cylindrical thin shells, while in the case of a purely electric field, no such shells are found for any value of σ.

Key-words: Gravitation; Cylindrical symmetry; Electromagnetism.

I. INTRODUCTION

Spacetimes with cylindrical symmetry have been studied intensively in the past twenty years or so, in the context of topological cosmic strings that may have formed in the early stages of the Universe [1], and in gravitational collapse [2]. Recently, the physical and geometrical interpretation of the Levi-Civita (LC) vacuum solutions, which represent the most general cylindrical static vacuum spacetimes, has attracted much attention [3]. In particular, it has been shown, among other things, that the LC vacuum solutions can be produced by cylindrically symmetric sources, which satisfy all the energy conditions, weak, strong, and dominant [4], only for $0 \leq \sigma \leq 1$ [5], where σ is one of the two parameters appearing in the LC vacuum solutions, which is related to, but in general not equal to the mass per unity length [3]. It has been also shown that when the LC solutions coupled with a cosmological constant, the spacetime structures are dramatically changed, and in some cases they give rise to black hole structures [6].

In this paper, our purpose is two fold. First, as we mentioned above, so far, physically acceptable and cylindrically symmetric sources for the LC vacuum solutions are found only for $0 \leq \sigma \leq 1$. Since $\sigma = 1$ does not represent any typical value in these solutions [5], it has troubled us for a long time why the solutions with $\sigma > 1$ cannot be realized by physically acceptable cylindrical sources. In this paper, we shall show that all the LC vacuum solutions can be produced by cylindrically symmetric thin shells that satisfy all the energy conditions, as long as $\sigma \geq 0$. The key observation that leads to such a conclusion is that, as $\sigma \to +\infty$, the solutions become (locally) Minkowski, again, but with x^2 becoming the angular coordinate. Thus, it is very plausible that, as σ increases to a certain value, the axial and angular coordinates may change their roles. By constructing cylindrically symmetric sources, in this

paper we shall confirm this claim, and argue that the change should happen at $\sigma = 1/2$, although in the range $1/4 \leq \sigma \leq 1$, physically acceptable sources for both of the two identifications are found. Thus, the interpretation of all the LC vacuum solutions with $\sigma \geq 0$ as representing cylindrically symmetric vacuum spacetimes is physically acceptable. It is interesting to note that several authors already speculated that as σ increases to the value $\sigma = 1/2$, the angular coordinate should be straighted out to infinite, so that the resultant spacetimes become plane symmetric [7]. Indeed, we have shown that the solution with $\sigma = 1/2$ can be produced by a massive plane with an uniform distribution of matter [8], while Philbin has shown that all the solutions with $|\sigma| > 1/2$ can be produced by massive planes [9]. However, as far as we know, this is the first time to be argued that, when $\sigma > 1/2$, the two spacelike coordinates x^2 and x^3 change their roles, and show that the LC vacuum solutions can be produced by cylindrical sources for $\sigma \geq 0$, after such an exchange of coordinates is taken place. The physics that provokes such an exchange has not worked out, yet. Here, we also extend our studies to the LC solutions coupled with electromagnetic fields and study the effects of the electromagnetic fields on the local and global structure of the spacetimes.

The paper is organized as follows: In Sec. II, we shall study the main properties of the LC solutions when coupled with an electromagnetic field, and take their vacuum limits. By doing so, we can find out the physically independent parameters and their physical and geometrical interpretations. In Sec. III, we shall consider cylindrically symmetric thin shells that produce the spacetimes described by the LC vacuum solutions or by the LC solutions coupled with an electromagnetic field. We use Israel's method [10] to obtain the general expression for the surface energy-momentum tensor of a thin shell, which separates two arbitrary cylindrical static regions. Then we apply these general formulae to the case where the shell separates a Minkowski-like internal region from an external region described by either the LC vacuum solutions or the LC solutions coupled with an electromagnetic field. Imposing the energy conditions, we show that only for some particular choices of the free parameters appearing in the solutions these conditions are fulfilled. The paper is closed with Sec. IV, where our main conclusions are presented.

II. LEVI-CIVITA SOLUTIONS COUPLED WITH ELECTROMAGNETIC FIELDS

The static spacetimes with cylindrical symmetry are described by the metric [11]

$$ds^2 = f(R)dT^2 - g(R)dR^2 - h(R)\left(dx^2\right)^2 - l(R)\left(dx^3\right)^2 , \tag{1}$$

where T and R are, respectively, the timelike and radial coordinate. In general, the spacetime possesses three Killing vectors, $\xi^{\mu}_{(0)} = \delta^{\mu}_0$, $\xi^{\mu}_{(2)} = \delta^{\mu}_2$, and $\xi^{\mu}_{(3)} = \delta^{\mu}_3$, where $\{x^{\mu}\} = \{T, R, x^2, x^3\}$. Clearly, the coordinate transformations

$$T = a\tilde{T}, \quad R = R(\tilde{R}), \quad x^2 = \alpha\tilde{x}^2, \quad x^3 = C^{-1}\tilde{x}^3, \tag{2}$$

preserve the form of metric, where a, α and C are arbitrary constants, and $R(\tilde{R})$ is an arbitrary function of the new radial coordinate $\tilde{R}$. A spacetime with cylindrical symmetry must obey several conditions [12,13]:

(i) *The existence of an axially symmetric axis*: The spacetime that has an axially symmetric axis is assured by the condition,

$$||\partial_{\varphi}|| = |g_{\varphi\varphi}| \rightarrow O(R^2), \tag{3}$$

as $R \rightarrow 0^+$, where we had chosen the radial coordinate such that the axis is located at $R = 0$, and φ denotes the angular coordinate with the hypersurfaces $\varphi = 0$ and $\varphi = 2\pi$ being identical. Since both $\xi^{\mu}_{(2)}$ and $\xi^{\mu}_{(3)}$ are spacelike Killing vectors, φ can be chosen to be either x^2 or x^3. This ambiguity always rises, since the Einstein field equations are differential equations, and consequently do not determine the global topology of the spacetime. This observation will be crucial in understanding the LC vacuum solutions to be discussed in Secs. II and III below. However, once φ is identified, the rescaling transformation of Eq.(2) for φ,

$$\varphi = \frac{\tilde{\varphi}}{D}, \tag{4}$$

maps the two identified hypersurfaces $\varphi = 0$ and $\varphi = 2\pi$, respectively, to $\tilde{\varphi} = 0$ and $\tilde{\varphi} = 2\pi D$. Consequently, it results in an angular defect in the coordinates $\{T, R, \tilde{\varphi}, z\}$, given by

$$\Delta\tilde{\varphi} = 2\pi(1 - D). \tag{5}$$

Thus, the coordinate transformation (4) in general yields physically different solutions. In particular, when the spacetime outside the axis is locally Minkowski, this angular defect can be associated with a cosmic string located on the axis [1].

(ii) *The elementary flatness on the axis*: This condition requires that the spacetime be locally flat on the axis, which in the present case can be expressed as

$$\frac{X_{,\alpha} X_{,\beta} g^{\alpha\beta}}{4X} = \frac{g_{\varphi\varphi,R}^2}{4g_{RR}g_{\varphi\varphi}} \to 1, \tag{6}$$

as $R \to 0^+$, where X is given by $X = ||\partial_\varphi|| = |g_{\varphi\varphi}|$, and $(\ )_{,R} \equiv \partial(\ )/\partial R$. Note that solutions that fail to satisfy this condition are sometimes accepted since the appearance of spacetime singularities on the axis can be considered as representing the existence of some kind of sources [3]. For example, when the left-hand side of Eq.(6) approaches a finite constant, the singularity on the axis can be related to a cosmic string [1].

(iii) *No closed timelike curves*: In the cylindrical spacetimes, closed timelike curves (CTCs) are rather easily introduced [4]. While the physics of the CTCs is not yet clear [14], we shall not consider this possibility in this paper and simply require that

$$g_{\varphi\varphi} < 0, \tag{7}$$

hold in all the region of the spacetime considered.

(iv) *Asymptotical flatness*: When the sources are confined within a finite region in the radial direction, one usually also requires that the spacetime be asymptotically flat as $\mathcal{R} \to +\infty$, where $\mathcal{R}$ denotes the geometric proper distance from the axis to a referred point in the radial direction.

