Gauging N=2 Supersymmetric Non-Linear σ-Models in the Atiyah-Ward Space-Time

by

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ABSTRACT

We build up a class of N=2 supersymmetric non-linear σ-models in an N=1 superspace based on the Atiyah-Ward space-time of (2+2)–signature metric. We also discuss the gauging of isometries of the associated hyper-Kählerian target spaces and present the resulting gauge-covariant supersymmetric action functional.

Key-words: σ-Models; Supersymmetry.

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1   Introduction

In the recent years much attention has been paid to the construction of new classical field models in the Atiyah-Ward space-time of (2+2)–signature metric [1]. As demonstrated in refs.[2, 3], this structure emerges in connection with a consistent N=2 superstring theory, whose underlying superconformal algebra requires a complex manifold as the relevant space-time background.

From the viewpoint of mathematics, the Atiyah-Ward space-time is also quite attractive when regarded as a four-dimensional arena in which one could introduce self-dual Yang-Mills connections [4, 5]. In fact, these objects are known to play a significant role as a field-theoretical tool in the Donaldson's programme on algebraic geometry [6] and, as conjectured by Ward [7], may be also of importance in the classification of lower-dimensional integrable models.

In view of these facts, it seems also interesting to build up and analyze supersymmetric Yang-Mills theories in the Atiyah-Ward space-time. Indeed, such models where first considered by Gates et al. in refs.[9], where a superspace formalism adapted to the (2+2)-signature was introduced: the so-called N=1 superspace of Atiyah-Ward. Other related aspects in this domain were further investigated in ref.[10]. Moreover, in ref.[11], one was able to present a supersymmetric non-linear σ-model also in the Atiyah-Ward superspace and to couple its associated scalar superfields to a super-Yang-Mills gauge sector through the gauging of isometries of the target manifold [12, 13, 14, 15, 16, 17, 18, 19, 20]. Clearly, the class of theories foccused here should be necessarily understood in the sense of the dimensional reduction framework used by Ward in [7]. In that scheme, one may eventually obtain new examples of integrable field models in two dimensions (see also ref.[8]).

This is the purpose of the present work: to give a detailed account on the construction and gauging of supersymmetric σ-models à la Atiyah-Ward. Specifically, we will be concerned here with hyper-Kählerian σ-models possessing N=2 supersymmetries – one of them being non-linerarly realized – and, subsequently, with the issue of performing their gauging by means of the approach developed in ref.[15].

Our paper is organized as follows: in Section 2 we describe in a self-contained fashion all the necessary steps needed to build up the gauged N=1 supersymmetric σ-model in D=2+2 dimensions (a problem already addressed in ref.[11]) and state the essential notions on hyper-Kähler geometry which are crucial for the N=2 extension of the following section; Section 3 is then devoted to the study of N=2 supersymmetry in the N=1 superspace of Atiyah-Ward and to the gauging of the hyper-Kählerian σ-model in the context of a certain Kählerian vector supermultiplet. In Section 4 we interpret our results and present our conclusions.

2   The hyper-Kählerian σ-Model in Superspace

We begin the present investigation by foccusing on the construction of gauged N=1 supersymmetric σ-models in the Atiyah-Ward space-time. The notation and conventions for a superspace with base space-time possessing a $(2 + 2)$–signature are the same as in [10]. To

build up the action functional for a class of Kählerian σ-models one will follow here the well-known method of Zumino [21] (see refs.[22, 23] for an extensive discussion on Kähler geometry). We introduce a set of complex chiral and antichiral superfields, Φ^i and Ξ^i (i=1,...,n), with their component field expansions written as[*]:

$$\Phi^i = A^i + i\theta\psi^i + i\theta^2 F^i + i\tilde{\theta}\tilde{\sigma}^\mu\theta\partial_\mu A^i + \tfrac{1}{2}\,\theta^2\tilde{\theta}\tilde{\sigma}^\mu\partial_\mu\psi^i - \tfrac{1}{4}\,\theta^2\tilde{\theta}^2\,\Box A^i, \tag{1}$$

$$\Xi^i = B^i + i\tilde{\theta}\tilde{\chi}^i + i\tilde{\theta}^2 G^i + i\theta\sigma^\mu\tilde{\theta}\partial_\mu B^i + \tfrac{1}{2}\,\tilde{\theta}^2\theta\sigma^\mu\partial_\mu\tilde{\chi}^i - \tfrac{1}{4}\,\theta^2\tilde{\theta}^2\,\Box B^i, \tag{2}$$

where A^i and B^i are complex scalar fields, ψ^i and $\tilde{\chi}^i$ are Majorana-Weyl spinors and F^i and G^i are complex scalar auxiliary fields. One has to observe that, differently to the Minkowskian situation, the scalar superfields at hand do not change their chirality properties under the complex conjugation operation:

