CBPF-NF-073/95

Superfluid Behaviour of a Quantum q-Gas

by

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Abstract

We explore some consequences of the inequivalent representations of a q-oscillator algebra on a highly deformed quantum q-gas. By a simple choice of the continuum limit of the background ν_0, the constant volume specific heat per mass C shows a λ-point transition and has a T^3 dependence for low temperatures. Choosing a particular value of the deformation parameter q, we are able to reproduce the experimental value of the $He\ II$ specific heat for $T < 0.5\ ^0K$.

Key-words: Statistical Mechanics; Quantum gases; Superfluidity; Liquid Helium; λ-Point transition; Quantum Groups; q-Oscillators.

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Bosonic q-oscillators [1] are a generalization of the Heisenberg algebra obtained by the introduction of a deformation parameter q. In the last few years their statistical properties have been studied mostly in the $q \simeq 1$ approximation [2–4]. In a series of papers [5–6] the highly deformed region ($q >> 1$) started to be investigated in connection with an ideal q-gas, showing the presence of the Bose-Einstein condensation phenomenum [7] with the specific heat, C_V, exhibiting a λ-point discontinuity.

A strong reason to discuss deformed quantum gases is the role played by the theory of ideal quantum gases in many different physical phenomena. Besides, the interest in the highly deformed region has very recently been sharpened by a result [8] showing the connection between spin-glasses and q-oscillators for q far from 1.

In the articles referred to above [1–8] the quantum q-gas was analysed in its "fundamental" representation of a q-oscillator algebra. More recently our interest was turned towards inequivalent representations of q-oscillator algebras [9, 10] and we have studied their consequences on a quantum q-gas [11]. In the present letter we show that in the case where $q >> 1$, for a particular choice of the background ν_0, the q-gas presents a superfluid behaviour.

The mutually adjoint operators a, a^+ and the self-adjoint operator N generate the algebra

$$\begin{aligned} &[N, a^+] = a^+ \quad , \quad [N, a] = -a \ , \\ &a a^+ - q a^+ a = q^{-N} \ , \end{aligned} \tag{1}$$

where $q \in I\!R$. Under the assumption that the spectrum is non-degenerate a series of inequivalent representations were built [9]. For $q > 1$, denoting the normalized basis vectors by $|n\rangle$, the following representations were obtained:

$$\begin{aligned} a^+|n\rangle &= q^{-\nu_0/2}[n+1]^{1/2}|n+1\rangle \ ; \\ a|n\rangle &= q^{-\nu_0/2}[n]^{1/2}|n-1\rangle \ ; \\ N|n\rangle &= (\nu_0 + n)|n\rangle \ ; \end{aligned} \tag{2}$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$ and ν_0 is a real free parameter which goes to zero when $q \to 1$. It is worthwhile noting that ν_0 can be zero for $q \neq 1$ and we call that case the "fundamental" representation of algebra (1). Moreover, only for $\nu_0 = 0$ (for arbitrary $q \in I\!R^+$), can N be interpreted as the usual particle number operator for the state $|n\rangle$;

for $\nu_0 \neq 0$, its eigenvalue is interpreted as the sum of the number of particles n, in the state $|n\rangle$, plus a background effect ν_0. We define here the operator $\hat{N} = N - \nu_0$, which is now the number operator, $\hat{N}|n\rangle = n|n\rangle$, for the representations in (2) characterized by ν_0.

In the "fundamental" representation, the relations

$$aa^+ - qa^+a = q^{-N} \quad , \quad aa^+ - q^{-1}a^+a = q^N \tag{3}$$

are simultaneously verified for the q-Fock representation given by eq. (2) with $\nu_0 = 0$. In the case we are going to consider ($q > 1$), ν_0 is the lowest bound of the spectrum of N and therefore classifies inequivalent representations of eq. (1) algebra [9]. In fact, it has been verified that [10]

$$\mathcal{C} = q^{-N}([N] - a^+a) \tag{4}$$

is a Casimir operator of the eq. (1) algebra and in the representation (2) one has

$$\mathcal{C}|n\rangle = q^{-\nu_0}[\nu_0]|n\rangle \ . \tag{5}$$

As the operator $\mathcal{C}$ (cf. eq. (4)) is different from zero only for $q \neq 1$, one sees from eq. (5) that for $q = 1$, ν_0 is necessarily zero.