For an electromagnetic field $A_\mu(R)$, the energy-momentum tensor (EMT) is given by

$$T_{\mu\nu} = \frac{2}{\kappa}\left(F_{\mu\alpha}F_{\beta\nu}g^{\alpha\beta} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \tag{8}$$

where $\kappa(\equiv 8\pi G/c^4)$ is the Einstein gravitational coupling constant, and

$$F_{\mu\nu} \equiv A_{\mu,\nu} - A_{\nu,\mu}. \tag{9}$$

Because of the symmetry, from Eq.(9) we find that

$$F_{02} = F_{03} = F_{23} = 0. \tag{10}$$

On the other hand, when the electromagnetic field is source-free, we have

$$F^{\mu\nu}{}_{;\mu} = F^{1\nu}{}_{,R} + \frac{1}{2}\left[\ln(fghl)\right]_{,R} F^{1\nu} = 0, \tag{11}$$

where the semicolon ";" denotes the covariant derivative. Clearly, the above equation has the general solution

$$F^{1\nu} = \frac{B^\nu}{(fghl)^{1/2}}, \ (\nu = 0, 1, 2, 3), \tag{12}$$

where B^ν are the integration constants with $B^1 = 0$. Substituting Eqs.(10) and (12) into Eq.(8) we find that

$$\begin{aligned}
T_{\mu\nu} = \frac{1}{\kappa fhl} &\left\{ f\left[f\left(B^0\right)^2 + h\left(B^2\right)^2 + l\left(B^3\right)^2\right]\delta_\mu^0\delta_\nu^0 - g\left[f\left(B^0\right)^2 - h\left(B^2\right)^2 - l\left(B^3\right)^2\right]\delta_\mu^1\delta_\nu^1\right. \\
&+ h\left[f\left(B^0\right)^2 + h\left(B^2\right)^2 - l\left(B^3\right)^2\right]\delta_\mu^2\delta_\nu^2 + l\left[f\left(B^0\right)^2 - h\left(B^2\right)^2 + l\left(B^3\right)^2\right]\delta_\mu^3\delta_\nu^3 \\
&\left. - 2B^0B^2fh\left(\delta_\mu^0\delta_\nu^2 + \delta_\mu^2\delta_\nu^0\right) - 2B^0B^3fl\left(\delta_\mu^0\delta_\nu^3 + \delta_\mu^3\delta_\nu^0\right) + 2B^2B^3hl\left(\delta_\mu^2\delta_\nu^3 + \delta_\mu^3\delta_\nu^2\right)\right\}. \tag{13}
\end{aligned}$$

When the electromagnetic field is the only source for the Einstein field equations, $G_{\mu\nu} = \kappa T_{\mu\nu}$, we find that the components T_{02}, T_{03} and T_{23} have to vanish, because the Einstein tensor $G_{\mu\nu}$ for the metric (1) has no non-diagonal terms. The vanishing of these terms yields,

$$B^0 B^2 = 0, \quad B^0 B^3 = 0, \quad B^2 B^3 = 0, \tag{14}$$

which have four different solutions,

$$A) \ B^0 = B^2 = B^3 = 0, \quad B) \ B^2 \neq 0, \quad B^0 = B^3 = 0,$$
$$C) \ B^3 \neq 0, \quad B^0 = B^2 = 0, \quad D) \ B^0 \neq 0, \quad B^2 = B^3 = 0. \tag{15}$$

In the following, let us consider them separately.

A. $B^0 = B^2 = B^3 = 0$

In this case, the electromagnetic field vanishes and the corresponding solution is the LC vacuum solution, given by [11]

$$ds^2 = a^2 R^{4\sigma} dT^2 - R^{4\sigma(2\sigma-1)} \left[dR^2 + \alpha^2 \left(dx^2 \right)^2 \right] - C^{-2} R^{2(1-2\sigma)} \left(dx^3 \right)^2, \tag{16}$$

where a, σ, α and C are the integration constants. Without loss of generality, one can always make $a = 1$ by rescaling the timelike coordinate, $\tilde{T} = aT$. In the following, we shall assume that this is always the case. The physical meaning of α and C depend on the choice of the angular coordinate φ. For example, if φ is chosen as x^3, then C will be related to the angular defect parameter D, and α has no physical meaning and can be transformed away by the rescaling $\tilde{x}^2 = \alpha x^2$. However, if φ is chosen as x^2, then the roles of α and C will be exchanged. The parameter σ is related, but not equal, to the mass per unit length [3], and physically acceptable sources have been found so far only for $0 \leq \sigma \leq 1$ [5]. When $\sigma = 0$, $1/2$, the corresponding solutions are flat in the region $0 < R < +\infty$. It was shown that in the case $\sigma = 1/2$, the (x^2, x^3)-plane can be extended to infinity, $-\infty < x^2$, $x^3 < +\infty$. Then, the resultant spacetime has plane symmetry and can be produced by a massive plane with uniform energy density [8]. Thus, in the vacuum case there are only two physically essential parameters, one is related to the mass per unit length, and the other is related to the angular defects.

Making the coordinate transformations,

$$\tilde{R} = \begin{cases} (2\sigma - 1)^{-2(2\sigma-1)^2/\mathcal{A}} R^{(2\sigma-1)^2}, & \sigma \neq 1/2, \\ \ln R, & \sigma = 1/2, \end{cases} \tag{17}$$

we find that the metric (16) can be written as,

$$ds^2 = \begin{cases} \tilde{R}^{4\sigma/(2\sigma-1)^2} \left(d\tilde{T}^2 - d\tilde{R}^2 \right) - \tilde{\alpha}^2 \tilde{R}^{4\sigma/(2\sigma-1)} \left(dx^2 \right)^2 - \tilde{C}^{-2} \tilde{R}^{2/(1-2\sigma)} \left(dx^3 \right)^2, & \sigma \neq 1/2, \\ e^{2\tilde{R}} \left(dT^2 - d\tilde{R}^2 \right) - \alpha^2 \left(dx^2 \right)^2 - C^{-2} \left(dx^3 \right)^2, & \sigma = 1/2, \end{cases} \tag{18}$$

where $\mathcal{A} \equiv 4\sigma^2 - 2\sigma + 1$, and

$$\tilde{T} \equiv (2\sigma - 1)^{4\sigma/\mathcal{A}} T, \quad \tilde{\alpha} \equiv \alpha(2\sigma - 1)^{4\sigma(2\sigma-1)/\mathcal{A}}, \quad \tilde{C} \equiv C(2\sigma - 1)^{2(2\sigma-1)/\mathcal{A}}. \tag{19}$$

It is interesting to note that, in the limit $\sigma \to 0$, the metric becomes locally Minkowski with x^3 as the angular coordinate and x^2 the axial coordinate, while as $\sigma \to +\infty$, the metric becomes also locally Minkowski but now with x^2 as the angular coordinate and x^3 the axial coordinate. This suggests that there may exist a critical value σ_c, when $\sigma < \sigma_c$, x^3 should be taken as the angular coordinate, and when $\sigma > \sigma_c$, x^2 should be taken as the coordinate. The analysis given below will confirm this speculation.

To study the spacetimes for the first three cases in Eq.(15), where the electromagnetic field is different from zero, let us consider them separately.

B. $B^2 \neq 0, \ \ B^0 = B^3 = 0$

In this case, since the components $F_{0\mu}$ vanish, the corresponding electromagnetic field is purely magnetic and produced by a current along the axis x^2 [15]. The corresponding EMT is given by

$$T_{\mu\nu} = \frac{\left(B^2\right)^2}{\kappa f l} \left(f\delta_\mu^0 \delta_\nu^0 + g\delta_\mu^1 \delta_\nu^1 + h\delta_\mu^2 \delta_\nu^2 - l\delta_\mu^3 \delta_\nu^3\right). \tag{20}$$

Solving the corresponding Einstein field equations, we find that the solutions are given by

$$f = g = R^{2m^2} G^2, \quad h = \frac{\alpha^2}{G^2}, \quad l = \frac{R^2 G^2}{C^2},$$

$$F^{\mu\nu} = \frac{CB^2}{\alpha R^{2m^2+1} G^2} \left(\delta_1^\mu \delta_2^\nu - \delta_2^\mu \delta_1^\nu\right), \quad B^2 = \pm \left(\frac{4c_1 c_2 m^2}{C^2}\right)^{1/2}, \tag{21}$$

where G is given by

$$G \equiv c_1 R^m + c_2 R^{-m}, \tag{22}$$

and α, C, c_1, c_2 and m are integration constants. Since $F_{\mu\nu}$ is real, we must have $c_1 c_2 \geq 0$ in the present case. These solutions are actually Witten's case 1 solutions [16].

When $m = 0$, the electromagnetic field vanishes, and the corresponding spacetime is locally Minkowski,

$$ds^2 = G^2 \left[dT^2 - dR^2 - \frac{\alpha^2}{G^4} \left(dx^2\right)^2 - \frac{R^2}{C^2} \left(dx^3\right)^2\right], \ (m = 0), \tag{23}$$

where $G = c_1 + c_2$. Clearly, now the angular coordinate φ should be chosen to be x^3, and the axial coordinate z to be x^2. Then, the constant C is related to the angular defect of the spacetime [1], and the constant α^2/A^4 can be made disappear by rescaling x^2, while the conformal factor A^2 can be transformed away by conformal transformations. Thus, in the above metric the only physically essential parameter is C.