$$\widetilde{D}_{\dot{\alpha}}\Phi^i = \widetilde{D}_{\dot{\alpha}}\Phi^{*i} = 0,$$
$$D_\alpha\Xi^i = D_\alpha\Xi^{*i} = 0, \tag{3}$$

with

$$D_\alpha = \partial_\alpha - i\widetilde{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}},$$
$$\widetilde{D}_{\dot{\alpha}} = \tilde{\partial}_{\dot{\alpha}} - i\theta^\alpha\tilde{\partial}_{\dot{\alpha}\alpha}, \tag{4}$$

and

$$\{D_\alpha, \widetilde{D}_{\dot{\alpha}}\} = -2i\,\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu, \quad \{D_\alpha, D_\beta\} = \{\widetilde{D}_{\dot{\alpha}}, \widetilde{D}_{\dot{\beta}}\} = 0,$$
$$[D_\alpha, \partial_\mu] = [\widetilde{D}_{\dot{\alpha}}, \partial_\mu] = 0.$$

Now, one writes down a rather specific supersymmetric action to govern the dynamics of the scalar superfields. We take[†]:

$$I = 2\int d^4x\,d^2\theta\,d^2\tilde{\theta}\,K(\Phi^i, \Xi^i; \Phi^{*i}, \Xi^{*i}), \tag{5}$$

where the Kähler potential K decomposes into two conjugated pieces as below:

$$K(\Phi^i, \Xi^i; \Phi^{*i}, \Xi^{*i}) = H(\Phi^i, \Xi^{*i}) + H^*(\Phi^{*i}, \Xi^i). \tag{6}$$

The pure scalar sector steming from the projection of (5) into component fields is given by:

$$I_{scalar} = 2\int d^4x\,\left(\frac{\partial^2 K}{\partial A^i\partial B^{*j}}\partial_\mu A^i\partial^\mu B^{*j} + \frac{\partial^2 K}{\partial A^{*i}\partial B^j}\partial_\mu A^{*i}\partial^\mu B^j\right). \tag{7}$$

Upon dimensional reduction and proper field truncations, I_{scalar} above will give rise to a sensible (ghost free) scalar kinetic term in D=1+2 space-time dimensions (see ref.[10]).

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[*]The Grassmann coordinates, θ and $\tilde{\theta}$, are Majorana-Weyl spinors.

[†] $\int d^4x\,d^2\theta\,d^2\tilde{\theta} \equiv \tfrac{1}{16}\int d^4x\,D^\alpha\widetilde{D}^{\dot{\alpha}}\widetilde{D}_{\dot{\alpha}}D_\alpha$

The possible target spaces associated to the action I in (5) do belong to a restricted class of 4n-dimensional Kähler manifolds, their Hermitian metric tensor appearing in a four-block structure as follows:

$$g_{\mathcal{I}\mathcal{J}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & g_{i\bar{j}} \\ \mathbf{0} & \mathbf{0} & g_{i\bar{j}} & \mathbf{0} \\ \mathbf{0} & g_{\bar{i}j} & \mathbf{0} & \mathbf{0} \\ g_{\bar{i}j} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{8}$$

with

$$g_{i\bar{j}} = \frac{\partial^2 H}{\partial \Phi^i \partial \Xi^{*j}}, \qquad g_{i\bar{j}} = \frac{\partial^2 H^*}{\partial \Xi^i \partial \Phi^{*j}}, \qquad g_{\bar{i}j} = \frac{\partial^2 H^*}{\partial \Phi^{*i} \partial \Xi^j}, \qquad g_{\bar{i}j} = \frac{\partial^2 H}{\partial \Xi^{*i} \partial \Phi^j}, \tag{9}$$

and

$$\mathcal{I}, \mathcal{J} = 1, \ldots 4n \text{ and } i, j = 1, \ldots n.$$

It is clear now that the particular form of $g_{\mathcal{I}\mathcal{J}}$ will entail a number of consequences for the geometry of our Kählerian target manifold. The most general type of Kähler transformation one can perform upon the potential K while keeping the action (5) invariant and the metric (8) unchanged is:

$$K \longrightarrow K' = K + \eta(\Phi) + \eta^*(\Phi^*) + \rho(\Xi) + \rho^*(\Xi^*), \tag{10}$$

with (η, η^*) and (ρ, ρ^*) standing for arbitrary chiral and antichiral functions respectively. Hence, every isometry transformation of the target manifold will be a simmetry of (5) provided its action on K writes into a form compatible with (10). The Killing vectors $(\kappa_a^i(\Phi), \tau_a^i(\Xi), \kappa_a^{*i}(\Phi^*), \tau_a^{*i}(\Xi))$ are the generators of the isometry group $\mathcal{G}$ and satisfy the usual Lie algebraic relations:

$$\kappa_a^i \kappa_{b,i}^j - \kappa_b^i \kappa_{a,i}^j = f_{ab}{}^c \kappa_c^j, \qquad \kappa_a^{*i} \kappa_{b,i}^{*j} - \kappa_b^{*i} \kappa_{a,i}^{*j} = f_{ab}{}^c \kappa_c^{*j},$$
$$\tau_a^i \tau_{b,i}^j - \tau_b^i \tau_{a,i}^j = f_{ab}{}^c \tau_c^j, \qquad \tau_a^{*i} \tau_{b,i}^{*j} - \tau_b^{*i} \tau_{a,i}^{*j} = f_{ab}{}^c \tau_c^{*j}, \tag{11}$$

where $f_{ab}{}^c$ are the structure constants. A global isometry transforms the target coordinates as:

$$\Phi'^i = \exp(L_{\lambda\cdot\kappa})\Phi^i, \qquad \Phi'^{*i} = \exp(L_{\lambda\cdot\kappa^*})\Phi^{*i},$$
$$\Xi'^i = \exp(L_{\lambda\cdot\tau})\Xi^i, \qquad \Xi'^{*i} = \exp(L_{\lambda\cdot\tau^*})\Xi^{*i}, \tag{12}$$

where λ is for a real parameter and $L_{\lambda\cdot\kappa}$ (resp. $L_{\lambda\cdot\tau}$) is the Lie derivative along the vector field $\lambda\cdot\kappa \equiv \lambda^a \kappa_a^i \partial_i$ (resp. $\lambda\cdot\tau \equiv \lambda^a \tau_a^i \partial_i$). The set of laws above may be related to some, Kähler transformation like (10), the chiral and antichiral functions being given as:

$$\begin{aligned} \eta_a(\Phi) &= \partial_i H(\Phi, \Xi^*) \, \kappa_a^i(\Phi) + Y_a(\Phi, \Xi^*), \\ \rho_a(\Xi) &= \partial_i H^*(\Xi, \Phi^*) \, \tau_a^i(\Xi) - Y_a^*(\Xi, \Phi^*), \\ \eta_a^*(\Phi^*) &= \partial_{\bar{i}} H^*(\Xi, \Phi^*) \, \kappa_a^{*i}(\Phi^*) + Y_a^*(\Xi, \Phi^*), \\ \rho_a^*(\Xi^*) &= \partial_{\bar{i}} H(\Phi, \Xi^*) \, \tau_a^{*i}(\Xi^*) - Y_a(\Phi, \Xi^*). \end{aligned} \tag{13}$$

By differentiating the first and last equations in (13) with respect to Ξ^{*j} and Φ^j respectively, one gets:

$$
\begin{aligned}
H_{i\bar{j}}\,\kappa_a^{\ i} &= -Y_{a\bar{j}}, \\
H_{\bar{i}j}\,\tau_a^{*i} &= Y_{aj},
\end{aligned}
\tag{14}
$$

which, in turn, allow one to write the identity:

$$
\kappa_a^{\ i}\,Y_{bi} + \tau_b^{*i}\,Y_{a\bar{i}} = 0.
\tag{15}
$$

From the algebra (11), and from (13), we have:

$$
H_i\,\kappa_{[a}^{\ j}\kappa_{b]j}^{\ i} + H_{\bar{i}}\,\tau_{[a}^{*j}\tau_{b]\bar{j}}^{*i} = f_{ab}^{\ \ c}(\eta_c + \rho_c^*),
\tag{16}
$$

which, by means of (15), can be rewriten as:

$$
\kappa_{[a}^{\ j}\eta_{b]j} + \tau_{[a}^{*j}\rho_{b]j}^* = f_{ab}^{\ \ c}(\eta_c + \rho_c^*).
\tag{17}
$$

From holomorphicity considerations, one may set:

$$
\begin{aligned}
\kappa_{[a}^{\ j}\eta_{b]j} &= f_{ab}^{\ \ c}\eta_c + ic_{ab}, \\
\tau_{[a}^{*j}\rho_{b]j}^* &= f_{ab}^{\ \ c}\rho_c^* - ic_{ab},
\end{aligned}
\tag{18}
$$

where $c_{ab} = -c_{ba}$ are real constants. In the restricted case of a semi-simple gauge group $\mathcal{G}$, we may remove the c_{ab}'s by simply imposing $c_{ab} = 0$ (in other cases they represent an obstruction to the gauging [15]). With this restriction, one writes the variation on the Killing potential as:

$$
\delta Y_a = \frac{1}{2}\lambda^b\left(\kappa_{[b}^i Y_{a]i} + \tau_{[b}^{*i}Y_{a]i}\right) = -\lambda^b f_{ab}^{\ \ c}Y_c,
\tag{19}
$$

where one has used (11), (13) and (18). Now, from (14) and (19) we obtain the complex potential Y_a:

$$
Y_a = 2f_{ab}^{\ \ c}\kappa_d^i\tau_c^{*j}\frac{\partial^2 H}{\partial\Phi^i\partial\Xi^{*j}}g^{bd},
\tag{20}
$$

in which g^{bd} is the inverse Killing metric.