Let us consider an ideal quantum q-gas in the representation (2) described by the Hamiltonian

$$H = \sum_i \ \omega_i a_i^+ a_i = \sum_i \ \omega_i([N_i] - q^{N_i}\mathcal{C}_i) \ , \tag{6}$$

where a_i and a_i^+ are interpreted as annihilation and creation operators of particles in levels i with energy ω_i and N_i is an operator that can be intepreted as the number operator of particles in levels i when $\nu_0^i = 0$. The operators a_i, a_i^+ and N_i satisfy the following algebra:

$$\begin{aligned} &a_i a_j^+ - q^{\delta_{ij}} a_j^+ a_i = \delta_{ij} q^{-N_i} \ ; \\ &[N_i, a_j] = -\delta_{ij} a_j \quad ; \quad [N_i, a_j^+] = \delta_{ij} a_j^+ \ . \end{aligned} \tag{7}$$

The grand canonical partition function is

$$Z = Tr \exp[-\beta(H - \mu\hat{N})] = \exp(-\beta\Omega) \ , \tag{8}$$

where μ is the chemical potential, $\hat{N} = \sum \hat{N}_i$, Ω is the grand canonical potential and $\beta = 1/kT$ with k the Boltzmann constant.

As Z factorizes for the above system, the grand canonical potential is given by a sum over single level partition functions [3]

$$\Omega = -\frac{1}{\beta} \sum_i \ln Z_i^0(w_i, \beta, \mu) \tag{9}$$

with

$$Z_i^0(\omega_i, \beta, \mu) = \sum_{n=0}^{\infty} e^{-\beta(\omega_i q^{-\nu_0^i}[n] - \mu n)} \ . \tag{10}$$

According to the usual procedure we enclose the system in a large 3-dimensional volume V and the sum over levels is replaced by an integral over $\vec{p}$-space. We assume that the energy spectrum of the q-particles follows the dispersion law $\omega_i \to p^2/2m$. In addition, we take a different ν_0^i for each level i such that in the continuum limit we have $q^{-\nu_0^i} \to \alpha(q) p^{-1}$. As we shall see later this choice will have interesting consequences. With these assumptions the grand canonical potential becomes

$$\Omega = \frac{-V}{h^3\beta} \int d^3p \ln \sum_{n=0}^{\infty} e^{-\beta(\frac{\alpha(q)}{2m} p[n] - \mu n)} \ . \tag{11}$$

The pressure $P = -\Omega/V$ and the density $n = (\partial P/\partial\mu)|_{T,V}$ are then:

$$\begin{aligned} P(T,z) &= \beta^{-1} \wedge_q^{-3} Y_q(z) \ ; \\ n(T,z) &= \wedge_q^{-3} y_q(z) \ ; \end{aligned} \tag{12}$$

where $z = \exp(\beta\mu)$ is the fugacity and

$$\wedge_q^{-3} = \frac{64\pi m^3 k^3 T^3}{h^3 \alpha^3(q)} \tag{13}$$

is the modified thermal wavelenght. The functions $Y_q(z)$ and $y_q(z)$ are respectively

$$\begin{aligned} Y_q(z) &= \frac{1}{6} \int_0^\infty dx\ x^3 \frac{\sum_{n=0}^{\infty} [n] z^n e^{-[n]x}}{\sum_{n=0}^{\infty} z^n e^{-[n]x}} \ , \\ y_q(z) &= \frac{1}{6} \int_0^\infty dx\ x^3 \left[\frac{\sum_{n=0}^{\infty} [n] n z^n e^{-[n]x}}{\sum_{n=0}^{\infty} z^n e^{-[n]x}} - \frac{\sum_{n,m=0}^{\infty} [n] m z^{n+m} e^{-([n]+[m])x}}{\left(\sum_{n=0}^{\infty} z^n e^{-[n]x}\right)^2} \right] \ , \end{aligned} \tag{14}$$

where $x = \frac{\beta\alpha(q)p}{2m}$.

Let us now study the Bose-Einstein condensation for the highly deformed case. It has been shown [12, 6] that in order to have a given accuracy in the integrals above the number of terms to be kept depends on q. As usual, when $z \to 1$ (or $T \to T_c, T_c$ being the critical temperature) one has to take into account the zero point energy and single out its contribution in (12). In addition, equation (10) shows that the effect of the deformation is cancelled when $\omega_i = 0$. As a consequence the series (10) cannot be approximated by a polynomial for the zero energy level. As usual [13], the critical temperature is defined as $n^{1/3}\wedge_c^q = y_q^{1/3}(1)$ which in the present case gives

$$T_c^q = \frac{\alpha(q)h\ n^{1/3}}{4\pi^{1/2}m\ k\ y_q^{1/3}(1)}\ . \tag{15}$$

As explained above, similarly to the non-deformed case [13], the basic equations are

$$\begin{aligned} P(T,z) &= \beta^{-1}\wedge_q^{-3} Y_q(z)\ , && (16.a)\\ n(T,z) &= \frac{1}{V}\ \frac{z}{1-z} + \wedge_q^{-3} y_q(z)\ . && (16.b)\end{aligned}$$

In eq. (16b) the first term on the right-hand side comes from the contribution of zero-energy and is relevant only for $T \le T_c^q$.