Case B.1) $c_1 = 0$, $c_2 \neq 0$. In this case, we find $F^{\mu\nu} = 0$ and the metric becomes,

$$ds^2 = c_2^2 \left[R^{2m(m-1)}(dT^2 - dR^2) - \frac{\alpha^2 R^{2m}}{c_2^4} \left(dx^2\right)^2 - \frac{R^{2(1-m)}}{C^2} \left(dx^3\right)^2\right], \ (c_1 = 0). \tag{24}$$

Thus, without loss of generality we can set $c_2 = 1$. Then, comparing Eq.(24) with Eq.(18), we find that

$$m = \frac{2\sigma}{2\sigma - 1}, \ \ (c_1 = 0, \ \sigma \neq 1/2). \tag{25}$$

It is interesting to note that in this case there is no direct limit of the LC solution with $\sigma = 1/2$.

Case B.2) $c_1 \neq 0$, $c_2 = 0$. In this case we have $F^{\mu\nu} = 0$, too, and the corresponding metric and the constant m can be obtained from Eqs.(24) and (25) by replacing c_2 by c_1 and changing the sign of m, i.e.,

$$ds^2 = c_1^2 \left[R^{2m(m+1)}(dT^2 - dR^2) - \frac{\alpha^2 R^{-2m}}{c_1^4} \left(dx^2\right)^2 - \frac{R^{2(1+m)}}{C^2} \left(dx^3\right)^2\right], \ (c_2 = 0), \tag{26}$$

and

$$m = -\frac{2\sigma}{2\sigma - 1}, \ \ (c_2 = 0, \ \sigma \neq 1/2). \tag{27}$$

Case B.3) $c_1 c_2 \neq 0$. In this case only one of the two parameters c_1 and c_2 is physically essential [17]. As a matter of fact, defining

$$c_1 = \beta^m \gamma, \quad c_2 = \frac{\gamma}{\beta^m}, \tag{28}$$

where $\beta > 0$, we find that the corresponding metric takes the form

$$ds^2 = \gamma^2 \left[R^{2m^2} G_+^{\ 2} \left(dT^2 - dR^2 \right) - \frac{\bar{\alpha}^2}{G_+^{\ 2}} \left(dx^2 \right)^2 - \frac{R^2 G_+^{\ 2}}{C^2} \left(dx^3 \right)^2 \right], \ (c_1 c_2 \neq 0), \tag{29}$$

where

$$G_+ \equiv (\beta R)^m + (\beta R)^{-m}, \quad \bar{\alpha} \equiv \frac{\alpha}{\gamma^2}. \tag{30}$$

Then, by a trivial conformal transformation, $\bar{g}_{\mu\nu} = \gamma^2 g_{\mu\nu}$, we can set $\gamma = 1$. In the following we shall assume that this is always the case whenever it is applicable. Therefore, when the LC solutions are coupled with electromagnetic fields they have only three physically independent parameters.

To study the singularity behavior of the solutions, we find that

$$F \equiv F^{\alpha\beta} F_{\alpha\beta} = \frac{8m^2}{R^{2(m^2+1)} G_+^4},$$

$$I \equiv R^{\alpha\beta\gamma\sigma} R_{\alpha\beta\gamma\sigma} = \frac{16m^2}{R^{4(m^2+1)} G_+^8} \times$$

$$\left\{ (m+1)^2 \left[m(m+1) + 1 \right] (\beta R)^{4m} + (m-1)^2 \left[m(m-1) + 1 \right] (\beta R)^{-4m} \right.$$

$$\left. -6m(m+1)^2 (\beta R)^{2m} + 6m(m-1)^2 (\beta R)^{-2m} - 2(m^4 - 12m^2 + 1) \right\}, \tag{31}$$

from which we find that

$$F = \begin{cases} \infty, & m \neq \pm 1 \\ 8\beta^4, & m = \pm 1 \end{cases}, \quad I = \begin{cases} \infty, & m \neq \pm 1 \\ -320\beta^8, & m = \pm 1 \end{cases}, \tag{32}$$

as $R \to 0^+$, and F and I all go to zero as $R \to +\infty$. One can show that all the fourteen scalars built from the Riemann tensor have similar behavior. Therefore, all these solutions are asymptotically flat as $R \to +\infty$ and singular at $R = 0$, except for the ones with $m = \pm 1$. The singularities at $R = 0$ are timelike and naked. The corresponding Penrose diagram for the solutions with $m \neq \pm 1$ is given by Fig. 1(a).

Case $B.3.a$) $m = +1$. When $m = +1$ the metric (29) takes the form

$$ds^2 = \beta^{-2} \left\{ \left[1 + (\beta R)^2 \right]^2 \left(dT^2 - dR^2 - dz^2 \right) - \frac{R^2}{\left[1 + (\beta R)^2 \right]^2} d\varphi^2 \right\}, \tag{33}$$

where

$$z \equiv \frac{x^3}{C}, \quad \varphi \equiv \alpha \beta^2 x^2. \tag{34}$$

Obviously, this solution is locally flat on the axis $R = 0$, asymptotically flat as $R \to +\infty$, and free of any kind of spacetime singularities in the whole spacetime. Thus, in this case the spacetime is geodesically complete and the corresponding Penrose diagram is given by Fig. 1(a), too, but now the vertical line $R = 0$ is free of spacetime singularities. Its applications to Cosmology were first studied by Melvin [18] and Thorne [19].

Case $B.3.b$) $m = -1$. The metric for $m = -1$ can be obtained from Eqs.(33) and (34) by replacing β by $1/\beta$.

C. $B^3 \neq 0, \ B^0 = B^2 = 0$

In this case, the only non-vanishing component of $F^{\mu\nu}$ is F^{13}, and the corresponding electromagnetic field is purely magnetic and produced by a current along the axis x^3. The EMT given by Eq.(13) now becomes

$$T_{\mu\nu} = \frac{\left(B^3\right)^2}{\kappa f h} \left(f\delta_\mu^0\delta_\nu^0 + g\delta_\mu^1\delta_\nu^1 - h\delta_\mu^2\delta_\nu^2 + l\delta_\mu^3\delta_\nu^3\right). \tag{35}$$

Solving the corresponding Einstein field equations, we find that the solutions are given by

$$f = g = R^{2m^2}G^2, \quad h = \frac{R^2G^2}{C^2}, \quad l = \frac{\alpha^2}{G^2},$$

$$F^{\mu\nu} = \frac{CB^3}{\alpha R^{2m^2+1}G^2} \left(\delta_1^\mu\delta_3^\nu - \delta_3^\mu\delta_1^\nu\right), \quad B^3 = \pm\left(\frac{4c_1c_2m^2}{C^2}\right)^{1/2}, \tag{36}$$

where G is still given by Eq.(22). These solutions are actually Witten's Case 2 solutions [16]. Comparing Eq.(21) with Eq.(36), we find that if we exchange the two coordinates x^2 and x^3, we shall get one solution from the other. Hence, the physical and geometrical properties of these solutions can be obtained from the ones given by Eq.(21) by exchanging the two coordinates x^2 and x^3.

D. $B^0 \neq 0$, $B^2 = B^3 = 0$

In this case, the only non-vanishing component of $F^{\mu\nu}$ is F^{01}, and the corresponding electromagnetic field is purely electric and produced by an axial charge distribution. The EMT given by Eq.(13) now becomes

$$T_{\mu\nu} = \frac{\left(B^0\right)^2}{\kappa h l} \left(f\delta_\mu^0\delta_\nu^0 - g\delta_\mu^1\delta_\nu^1 + h\delta_\mu^2\delta_\nu^2 + l\delta_\mu^3\delta_\nu^3\right), \tag{37}$$

and the corresponding solutions are given by

$$f = G^{-2}, \quad l = \frac{R^2G^2}{C^2}, \quad g = \alpha^{-2}h = R^{2m^2}G^2,$$

$$F^{\mu\nu} = \frac{CB^0}{\alpha R^{2m^2+1}G^2} \left(\delta_0^\mu\delta_1^\nu - \delta_1^\mu\delta_0^\nu\right), \quad B^0 = \pm\left(\frac{-4\alpha^2c_1c_2m^2}{C^2}\right)^{1/2}. \tag{38}$$

Clearly, to have $F^{\mu\nu}$ be real, now we must require $c_1c_2 \leq 0$. The above solutions are Witten's Case 3 solutions [16]. This class of solutions have been studied by several authors using different forms of metric [20]. The form used above is the same as in [11].