To proceed to the covariantization of the action (5) with respect to gauged isometries, i.e. the local version of the set of field transformations (12), one introduces a couple (Λ, Γ) of real chiral and antichiral superfield parameters respectively [11]. The local isometry transformations are defined as:

$$
\Phi' = \exp\left(L_{\Lambda\cdot\kappa}\right)\Phi, \qquad \Xi' = \exp\left(L_{\Gamma\cdot\tau}\right)\Xi.
\tag{21}
$$

The gauge sector is built up from the prepotential V, a real superfield transforming such as:

$$
\exp\left(L_{V'\cdot\tau}\right) = \exp\left(L_{\Lambda\cdot\tau}\right)\exp\left(L_{V\cdot\tau}\right)\exp\left(-L_{\Gamma\cdot\tau}\right).
\tag{22}
$$

We modify then the action (5) by replacing the antichiral superfields (Ξ, Ξ^*) with the redefined quantities $(\tilde{\Xi}, \tilde{\Xi}^*)$ given below:

$$\tilde{\Xi}^i \equiv \exp\left(L_{V \cdot \tau}\right)\Xi^i, \qquad \tilde{\Xi}^{*i} \equiv \exp\left(L_{V \cdot \tau^*}\right)\tilde{\Xi}^{*i}. \tag{23}$$

Infinitesimally one has the following isometry transformation laws for the superfields:

$$\delta \Phi^i = \Lambda^a \kappa_a^i, \qquad \delta \Phi^{*i} = \Lambda^a \kappa_a^{*i},$$
$$\delta \tilde{\Xi}^i = \Lambda^a \tau_a^i, \qquad \delta \tilde{\Xi}^{*i} = \Lambda^a \tau_a^{*i}. \tag{24}$$

It turns out moreover that the correct covariantization of (5) still demands the introduction of a complex conjugated pair of antichiral superfields (v, v^*) transforming as:

$$\delta v = \lambda^a \rho_a(\Xi),$$
$$\delta v^* = \lambda^a \rho_a^*(\Xi^*). \tag{25}$$

The isometry-covariant action functional is then taken to be:

$$I_{cov} = 2 \int d^4x d^2\theta d^2\tilde{\theta} \left[H(\Phi, \tilde{\Xi}^*) + H^*(\Phi^*, \tilde{\Xi}) - \tilde{v} - \tilde{v}^* \right], \tag{26}$$

which, in terms of the original variables, writes as:

$$I_{cov} = 2 \int d^4x d^2\theta d^2\tilde{\theta} \left\{ H(\Phi, \Xi^*) + H^*(\Phi^*, \Xi) + 2 \, Re\left[\frac{e^L - 1}{L} \, V^a Y_a^*(\Phi^*, \Xi) \right] \right\}, \tag{27}$$

with $L \equiv L_{V \cdot \tau}$.

   As mentioned in the introduction, it will be our aim hereafter to extend the construction leading to I_{cov} in (27) above to the more general task of analyzing the gauging of N=2 supersymmetric σ-model in the N=1 superspace of Atiyah-Ward. With this purpose in mind, one is enforced here to consider the more restricted class of hyper-Kählerian σ-models in order to introduce a second set of supersymmetry field transformations, following in much the same way what was envisaged already in the last decade by Alvarez-Gaumé and Freedman [24]. The Kählerian target space of our σ-model can also be taken as a hyper-Kähler manifold as long as its metric tensor $g_{\mathcal{I}\mathcal{J}}$ in (8) is hermitian with respect to a quaternionic structure $\{J_{\mathcal{I}}^{(1)\mathcal{J}}, J_{\mathcal{I}}^{(2)\mathcal{J}}, J_{\mathcal{I}}^{(3)\mathcal{J}}\}$. The tensors $J_{\mathcal{I}}^{(x)\mathcal{J}}$ are covariantly constant and generate the SU(2) algebra:

$$J_{\mathcal{I}}^{(x)\mathcal{J}} J_{\mathcal{J}}^{(y)\mathcal{K}} = -\delta^{xy}\delta_{\mathcal{I}}^{\mathcal{K}} + \epsilon^{xyz} J_{\mathcal{I}}^{(z)\mathcal{K}}.$$