The constant volume specific heat per mass, C, defined as

$$C = \frac{1}{mn}\ \frac{\partial \tilde{e}}{\partial T}\bigg|_n\ , \tag{17}$$

is

$$\begin{aligned} C &= 12k\wedge_q^{-3} n^{-1}m^{-1}\ Y_q(z) - 9km^{-1}z^{-1}\ \frac{y_q(z)}{y_q'(z)}\quad ,\quad T > T_c^q && (18.a)\\ C &= 12k\wedge_q^{-3} n^{-1}m^{-1}\ Y_q(1) \qquad\qquad T < T_c^q\ , && (18.b)\end{aligned}$$

where $\tilde{e}$ is the energy density (internal energy per volume) and $y_q'(z) = \frac{\partial}{\partial z}\ y_q(z)$.

The above specific heat deserves some comments. To start with, it shows a λ-point transition since the second term on the right-hand side of eq. (18a) is different from zero. This is a feature of interesting phenomena, including superfluidity. Moreover, it is remarkable that $C \propto T^3$ for $T < T_c^q$, thus presenting the low-temperature behavior of a superfluid [14, 16]. This is a consequence of the assumption we have made for the continuum limit of the background effec ν_0.

Equation (18b) can be used for a direct comparison with the experimentally determined value of $He\ II$ specific heat [14, 15] for temperatures lower than 0.5 ^0K. In this region, the saturated vapour pressure specific heat C_s, which is the quantity found from calorimetric measurements [14], can be considered equal to the constant volume specific heat per mass C. From (18b) and (13) we have

$$C = \frac{3 \times 2^8 \pi m^2 k^4 Y_q(1)}{h^3 \alpha^3(q) n} \, T^3 \, . \tag{19}$$

We take $m = m_{{}^4_2He} = 6.65 \times 10^{-24} g$, $n = 2.2 \times 10^{-22} cm^{-3}$ [13] and choose $q = 3$, which leads to $Y_q(1) = 0.9658$. In this case the experimental value of the specific heat of $He\ II$, for $T < 0.5\ {}^0K$, ($C_s = 2.04\ (\pm 0.04) \times 10^{-2} T^3\ J/q\ deg$) [14–15] is reproduced for $\alpha(q = 3) = 2.45\ ergs\ sec.\ cm^{-1}$. We note that the integrals (14) and $y_q'(z)$ converge within the Mathematica software accuracy for $z = 1$ keeping only five terms in the series (Table I).

Although the specific heat (19) reproduces the experimental results for $He\ II$ for $T < 0.5\ {}^0K$ and presents a λ-point transition, the particular simple choice we have made for the continuum limit of the background effect ν_0, does not provide a good model for $He\ II$. Indeed, for temperatures higher than $0.5\ {}^0K$, the specific heat does not have a T^3 dependence anymore [15–16]. As a consequence, the critical temperature obtained from (15) is completely different from the experimental value of the $He\ II$ critical temperature ($T_\lambda = 3.17\ {}^0K$) for any value of q. We believe that with a less simplistic choice for the continuum limit of the background effect ν_0 we might be able to describe the specific heat of $He\ II$. Finally, it is known that there is no completely satisfactory theory for the superfluidity phenomenum [15, 16] yet. Our result indicates that the solution to this problem can somehow be related to deformed algebras.

Acknowledgements: The authors thank C. Tsallis for calling their attention to ref. [7] and M.R-M. thanks S. Sciuto for discussions.

TABLE I

q=3

n	4	5	6	10
$Y_3(1)$	0.965787	0.965799	0.965799	0.965799
$y_3(1)$	0.933122	0.933123	0.933123	0.933123
$y_3'(1)$	0.871144	0.871148	0.871148	0.871148

Numerical results for $Y_q(1)$, $y_q(1)$ and $y_q'(1)$ for $q = 3$. n is the number of terms in series (14). The Mathematica software accuracy is obtained for $n = 5$.

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