Case D.1) $c_1c_2 \neq 0$. In this case, the solutions have three physically independent parameters, too. In fact, introducing two parameters β and γ via the relations,

$$c_1 = \beta^m\gamma, \quad c_2 = -\frac{\gamma}{\beta^m}, \tag{39}$$

where $\beta > 0$, we find that the corresponding metric takes the form

$$ds^2 = \gamma^2\left[G_-^{-2}\left(\frac{dT}{\gamma^2}\right)^2 - R^{2m^2}G_-^2 dR^2 - \alpha^2 R^{2m^2}G_-^2\left(dx^2\right)^2 - \frac{R^2G_-^{-2}}{C^2}\left(dx^3\right)^2\right], \quad (c_1c_2 \neq 0), \tag{40}$$

where

$$G_- \equiv (\beta R)^m - (\beta R)^{-m}. \tag{41}$$

Similarly, the constant γ can be transformed away by a constant conformal transformation. Therefore, without loss of generality, in the following we shall set $\gamma = 1$. Then, it can be shown that the corresponding quantities F and I are given by

$$F \equiv F^{\alpha\beta}F_{\alpha\beta} = -\frac{8m^2}{R^{2(m^2+1)}G_-^4},$$

$$I \equiv R^{\alpha\beta\gamma\sigma}R_{\alpha\beta\gamma\sigma} = \frac{16m^2}{R^{4(m^2+1)}G_-^8} \times$$

$$\left\{(m+1)^2\left[m(m+1)+1\right](\beta R)^{4m} + (m-1)^2\left[m(m-1)+1\right](\beta R)^{-4m}\right.$$

$$\left. +6m(m+1)^2(\beta R)^{2m} - 6m(m-1)^2(\beta R)^{-2m} - 2(m^4 - 12m^2 + 1)\right\}, \tag{42}$$

from which we find that

$$F = \begin{cases} -\infty, & m \neq \pm 1 \\ -8\beta^4, & m = \pm 1 \end{cases}, \qquad I = \begin{cases} \infty, & m \neq \pm 1 \\ -320\beta^8, & m = \pm 1 \end{cases},$$ (43)

as $R \rightarrow 0^+$, and F and I all go to zero as $R \rightarrow +\infty$. Therefore, all these solutions are asymptotically flat as $R \rightarrow +\infty$ and singular at $R = 0$, except for the ones with $m = \pm 1$. In addition to the singularities at $R = 0$, the solutions are also singular at $R = \beta^{-1}$ where $G_- = 0$. Thus, when $m \neq \pm 1$, the semi-axis $R \geq 0$ is divided into two parts, $0 \leq R \leq \beta^{-1}$ and $\beta^{-1} \leq R < +\infty$, by the singularities located, respectively, at $R = 0$ and $R = \beta^{-1}$. While the physics of the spacetime in the region $0 \leq R \leq \beta^{-1}$ is not clear, one can introduce a new coordinate R' by $R' = R - \beta^{-1}$ in the region $\beta^{-1} \leq R < +\infty$, so the solutions are singular at $R' = 0$ and asymptotically flat as $R' \rightarrow +\infty$. The spacetime is maximal in this region and the corresponding Penrose diagram is given by Fig. 1(a).

Case D.1.a) $m = +1$. When $m = 1$, the metric takes the form,

$$ds^2 = \frac{(\beta R)^2}{\left[1 - (\beta R)^2\right]^2} dT^2 - \frac{\left[1 - (\beta R)^2\right]^2}{\beta^2} \left[dR^2 + \alpha^2 \left(dx^2\right)^2 + C^{-2} \left(dx^3\right)^2\right], \ (c_1 c_2 \neq 0, m = 1),$$ (44)

Introducing a new coordinate r via the relation

$$r = \frac{1}{2} \left(\beta^{-2} - R^2\right),$$ (45)

we find that the metric takes the form

$$ds^2 = 2\beta^2 \left\{ f(r)dt^2 - f^{-1}(r)dr^2 - 2r^2 \left[\alpha^2 \left(dx^2\right)^2 + C^{-2} \left(dx^3\right)^2\right] \right\}, \ (c_1 c_2 \neq 0, m = 1),$$ (46)

where $t = (\beta^2 T)/2$ and

$$f(r) \equiv \frac{r_0 - r}{r^2},$$ (47)

with $r_0 \equiv \beta^{-2}/2$. From Eq.(45) we can see that the spacetime singularity at $R = \beta^{-1}$ is mapped to $r = 0$, and the hypersurface $R = 0$ is mapped to $r = r_0$. The region $\beta^{-1} \leq R < +\infty$ is mapped into the region $r \leq 0$. In this region the singularity at $r = 0$ is naked and timelike, and the corresponding Penrose diagram is given by Fig. 1(a). The region $0 \leq R \leq \beta^{-1}$ is mapped into $0 \leq r \leq r_0$, which will be referred as to Region I. The metric is singular at r_0. As showed above, this singularity is not a curvature one, and we need to extend the spacetime beyond it. Since the part

$$r^2 \left[\alpha^2 \left(dx^2\right)^2 + C^{-2} \left(dx^3\right)^2\right],$$

is regular across $r = r_0$, we need to consider the extension only for the part

$$d\sigma^2 = f(r)dt^2 - f^{-1}(r)dr^2,$$ (48)

which is similar to the Schwarzschild solution with spherical symmetry [4]. Following the same procedure for the extension of the Schwarzschild solution, we find that the corresponding Penrose diagram now is given by Fig. 1(b). In this diagram there are three extended regions, I', II and II', where Region I' is symmetric with Region I, while Region II' is symmetric with Region II, where in Region II we have $r_0 < r < +\infty$. The spacetime is asymptotically flat as $r \rightarrow +\infty$. The nature of the hypersurfaces $r = r_0$, however, is different from $r = 2m$ in the Schwarzschild case, representing now Cauchy horizons.

Case D.1.b) $m = -1$. When $m = -1$ the solution can be obtained from Eq.(46) by replacing β by $-1/\beta$.

Case D.2) $c_1 = 0$, $c_2 \neq 0$. In this case , the corresponding electromagnetic field vanishes, and the solutions reduce to

$$ds^2 = c_2^2 \left\{ R^{2m} \left(\frac{dT}{c_2^2}\right)^2 - R^{2m(m-1)} \left[dR^2 + \alpha^2 \left(dx^2\right)^2\right] - C^{-2} R^{2(1-m)} \left(dx^3\right)^2 \right\}, \ (c_1 = 0).$$ (49)

Comparing it with Eq.(16) we find that

$$m = 2\sigma, \ (c_1 = 0). \tag{50}$$

Case D.3) $c_1 \neq 0$, $c_2 = 0$. In this case, the corresponding solutions can be obtained from Eq.(49) by replacing c_2 by c_1. Then, the relation of m with σ is now given by

$$m = -2\sigma, \ (c_2 = 0). \tag{51}$$

III. SOURCES OF THE LC SOLUTIONS WHEN COUPLED WITH ELECTROMAGNETIC FIELDS

As showed in the last section, all the solutions are singular at the axis, except for the cases $m = \pm 1$. These singularities are usually considered as representing sources. However, these sources to be physically acceptable have to satisfy certain conditions, such as, the weak, dominant, and/or strong energy conditions [4].

In this paper, we shall consider shell-like sources for the spacetimes studied in Section II. Assume that the shell, located on the hypersurface Σ, divides the whole spacetime into two regions, $V^{\pm}$. Let V^+ denote the region outside the shell, and V^- denote the region inside the shell. In V^+, the metric takes the form

$$ds_+^2 = f^+(R)dT^2 - g^+(R)dR^2 - h^+(R)\left(dx^2\right)^2 - l^+(R)\left(dx^3\right)^2, \ (R \geq R_0), \tag{52}$$

where $\{x^{+\mu}\} = \{T, \ R, \ x^2, \ x^3\}$, and $R = R_0 = Const.$ is the location of the shell in the coordinates $x^{+\mu}$. In V^-, the metric takes the form

$$ds_-^2 = f^-(r)dt^2 - g^-(r)dr^2 - h^-(r)\left(dx^2\right)^2 - l^-(r)\left(dx^3\right)^2, \ (r \leq r_0), \tag{53}$$

where $\{x^{-\mu}\} = \{t, \ r, \ x^2, \ x^3\}$, and the hypersurface $r = r_0 = Const.$ is the location of the shell in the coordinates $x^{-\mu}$. On the shell, the intrinsic coordinates will be chosen as $\{\xi^a\} = \{\tau, \ x^2, \ x^3\}$, $(a = 1, 2, 3)$, where τ denotes the proper time of the shell. In terms of ξ^a, the metric on the shell takes the form

$$ds^2\big|_{\Sigma} = \gamma_{ab}d\xi^a d\xi^b = d\tau^2 - h\left(dx^2\right)^2 - l\left(dx^3\right)^2, \tag{54}$$

where γ_{ab} denotes the induced metric on the hypersurface. The first junction condition requires that the metrics in both sides of the shell reduce to the same metric (54), that is,

$$\left[f^+(R_0)\right]^{1/2}dT = \left[f^-(r_0)\right]^{1/2}dt = d\tau,$$
$$h^+(R_0) = h^-(r_0) = h, \quad l^+(R_0) = l^-(r_0) = l. \tag{55}$$