The complex structures are parametrized here as follows:

$$J_{\mathcal{I}}^{(1)\mathcal{J}} = \begin{pmatrix} i\delta_i^j & 0 & 0 & 0 \\ 0 & i\delta_{\hat{i}}^{\hat{j}} & 0 & 0 \\ 0 & 0 & -i\delta_{\bar{i}}^{\bar{j}} & 0 \\ 0 & 0 & 0 & -i\delta_{\hat{\bar{i}}}^{\hat{\bar{j}}} \end{pmatrix}, \tag{28}$$

$$J_{\mathcal{I}}^{(2)\mathcal{J}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & J_i^{\bar{\hat{j}}} \\ \mathbf{0} & \mathbf{0} & J_{\hat{i}}^{\bar{j}} & \mathbf{0} \\ \mathbf{0} & J_{\bar{i}}^{\hat{j}} & \mathbf{0} & \mathbf{0} \\ J_{\bar{\hat{i}}}^{j} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} , \tag{29}$$

and

$$J_{\mathcal{I}}^{(3)\mathcal{J}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & iJ_i^{\bar{\hat{j}}} \\ \mathbf{0} & \mathbf{0} & iJ_{\hat{i}}^{\bar{j}} & \mathbf{0} \\ \mathbf{0} & -iJ_{\bar{i}}^{\hat{j}} & \mathbf{0} & \mathbf{0} \\ -iJ_{\bar{\hat{i}}}^{j} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} . \tag{30}$$

It is the very existence of such a quaternionic structure what enables one to introduce a non-linearly realized supersymmetry in the theory. In fact, we shall see in the next section that the action (27) can be conveniently supplemented with new interaction terms which will render it invariant under N=2 supersymmetries, while preserving its covariance under the gauged isometries (24).

3   The N=2 Supersymmetric Extension

In this section we analyze the N=2 supersymmetric extension of our gauged σ-model in the Atiyah-Ward superspace. By following a reasoning similar to that of [15], one defines the second supersymmetry in terms of two sets of complex functions of the target coordinates, the potentials $\Omega^i \equiv \Omega^i(\Phi, \Xi^*)$ and $\Upsilon^i \equiv \Upsilon^i(\Xi, \Phi^*)$ $(i = 1, ..., n)$, the field transformation laws being given by:

$$\delta\Phi^i = i\widetilde{D}^2(\epsilon\,\Omega^i), \qquad \delta\Phi^{*i} = i\widetilde{D}^2(\epsilon\,\Omega^{*i}),$$
$$\delta\Xi^i = iD^2(\zeta\,\Upsilon^i), \qquad \delta\Xi^{*i} = iD^2(\zeta\,\Upsilon^{*i}), \tag{31}$$

where ζ and ϵ are real constant chiral and antichiral scalar superfields respectively, i.e.

$$D_\alpha\epsilon = \partial_\mu\epsilon = 0, \qquad \widetilde{D}_{\dot{\alpha}}\zeta = \partial_\mu\zeta = 0, \tag{32}$$

and moreover

$$\widetilde{D}^2\epsilon = D^2\zeta = 0. \tag{33}$$

The on-shell closure of the algebra of transformations in (31) imposes the following constraints on the potentials:

$$\Omega^i{}_{,\overline{jk}}\,\Upsilon^{*j}{}_{,n} + \Omega^i{}_{,\overline{j}}\,\Upsilon^{*j}{}_{,n\overline{k}} = 0 \ , \qquad \Upsilon^i{}_{,\overline{jk}}\,\Omega^{*j}{}_{,\hat{n}} + \Upsilon^i{}_{,\overline{j}}\,\Omega^{*j}{}_{,\hat{n}\overline{k}} = 0 \ ,$$
$$\Omega^i{}_{,\overline{j}}\,\Upsilon^{*j}{}_{,n} = -\delta^i_n \ , \qquad \Upsilon^i{}_{,\overline{j}}\,\Omega^{*j}{}_{,\hat{n}} = -\delta^{\hat{i}}_{\hat{n}} \ ,$$
$$\Omega^i{}_{,j[\overline{k}}\,\Omega^j{}_{,\overline{n}]} = 0 \ , \qquad \Upsilon^i{}_{,j[\overline{k}}\,\Upsilon^j{}_{,\overline{n}]} = 0 \ ,$$
$$\widetilde{D}^2\Omega^i = 0 \ , \qquad D^2\Upsilon^i = 0 \ , \tag{34}$$