Note that in writing the above expressions, we have chosen, without loss of generality, dT, dt and $d\tau$ to have the same sign. The normal vector to the hypersurface Σ is given in V^+ and V^-, respectively, by

$$n_{\mu}^+ = \left[g^+(R_0)\right]^{1/2}\delta_{\mu}^R, \ n_{\mu}^- = \left[g^-(r_0)\right]^{1/2}\delta_{\mu}^r. \tag{56}$$

On the hypersurface Σ, let us introduce the vectors, $e_{(a)}^{\pm\mu}$, defined by $e_{(a)}^{\pm\mu} \equiv \partial x^{\pm\mu}/\partial\xi^a$, we find that

$$e_{(1)}^{+\mu} = \left[f^+(R_0)\right]^{-1/2}\delta_T^{\mu}, \quad e_{(2)}^{+\mu} = \delta_2^{\mu}, \quad e_{(3)}^{+\mu} = \delta_3^{\mu},$$
$$e_{(1)}^{-\mu} = \left[f^-(r_0)\right]^{-1/2}\delta_t^{\mu}, \quad e_{(2)}^{-\mu} = \delta_2^{\mu}, \quad e_{(3)}^{-\mu} = \delta_3^{\mu}. \tag{57}$$

Then, the extrinsic curvatures $K_{ab}^{\pm}$, defined by [1]

---

[1] Note that in this paper the definition for the extrinsic curvature tensor is the same as that given in [21] but different from Israel's by a sign [10].

$$K_{ab}^{\pm} = -e_{(a)}^{\pm\,\alpha} e_{(b)}^{\pm\,\beta} \left\{ \frac{\partial^2 n_{\alpha}^{\pm}}{\partial \xi^a \partial \xi^b} - \Gamma_{\alpha\beta}^{\pm\,\lambda} n_{\lambda}^{\pm} \right\}, \tag{58}$$

have the following non-vanishing components,

$$K_{11}^{+} = \frac{f_{,R}^{+}}{2f^{+}(g^{+})^{1/2}}, \quad K_{22}^{+} = -\frac{h_{,R}^{+}}{2(g^{+})^{1/2}}, \quad K_{33}^{+} = -\frac{l_{,R}^{+}}{2(g^{+})^{1/2}},$$

$$K_{11}^{-} = \frac{f_{,r}^{-}}{2f^{-}(g^{-})^{1/2}}, \quad K_{22}^{-} = -\frac{h_{,r}^{-}}{2(g^{-})^{1/2}}, \quad K_{33}^{-} = -\frac{l_{,r}^{-}}{2(g^{-})^{1/2}}. \tag{59}$$

In terms of $K_{ab}^{\pm}$, the surface energy-momentum tensor, τ_{ab}, is given by [10],

$$\tau_{ab} = \frac{1}{\kappa} \left\{ [K_{ab}] - \gamma_{ab} [K] \right\}, \tag{60}$$

where $[K_{ab}] \equiv K_{ab}^{+} - K_{ab}^{-}$ and $[K] \equiv \gamma^{ab} [K_{ab}]$. Substituting Eq.(59) into Eq.(60) we find that τ_{ab} can be written in the form,

$$\tau^{ab} = \rho w^a w^b + p_2 e_{(2)}^a e_{(2)}^b + p_3 e_{(3)}^a e_{(3)}^b, \tag{61}$$

where $w^a = \delta_{\tau}^a$, $e_{(2)}^a = h^{-1/2} \delta_2^a$, $e_{(3)}^a = l^{-1/2} \delta_3^a$, and

$$\rho = -\frac{1}{2\kappa} \left\{ \frac{1}{h} \left[\frac{h_{,R}^{+}}{(g^{+})^{1/2}} - \frac{h_{,r}^{-}}{(g^{-})^{1/2}} \right] + \frac{1}{l} \left[\frac{l_{,R}^{+}}{(g^{+})^{1/2}} - \frac{l_{,r}^{-}}{(g^{-})^{1/2}} \right] \right\},$$

$$p_2 = \frac{1}{2\kappa} \left\{ \left[\frac{f_{,R}^{+}}{f^{+}(g^{+})^{1/2}} - \frac{f_{,r}^{-}}{f^{-}(g^{-})^{1/2}} \right] + \frac{1}{l} \left[\frac{l_{,R}^{+}}{(g^{+})^{1/2}} - \frac{l_{,r}^{-}}{(g^{-})^{1/2}} \right] \right\},$$

$$p_3 = \frac{1}{2\kappa} \left\{ \left[\frac{f_{,R}^{+}}{f^{+}(g^{+})^{1/2}} - \frac{f_{,r}^{-}}{f^{-}(g^{-})^{1/2}} \right] + \frac{1}{h} \left[\frac{h_{,R}^{+}}{(g^{+})^{1/2}} - \frac{h_{,r}^{-}}{(g^{-})^{1/2}} \right] \right\}. \tag{62}$$

Thus, the surface EMT given above can be considered as representing a fluid with its velocity w^a, energy density ρ and pressures p_2 and p_3 in the two principal directions $e_{(2)}^a$ and $e_{(3)}^a$, respectively, provided that the fluid satisfies some energy conditions [4].

Once we have the general formulae for the matching of two static cylindrical regions, let us consider some specific models, where the solutions given in the last section are taken as valid only in the region V^{+} defined above. To make sure that the spacetimes indeed possess cylindrical symmetry, and that the LC vacuum solutions or the LC solutions coupled with an electromagnetic field are produced by a cylindrically symmetric source, in the region V^{-} we shall choose the metric as that of Minkowski,

$$ds_{-}^2 = dt^2 - dr^2 - dz^2 - r^2 d\varphi^2, \ (r \le r_0), \tag{63}$$

so that the spacetime and its symmetry inside the shell is well defined and free of any kind of spacetime singularities on the axis $r = 0$. Obviously, for such a matching a matter shell in general appears on the hypersurface $r = r_0$. Since inside the shell, the spacetime is flat and free of any kind of sources, the spacetime outside the shell is produced solely by the shell. To further study the problem, let us consider the four cases defined by Eq.(15) separately.

A. $B^0 = B^2 = B^3 = 0$

In this case, the spacetime outside the shell is described by the LC solutions, which are given by Eq.(16) or (18). Without loss of generality, in the following we shall consider only the metric of Eq.(16), which is valid for any σ.

Because of the ambiguity of specifying the angular and axial coordinates, let us first consider the possible identification $z = x^2$ and $\varphi = x^3$. In this case the first junction condition of Eq.(55) yields

$$\alpha = R_0^{-2\sigma(2\sigma-1)}, \quad C = \frac{R_0^{1-2\sigma}}{r_0}. \tag{64}$$

Then, from Eq.(62) we find that

$$\rho = \frac{1}{\kappa}\left[\frac{1}{r_0} - \frac{(2\sigma-1)^2}{R_0^{\mathcal{A}}}\right],$$

$$p_z = \frac{1}{\kappa}\left(\frac{1}{R_0^{\mathcal{A}}} - \frac{1}{r_0}\right), \quad p_\varphi = \frac{4\sigma^2}{\kappa R_0^{\mathcal{A}}}. \tag{65}$$

From the above expressions, it can be shown that the weak and strong energy conditions will be fulfilled when

$$r_0 \le \frac{R_0^{\mathcal{A}}}{(2\sigma-1)^2}, \quad 0 \le \sigma \le 1, \tag{66}$$

while the dominant energy condition requires

$$r_0 \le \begin{cases} \frac{R_0^{\mathcal{A}}}{(2\sigma-1)^2+1}, & 0 \le \sigma \le 1/3 \\ \frac{R_0^{\mathcal{A}}}{(2\sigma-1)^2+4\sigma^2}, & \sigma > 1/3 \end{cases}, \quad 0 \le \sigma \le 1. \tag{67}$$

Clearly, by properly choosing the constant r_0 the three energy conditions, weak, dominant and strong, can be all satisfied, provided that

$$0 \le \sigma \le 1. \tag{68}$$

That is, the solutions of Eq.(16) with $z = x^2$ and $\varphi = x^3$ can be produced by physically reasonable sources for $0 \le \sigma \le 1$. This is consistent with the conclusions obtained in [5].