with the lower indices standing for derivatives with respect to the target coordinates. Moreover, by requiring the invariance of the action (5) under (31) we arrive at the additional conditions upon the functions Ω^i and Υ^i:

$$
\begin{aligned}
H_{i\bar{j}}\Omega^i{}_{,\bar{n}} + H_{i\bar{n}}\Omega^i{}_{,\bar{j}} = 0, &\qquad H^*_{ij}\Upsilon^i{}_{,\overline{n}} + H^*_{\overline{in}}\Upsilon^i{}_{,\overline{j}} = 0, \\
H_{i\bar{n}}\Omega^i{}_{,\overline{jk}} + H_{i\bar{j}\hat{k}}\Omega^i{}_{,\bar{n}} = 0, &\qquad H^*_{\overline{in}}\Upsilon^i{}_{,\overline{jk}} + H^*_{\overline{ij}k}\Upsilon^i{}_{,\overline{n}} = 0, \\
H_{i\hat{j}}\Omega^i{}_{,\hat{n}k} + H_{ijk}\Omega^i{}_{,\bar{n}} = 0, &\qquad H^*_{\overline{ij}}\Upsilon^i{}_{,\overline{n}k} + H^*_{\overline{ij}\hat{k}}\Upsilon^i{}_{,\overline{n}} = 0,
\end{aligned}
\tag{35}
$$

together with their complex conjugated counterparts. At this point, by means of a careful inspection of eqs. (34) and (35), one observes that the functions Ω^i and Υ^i are encompassing in their structure all the important features of the hyper-Kählerian geometry [15, 16]. Indeed, this property can be made even more apparent if we introduce the identifications:

$$
J_{\hat{i}}{}^{\bar{j}} = \Omega^j{}_{,\bar{i}} \ , \qquad J_{\hat{i}}{}^{\hat{j}} = \Upsilon^j{}_{,\bar{i}} \ , \qquad J_i{}^{\bar{j}} = \Omega^{*j}{}_{,\hat{i}} \ , \qquad J_i{}^{\bar{j}} = \Upsilon^{*j}{}_{,i} \ ,
\tag{36}
$$

in the complex structures (29) and (30).

Furthermore, from the assumption of triholomorphicity of the Killing vectors with respect to the quaternionic structure, one can define the potentials $P^{(+)}_a \equiv P^{(+)}_a(\Phi, \Xi)$ and $P^{(-)}_a \equiv P^{(-)}_a(\Phi^*, \Xi^*)$ such that $P^{(-)}_a = (P^{(+)}_a)^*$ and

$$
k^i_a \omega^{(+)}_{ij} = -P^{(+)}_a{}_{,j} \ , \qquad k^{*i}_a \omega^{(-)}_{\overline{ij}} = -P^{(-)}_a{}_{,\bar{j}} \ ,
\tag{37}
$$

$$
\tau^{\hat{i}}_a \omega^{(+)}_{\hat{i}j} = -P^{(+)}_a{}_{,\hat{j}} \ , \qquad \tau^{*\hat{i}}_a \omega^{(-)}_{\overline{\hat{i}j}} = -P^{(-)}_a{}_{,\overline{\hat{j}}} \ ,
\tag{38}
$$

with

$$
\omega^{(+)}_{ij} = -2H_{j\bar{k}}\Upsilon^{*k}{}_{,i} \ , \quad \omega^{(-)}_{\overline{ij}} = -2H^*_{\overline{j}k}\Upsilon^k{}_{,\hat{i}} \ , \quad \omega^{(+)}_{\hat{i}j} = -2H^*_{\hat{j}k}\Omega^{*k}{}_{,\hat{i}} \ , \quad \omega^{(-)}_{\overline{\hat{i}j}} = -2H_{\overline{j}k}\Omega^k{}_{,\bar{i}}. \tag{39}
$$

From eqs. (13) above and from the formulae expressing the chiral and antichiral functions (37,38) we derive some useful relations involving the Killing potentials $Y_a(\Phi, \Xi^*)$:

$$
\begin{aligned}
P^{(+)}_{a,j}\Omega^{j\bar{i}}{}_{,\bar{i}} = -2Y_{a,\bar{i}} &\quad \rightleftharpoons \quad P^{(+)}_{a,i} = 2Y_{a,\bar{j}}\Upsilon^{*j}{}_{,i} \\
P^{(+)}_{a,\hat{j}}\Upsilon^j{}_{,\bar{i}} = 2Y^*_{a,\bar{i}} &\quad \rightleftharpoons \quad P^{(+)}_{a,\hat{i}} = -2Y^*_{a,\bar{j}}\Omega^{*j}{}_{,\hat{i}} \\
P^{(-)}_{a,\bar{j}}\Omega^{*j}{}_{,\hat{i}} = -2Y^*_{a,\hat{i}} &\quad \rightleftharpoons \quad P^{(-)}_{a,\bar{i}} = 2Y^*_{a,j}\Upsilon^j{}_{,\bar{i}} \\
P^{(-)}_{a,\hat{\bar{j}}}\Upsilon^{*j}{}_{,i} = 2Y_{a,i} &\quad \rightleftharpoons \quad P^{(-)}_{a,\hat{i}} = -2Y_{a,j}\Omega^j{}_{,\bar{i}}.
\end{aligned}
\tag{40}
$$

To obtain $P^{(+)}_a$ and $P^{(-)}_a$ one observes that the complex functions

$$
U_a = P^{(-)}_a - P^{(+)}_a - 2iY_a + 2iY^*_a
\tag{41}
$$

do satisfy the following differential equations:

$$
\left(\partial_i + i\Upsilon^{*j}{}_{,i}\partial_{\bar{j}}\right) U_a = 0,
\tag{42}
$$

$$
\left(\partial_{\hat{i}} + i\Omega^{*j}{}_{,i}\partial_{\bar{j}}\right) U_a = 0,
\tag{43}
$$

the complex conjugated, U_a^*, obeying the complexified analogs thereof. Actually, eqs. (42), (43) are specifying the U_a's (resp. U_a^*'s) as holomorphic functions (resp. antiholomorphic functions) relatively to a non-canonical complex structure [15]. From the definition given in (41), one can write:

$$U + U^* = 4(-iY + iY^*) \tag{44}$$

and

$$U - U^* = 2(P^{(-)} - P^{(+)}). \tag{45}$$

Now, from the holomorphicity and the gauge transformation of Y_a:

$$\delta Y_a = -\lambda^b f_{ab}{}^c\, Y_c, \tag{46}$$

one arrives at

$$\delta P_a^{(+)} \;=\; -\lambda^b f_{ab}{}^c\; P_c^{(+)}, \tag{47}$$

and

$$\delta P_a^{(-)} \;=\; -\lambda^b f_{ab}{}^c\; P_c^{(-)}, \tag{48}$$

where the gauge group was assumed to be semi-simple, which implies the absence of obstructions in (46), (47) and (48) above. On the other hand we have:

$$\delta P_a^{(+)} = \lambda^b \left(k_b^i P_a^{(+)},_i + \tau_b^i P_a^{(+)},_{\hat{i}} \right), \tag{49}$$

which, by comparison with (47) and use of the first equations of (37) and (38), gives us the following:

$$P_a^{(+)} = f_a{}^{bc} \left(k_c^i k_b^j \omega_{ji}^{(+)} + \tau_c^i \tau_b^j \omega_{\hat{j}\hat{i}}^{(+)} \right). \tag{50}$$

Through complex conjugation, one also has:

$$P_a^{(-)} = f_a{}^{bc} \left(k_c^{*i} k_b^{*j} \omega_{\overline{ji}}^{(-)} + \tau_c^{*i} \tau_b^{*j} \omega_{\overline{\hat{j}\hat{i}}}^{(-)} \right). \tag{51}$$

We now turn to the construction of the N=2 supersymmetric gauge sector in the N=1 superspace of Atiyah-Ward. In [25], Gates et al. succeeded in writting down a set of non-linear supersymmetry transformations for a certain N=2 gauge-supermultiplet in N=1 Minkowski superspace. We adopt a similar approach here: our Kählerian gauge supermultiplet consists of a chiral scalar superfield S and an antichiral scalar superfield T, together with the vector superfield V of the previous section. All the three superfields are real and take values in the adjoint representation of the isometry gauge group $\mathcal{G}$. We propose the non-linear supersymmetry transformations on gauge superfields as below:

$$\begin{aligned}
\delta S &= iW^\alpha D_\alpha \zeta, \\
\delta T &= i\tilde{W}^{\dot\alpha} \widetilde{D}_{\dot\alpha} \epsilon, \\
e^{-iV}\delta e^{iV} &= \epsilon\, e^{-iV} S e^{iV} - \zeta\, T,
\end{aligned} \tag{52}$$

where the real scalar superfield parameters (ϵ, ζ) are the ones appearing in the supersymmetry transformations (31) for the matter sector; the gauge superfield-strengths are defined to be:

$$W_\alpha \;\equiv\; i\widetilde{D}^2\left(e^{iV}D_\alpha e^{-iV}\right),$$

$$\tilde{W}_{\dot\alpha} \;\equiv\; iD^2\left(e^{-iV}\widetilde{D}_{\dot\alpha}e^{iV}\right), \tag{53}$$

they are covariant under gauge transformations of the type

$$e^{-iV}\delta_g e^{iV} \;=\; i\left(e^{-iV}\Lambda e^{iV} - \Gamma\right). \tag{54}$$