When σ is larger, the two coordinates x^2 and x^3 change the roles, as we pointed in the last section. In the following, we shall show that this is indeed the case. Choosing $z = x^3$ and $\varphi = x^2$ in Eq.(16) we find that the first junction condition of Eq.(55) becomes

$$\alpha = \frac{r_0}{R_0^{2\sigma(2\sigma-1)}}, \quad C = R_0^{1-2\sigma}, \tag{69}$$

while Eq.(62) yields

$$\rho = \frac{1}{\kappa}\left[\frac{1}{r_0} - \frac{(2\sigma-1)^2}{R_0^{\mathcal{A}}}\right],$$

$$p_z = \frac{1}{\kappa R_0^{\mathcal{A}}}, \quad p_\varphi = \frac{1}{\kappa}\left(\frac{4\sigma^2}{R_0^{\mathcal{A}}} - \frac{1}{r_0}\right). \tag{70}$$

From these expressions, it can be shown that the weak energy condition will be satisfied when

$$r_0 \le \frac{R_0^{\mathcal{A}}}{(2\sigma-1)^2}, \quad \sigma \ge \frac{1}{4}, \tag{71}$$

and that the dominant energy condition will be satisfied when

$$r_0 \le \begin{cases} \frac{R_0^{\mathcal{A}}}{(2\sigma-1)^2+1}, & 1/4 \le \sigma \le 3/4 \\ \frac{R_0^{\mathcal{A}}}{4\sigma(2\sigma-1)+1}, & \sigma > 3/4 \end{cases}, \quad \sigma \ge \frac{1}{4}, \tag{72}$$

while the strong energy condition is fulfilled when Eq.(66) holds. Thus, by properly choosing the constant r_0 the three energy conditions can be all satisfied, for

$$\sigma \ge \frac{1}{4}. \tag{73}$$

That is, the solutions of Eq.(16) with $z = x^3$ and $\varphi = x^2$ are physically acceptable in the sense that they can be produced by cylindrical matter shells that satisfy all the three energy conditions, provided that $\sigma \geq 1/4$.

This confirms our early claim that when σ is large, the coordinate x^2 should be taken as the angular coordinate. From Eqs.(68) and (73) we can see that there exists a common range $1/4 \leq \sigma \leq 1$, in which the angular coordinate can be chosen as either x^2 or x^3. For each of such choices, the solutions can be produced by cylindrical matter shells that satisfy all the three energy conditions. However, considering Eq.(3), we can see that when $\sigma > 1/2$ the coordinate x^2 is more likely to play the role of the angular coordinate, while when $\sigma < 1/2$ the coordinate x^3 is more likely. When $\sigma = 1/2$, the corresponding solution becomes (locally) Minkowski and the metric coefficients g_{22} and g_{33} are constant. In [8] it was shown that it can be considered as representing the gravitational field produced by a massive plane with a uniform matter distribution. In this case, the ranges of the two coordinates x^2 and x^3 were extended to $-\infty < x^2,\ x^3 < +\infty$. The above considerations, on the other hand, show that the same solution can be also considered as representing the gravitational field produced by a cylindrical shell that satisfies all the energy conditions, but in the latter case one has to identify the hypersurface $x^2 = 0$ $(x^3 = 0)$ with the one $x^2 = 2\pi$ $(x^3 = 2\pi)$.

In any case, the above analysis shows clearly that *all the LC solutions with $\sigma \geq 0$ are physically acceptable*, in the sense that they can be produced by cylindrically symmetric sources that satisfy all the three energy conditions [4].

We would like to note that if the form of the metric Eq.(18) of the LC solutions is used as the exterior of the shell, we shall obtain the same conclusions, that is, for the solutions to be produced by a cylindrical shell that satisfies all the three energy conditions, we have to choose x^3 to be the angular coordinate φ for $0 \leq \sigma \leq 1$, and to choose x^2 to be the angular coordinate for $\sigma \geq 1/4$.

B. $B^2 \neq 0$, $B^0 = B^3 = 0$

In this case, the metric outside the shell is given by Eqs.(29) and (30) with $\gamma = 1$. In the following, let us first consider the case where $x^2 = z$ and $x^3 = \varphi$. Then, we can see that the first junction conditions Eq.(55) require

$$\alpha = G_+(R_0), \quad C = \frac{R_0 G_+(R_0)}{r_0}, \tag{74}$$

while Eq.(62) gives

$$\rho = \frac{1}{\kappa}\left(\frac{1}{r_0} - \frac{2}{\alpha R_0^{1+m^2}}\right), \quad p_\varphi = \frac{m^2}{\kappa \alpha R_0^{1+m^2}},$$

$$p_z = \frac{1}{\kappa}\left\{\frac{2m\left[(\beta R_0)^m - (\beta R_0)^{-m}\right]}{\alpha^2 R_0^{1+m^2}} + \frac{1+m^2}{\alpha R_0^{1+m^2}} - \frac{1}{r_0}\right\}. \tag{75}$$

From the above expressions we can show that all the three energy conditions can be satisfied by properly choosing the two constants r_0 and R_0, provided that

$$m < -(\sqrt{2} - 1), \quad \text{or} \quad m > \sqrt{2} - 1. \tag{76}$$

That is, in this case if we make the identification $x^2 = z$ and $x^3 = \varphi$, all the solutions with $m < -(\sqrt{2}-1)$, or with $m > \sqrt{2} - 1$ can be produced by cylindrically symmetric shells that satisfy all the three energy conditions.

As shown in the last section, the parameter m is related to σ via the relation,

$$m = \pm\frac{2\sigma}{2\sigma - 1}, \tag{77}$$

where the signs "$\pm$" depend on the way how to take the vacuum limits. However, in any case, in terms of σ, Eq.(76) takes the form,

$$\sigma < -\frac{1}{\sqrt{8}}, \quad \text{or} \quad \sigma > +\frac{1}{\sqrt{8}}. \tag{78}$$

Comparing this result with the corresponding one for the LC solutions obtained in the last subsection, we can see that the coupling of the electromagnetic field with the gravitational field of the LC solutions extends the range $0 \le \sigma \le 1$ to the range $\sigma > 1/\sqrt{8}$ or to $\sigma < -1/\sqrt{8}$. The extension to the negative values of σ is particularly interesting, as in the vacuum case $\sigma < 0$ corresponds to the situation where the solutions are produced by negative mass [3,5].

Now let us turn to consider the identification $x^2 = \varphi$ and $x^3 = z$. Then, the first junction condition Eq.(55) requires

$$\alpha = r_0 G_+(R_0), \quad C = R_0 G_+(R_0), \tag{79}$$

while from Eq.(62) we find that

$$\rho = \frac{1}{\kappa} \left[\frac{1}{r_0} - \frac{1}{G_+(R_0)R_0^{1+m^2}} \right], \quad p_\varphi = \frac{1}{\kappa} \left[\frac{m^2}{G_+(R_0)R_0^{1+m^2}} - \frac{1}{r_0} \right],$$

$$p_z = \frac{1}{\kappa G_+(R_0)R^{1+m^2}} \left\{ (1+m^2) + \frac{2m}{G_+(R_0)} \left[(\beta R_0)^m - (\beta R_0)^{-m} \right] \right\}. \tag{80}$$

From these expressions it can be shown that by properly choosing the constant r_0, all the three energy conditions can be satisfied, provided that

$$m^2 \ge 1, \quad \text{or} \quad \sigma \ge +\frac{1}{4}. \tag{81}$$

This is the same condition as that given in the corresponding vacuum solutions, given by Eq.(73).

C. $B^3 \ne 0$, $B^0 = B^2 = 0$

As we noted previously, the case with $B^3 \ne 0$, $B^0 = B^2 = 0$ can be obtained from the case $B^2 \ne 0$, $B^0 = B^3 = 0$ by exchanging the two spacelike coordinates z and φ. Since in the above, both possibilities of $(x^2, x^3) = (z, \varphi)$ and $(x^2, x^3) = (\varphi, z)$ were considered, the above analysis in fact already included the case $B^3 \ne 0$, $B^0 = B^2 = 0$, so in the following we shall not consider it any more.

D. $B^0 \ne 0$, $B^2 = B^3 = 0$

In this case, the metric outside the shell is given by Eqs.(40) and (41) with $\gamma = 1$. As we showed in the last section, the spacetimes are singular at both $R = 0$ and $R = \beta^{-1}$. The physics of the spacetimes in the region $0 \le R \le \beta^{-1}$ is not clear (if there is any), while the spacetimes in the region $\beta^{-1} < R < +\infty$ are maximal with a naked singularity at $R = \beta^{-1}$. Thus, in this case in order to avoid the presence of spacetime singularities outside the shell, we shall assume that $R_0 > \beta^{-1}$, where R_0 denotes the location of the shell in the coordinates T, R, z, and φ, and choose, without loss of generality, the function $G(R)$ as a monotonically increasing function,

$$G(R) = (\beta R)^{|m|} - (\beta R)^{-|m|}. \tag{82}$$

As in the previous cases, now we have two possibilities of identifying the axial and angular coordinates z and φ. Let us first consider the identification $x^2 = z$ and $x^3 = \varphi$. Then, we find that the first junction condition Eq.(55) requires

$$\alpha = \frac{1}{R_0^{m^2} G_-(R_0)}, \quad C = \frac{R_0 G_-(R_0)}{r_0}, \tag{83}$$

and that Eq.(62) gives

$$\rho = \frac{1}{\kappa} \left\{ \frac{1}{r_0} - \frac{1+m^2}{R_0^{1+m^2} G_-(R_0)} - \frac{2|m|}{R_0^{1+m^2} G_-^2(R_0)} \left[(\beta R_0)^{|m|} + (\beta R_0)^{-|m|} \right] \right\},$$

$$p_z = \frac{1}{\kappa} \left\{ \frac{1}{R_0^{1+m^2} G_-(R_0)} - \frac{1}{r_0} \right\}, \quad p_\varphi = \frac{m^2}{\kappa R_0^{1+m^2} G_-(R_0)}. \tag{84}$$

From these expressions it can be shown that none of the three energy conditions is satisfied. Thus, unlike the last subcase, now due to the presence of the purely electric field, all the solutions in this case cannot be produced by physically acceptable thin shells.

If we choose $x^2 = \varphi$ and $x^3 = z$, then we find that

$$\alpha = \frac{r_0}{R_0^{m^2} G_-(R_0)}, \quad C = R_0 G_-(R_0), \tag{85}$$

and that

$$\rho = \frac{1}{\kappa} \left\{ \frac{1}{r_0} - \frac{1+m^2}{R_0^{1+m^2} G_-(R_0)} - \frac{2|m|}{R_0^{1+m^2} G_-^2(R_0)} \left[(\beta R_0)^{|m|} + (\beta R_0)^{-|m|} \right] \right\},$$

$$p_z = \frac{1}{\kappa R_0^{1+m^2} G_-(R_0)}, \quad p_\varphi = \frac{1}{\kappa} \left\{ \frac{m^2}{R_0^{1+m^2} G_-(R_0)} - \frac{1}{r_0} \right\}, \tag{86}$$

from which it can be shown that, similar to the last case, none of the three energy conditions is satisfied for any choice of the free parameters involved.

Therefore, due to the presence of the purely electric field produced by a charge distribution along one of the space-like coordinates z and φ, the solutions cannot be produced by physically acceptable cylindrical shell-like sources.

IV. CONCLUDING REMARKS

The Levi-Civita solutions coupled with an electromagnetic field are usually classified into three different families. In the first two, the electromagnetic fields are purely magnetic and produced by a current along, respectively, the spacelike axis x^2 and x^3, while in the last family it is purely electric and produced by a charge distribution along the axial axis. In this paper, the local and global properties of all these solutions have been studied, and in particular found that all the solutions have a naked singularity at $R = 0$, except for the ones with $m = \pm 1$. In the latter case, two solutions are distinguishable, one, given by Eq.(33), is free of any kind of spacetime singularities, and the corresponding spacetime is geodesically complete. The other, given by Eq.(44), has a coordinate singularity at $r = r_0$. After being maximally extended beyond this hypersurface, it has been found that this hypersurface actually represents Cauchy horizons. It has been also found that the solutions that represent the purely electric fields are also singular at a finite radial distance $R = \beta^{-1}$. In this case one can introduce a new radial coordinate $R' = R - R_0$, so that in terms of R' these singularities occur at $R' = 0$. Then, one can consider $R' = 0$ as the new axis, and the resultant spacetimes are asymptotically flat as $R' \to +\infty$ and maximal in the region $0 \leq R' < +\infty$ with a naked singularity on the axis.

The limits of these solutions to vacuum case have been also studied, and found that such limits are not unique. As a matter of fact, at least there exist two different ways to take such limits. For each limit one will get the Levi-Civita vacuum solutions with different range of values for the parameter σ. From such limiting process, we have found that when $\sigma \to +\infty$, the metric becomes Minkowski with x^2 as the angular coordinate [9], while when $\sigma \to 0^+$, the metric becomes Minkowski, too, but now with x^3 being the angular coordinate. This observation leads us to believe that at certain value of σ, the two spacelike coordinates x^2 and x^3 change their roles. By constructing cylindrical thin shells, we have been able to confirm our above expectation, that is, we have found that, if we make the identification $(x^2, x^3) = (z, \varphi)$, the corresponding Levi-Civita vacuum solutions can be produced by physically acceptable thin shells only when $0 \leq \sigma \leq 1$. However, if we make the identification $(x^2, x^3) = (\varphi, z)$, the corresponding Levi-Civita vacuum solutions can be produced by physically acceptable thin shells for $\sigma \geq 1/4$.

Cylindrically symmetric thin shells for the Levi-Civita solutions coupled with electromagnetic fields have been also studied, and found that, in the case of purely magnetic field, due to the coupling of the

magnetic field with the gravitational field, the range, $0 \leq \sigma \leq 1$, of the corresponding vacuum case, has been extended to $\sigma > 1/\sqrt{8}$ or to $\sigma < -1/\sqrt{8}$. The latter extension is very remarkable, as in the vacuum case it corresponds to the spacetimes that are produced by negative masses [3,5]. However, in the case of purely electric field, it has been found that the solutions resulted from both of the two identifications, $(x^2, x^3) = (z, \varphi)$ and $(x^2, x^3) = (\varphi, z)$, cannot be produced by any physically acceptable cylindrical thin shells. This causes some caution on the physical reality of these solutions. Although the sources considered in this paper are the most general cylindrical thin shells, one may still argue that they are still not general enough to give a definitive answer to this problem, and therefore, it would be very interesting to look for other kinds of non-shell-like sources for these solutions.

ACKNOWLEDGEMENTS

[1] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and other Topological Defects*, (Cambridge University Press, Cambridge, 1994); M.B. Hindmarsh and T.W.B. Kibble, Rep. Prog. Phys. **58**, 477 (1995).

[2] T.A. Apostolatos and K.A. Thorne, Phys. Rev. **D46**, 2433 (1992); P.R.C.T. Pereira and A.Z. Wang, "*Gravitational Collapse of Cylindrical Shells made of Counter-Rotating Dust particles*," gr-qc/0007077, Phys. Rev. **D62**, in press (2000) and references therein.

[3] W.B. Bonnor, Gen. Rel. Grav. **24** 551 (1992); W.B. Bonnor, J.B. Griffiths and M.A.H. MacCallum, *ibid.* **26**, 687 (1994); M.F.A. da Silva, L. Herrera, F.M. Paiva, and N.O. Santos, J. Math. Phys. **36**, 3625 (1995); S. Haggag and F. Dsokey, Class. Quantum Grav. **13**, 3221 (1996).

[4] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime*, (Cambridge University Press, Cambridge, 1973).

[5] A.Z. Wang, M.F.A. da Silva, and N.O. Santos, Class. Quantum Grav. **14**, 2417 (1997).

[6] M.F.A. da M.F.A. da Silva, A.Z. Wang, F.M. Paiva, and N.O. Santos, Phys. Rev. **D61**, 044003 (2000).

[7] R. Gautreau and R.B. Hoffman, Nuovo Cimento, **B61**, 411 (1969); W.B. Bonnor, "The Static Cylinder in General Relativity," in *On Einstein's Path*, edt. by Alex Harvey, (New York, Springer, 1999), p. 113; L. Herrera, J. Ruifernández, and N.O. Santos, "*The Static Cylinder, Gyroscopes, the C-metric, and All That*," gr-qc/0010017, Gen. Rel. Grav. in press (2000).

[8] M.F.A. da Silva, A.Z. Wang, and N.O. Santos, Phys. Lett. **A 244**, 462 (1998).

[9] T.G. Philbin, Class. Quantum Grav. **13**, 1217 (1996).

[10] W. Israel, Nuovo Cimento, **B44**, 1 (1966); *ibid.* **48B**, 463(E) (1967).

[11] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein's Field Equations,* (Cambridge University Press, Cambridge, 1980).