One has also to consider gauge transformation laws for the scalar gauge superfields:

$$\delta_g S = i[\Lambda, S], \qquad \delta_g T = i[\Gamma, T]. \tag{55}$$

At this stage, we are ready to present the fully gauged N=2 supersymmetric non-linear σ-model in terms of N=1 superfields of the Atiyah-Ward superspace. As stated previously, this task is accomplished by suplementing the action (27) with new interaction pieces such as to render the second supersymmetry, i.e. (31) and (52), a further invariance of the model [15]. Our main result is:

$$I_{cov} = 2\int d^4x d^2\theta d^2\widetilde{\theta}\ \left\{H(\Phi,\Xi^*) + H^*(\Phi^*,\Xi) + 2\ Re\left[\tfrac{e^{\hat{\mathcal{L}}}-1}{\hat{\mathcal{L}}}\ V^a Y_a^*(\Phi^*,\Xi)\right] - \tfrac{1}{2}S^a\tilde{T}_a\right\} +$$

$$-\tfrac{1}{16}\int d^4x d^2\theta\ \left\{g_{ab}W^{a\alpha}W_\alpha^b - 4iS^a\left[F_a(\Phi) + F_a^*(\Phi^*)\right]\right\} +$$

$$-\tfrac{1}{16}\int d^4x d^2\widetilde{\theta}\ \left\{g_{ab}\tilde{W}^{a\dot\alpha}\tilde{W}_{\dot\alpha}^b - 4iT^a\left[G_a(\Xi) + G_a^*(\Xi^*)\right]\right\}, \tag{56}$$

where we have made implicit use of the splittings in the functions $P_a^{(+)}$ and $P_a^{(-)}$ in (50) and (51):

$$P_a^{(+)} = F_a(\Phi) + G_a(\Xi), \qquad P_a^{(-)} = F_a^*(\Phi^*) + G_a^*(\Xi^*). \tag{57}$$

Finally, it is straightforward to check the invariance of (56) under (52) and the (gauge covariant) supersymmetry transformations for the matter superfields:

$$\delta\Phi^i = i\widetilde{D}^2\left(\epsilon\ \Omega^i(\Phi, e^{2\hat{\mathcal{L}}^*}\Xi^*)\right), \qquad \delta\Phi^{*i} = i\widetilde{D}^2\left(\epsilon\ \Omega^{*i}(\Phi^*, e^{2\hat{\mathcal{L}}}\Xi)\right),$$

$$\delta\Xi^i = iD^2\left(\zeta\ \Upsilon^i(e^{-2\mathcal{L}^*}\Phi^*, \Xi)\right), \qquad \delta\Xi^{*i} = iD^2\left(\zeta\ \Upsilon^{*i}(e^{-2\mathcal{L}}\Phi, \Xi^*)\right), \tag{58}$$

in which

$$\mathcal{L} = V^a k_a^i\frac{\partial}{\partial\Phi^i}, \qquad \hat{\mathcal{L}} = V^a\tau_a^i\frac{\partial}{\partial\Xi^i}. \tag{59}$$

Indeed, one may impose the Wess-Zumino gauge condition, i.e. $V^3 = 0$, and to verify the invariance of I_{cov} under the non-linear supersymmetry transformations at each order in the prepotential V. It should be observed once more that due to the presence of some gauge-algebraic obstructions [15, 26], the supersymmetric gauging expressed in (56) will only hold for semisimple gauge groups $\mathcal{G}$, in which case one can always determine the potentials (20), (50) and (51).

4 Concluding Remarks

We have explicitly constructed a class of N=2 supersymmetric non-linear σ-models coupled to a super-Yang-Mills gauge sector in the N=1 superspace of Atiyah-Ward. In order to perform this gauge coupling, one makes use of a general formalism introduced by Hull et al. in [15], gauging the isometries of the associated (hyper-Kähler) target manifold. We observe then that, also in the Atiyah-Ward superspace, it is possible to obtain the specific potentials needed for the referred gauging of the hyper-Kählerian σ-model, namely the Killing potential (20) (which is complex here) and the so-called momentum maps (50) and (51).

The gauge-invariant supersymmetric σ-model obtained in the previous section may have some interesting applications in connection with the study of gauge dynamics of supersymmetric gauge theories in lower dimensions. In fact, by suppressing one time coordinate in the action (56), one may in principle arrive at new supersymmetric field models in three Minkowskian dimensions. The latter type of theories could then be regarded as an alternative scenario for checking the consequences of the duality hypothesis of four dimensions, following in much the same way what has been proposed in the recent literature [27].

Acknowledgements

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