[12] P. R. C. T. Pereira, N. O. Santos and A.Z. Wang, Class Quantum Grav. **13**, 1641 (1996).

[13] M.A.H. MacCallum and N.O. Santos, Class. Quantum Grav. **15**, 1627 (1998).

[14] S.W. Hawking, Phys. Rev. **D46**, 603 (1992).

[15] W.B. Bonnor, Proc. Phys. Soc. Lond. **A67**, 225 (1954).

[16] L. Witten, in *Gravitation: an Introduction to current Research*, ed. by L. Witten (John Wiley & Sons, New York, 1962), pp.382-437.

[17] J.L. Safko and L. Witten, J. Math. Phys. **12**, 257 (1971).

[18] M.A. Melvin, Phys. Lett. **8**, 65 (1964); Phys. Rev. **B139**, 225 (1965).

[19] K. S. Thorne, Phys. Rev. **B138**, 251 (1965); *ibid.*, **B139**, 244 (1965).

[20] B.C. Mukherji, Bull. Cal. Math. Soc. **30**, 95 (1938); W.B. Bonnor, Proc. Phys. Soc. **A66**, 145 (1953); A.K. Raychaudhuri, Ann. Phys. (N.Y.), **11**, 501 (1960).

[21] P.R.C.T. Pereira and A.Z. Wang, "*Gravitational Collapse of Cylindrical Shells made of Counter-Rotating Dust particles*," gr-qc/0007077, Phys. Rev. D, in press (2000).

R $= 0$

R $= +\infty, T = -\infty$

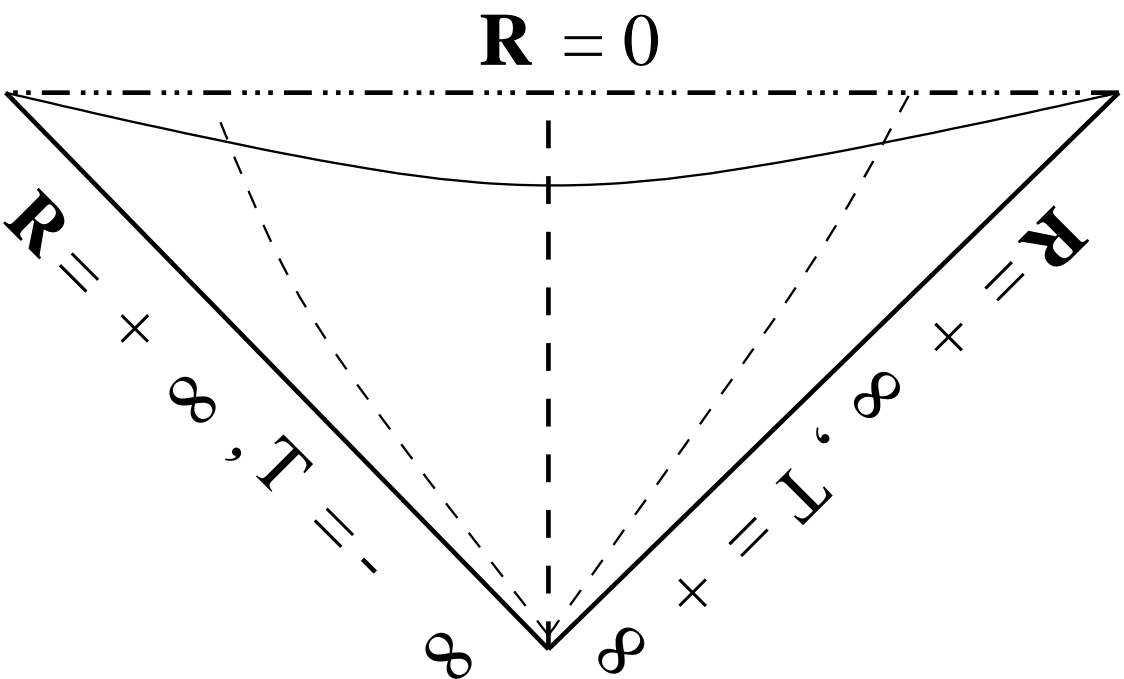**R** $= +\infty, T = +\infty$

Fig. 1 The Penrose diagram. The vertical line $R = 0$ in general represents a naked spacetime singularity. The dashed lines represent the hypersurfaces $T = Const.$, and the vertical curved line represents the hypersurface $R = Const.$

r = + ∞ , t = + ∞

II

$r = r_o$ $r = r_o$

r = 0   I'   I   r = 0
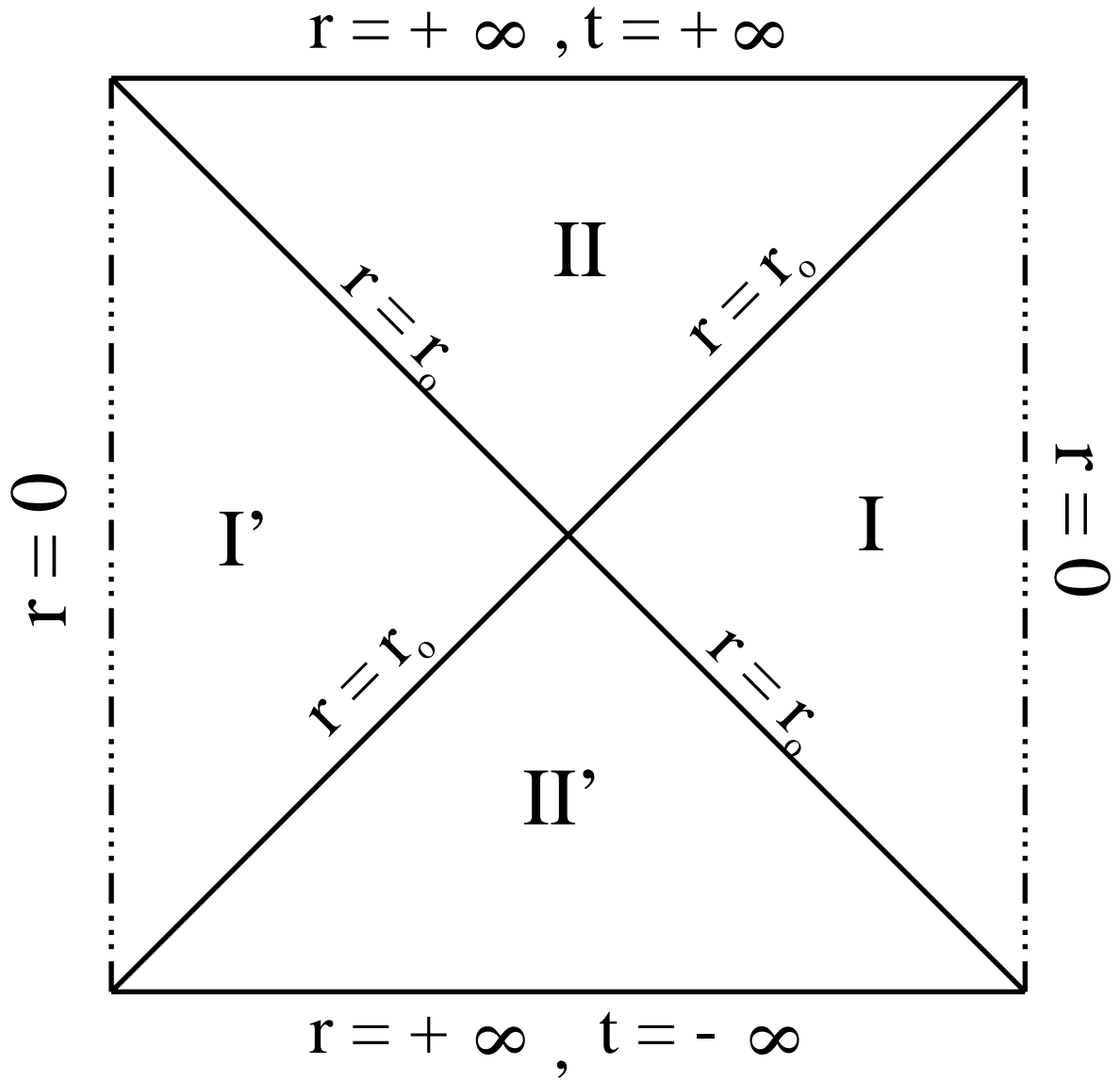
$r = r_o$ $r = r_o$

II'

r = + ∞ , t = - ∞

Fig. 2 The corresponding Penrose diagram for the solution given by Eq.(46). The vertical lines $r = 0$ represent the spacetime singularities that are timelike, and the ones $r = r_0$ represent Cauchy horizons. Region I (II) is symmetric to Region I (II'), and Regions II and II' are asymptotically flat.