Hidden symmetries in one-dimensional quantum Hamiltonians

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Abstract

We construct a Heisenberg-like algebra for the one dimensional infinite square-well potential in quantum mechanics. The number-type and ladder operators are realized in terms of physical operators of the system as in the harmonic oscillator algebra. These physical operators are obtained with the help of variables used in a recently developed non commutative differential calculus. This "square-well algebra" is an example of an algebra in a large class of generalized Heisenberg algebras recently constructed. This class of algebras also contains q-oscillators as a particular case. We also show here how this general algebra can address hidden symmetries present in several quantum systems.

Key-words: Heisenberg algebra; quantum algebras; q-oscillators; non-linearity; non commutative calculus.

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1 Introduction

The harmonic oscillator is a paradigmatic system in physics for several well-known reasons. The algebra related to it, the Heisenberg algebra, is a reference tool in second quantization, and its structure, based on creation and annihilation operators and its particle interpretation, is used everywhere having, up to now, no analogous interpretation in any other system.

In the last years, several attempts have been made to generalize Heisenberg algebra and a particular generalization, known as q-oscillators [1], and their applications [2, 3, 4] have attracted considerable attention. Nevertheless, in all generalizations of Heisenberg algebra, a clear comprehension of the physical problem under consideration is always lacking. The special role played by the harmonic oscillator system is then indisputable although it is not understood why other systems could not have similar algebraic structures.

Recently, it was constructed a generalization of the Heisenberg algebra depending on a general functional of one generator of the algebra, $f(J_0)$ [5, 6]. For linear f it was shown that the algebra corresponds to q-oscillators, the Heisenberg algebra being obtained in the limit when the deformation parameter $q \rightarrow 1$. The representations of the algebra, when f is any analytical function, was shown to be obtained through the study of the stability of the fixed points of f and of their composed functions, exhibiting an unsuspected link between algebraic and dynamical system formalisms.

We show here that this generalization of the Heisenberg algebra together with a non-commutative differential calculus, developed to be used in space-time discrete networks [7, 8, 9], are appropriate to describe hidden algebraic aspects of a simple quantum mechanical system: the one-dimensional infinite square-well potential. All generators of the algebra, ladder operators plus a number-type one, are written in terms of physical operators of the system in a similar way to what happens in the harmonic oscillator.

The introduction of creation and annihilation operators realized in terms of physical operators and a Fock space representation for this simple problem, opens the possibility of applying the formalism of second quantization to a large amount of quantum systems with possible applications ranging from condensed matter to quantum field theories. We also stress in this paper hidden symmetries in many quantum mechanical systems presented in the sequence of energy eigenvalues, a symmetry unsuspected up to now [5, 6].

The generalization of the Heisenberg algebra recently developed in [5, 6] can be described by the generators J_0, $J_\pm$ satisfying the relations:

$$J_0\,J_+ = J_+\,f(J_0)\,, \quad (1)$$
$$J_-\,J_0 = f(J_0)\,J_-\,, \quad (2)$$
$$[J_+, J_-] = J_0 - f(J_0)\,, \quad (3)$$

where $J_- = J_+^\dagger$, $J_0^\dagger = J_0$ and $f(J_0)$ is a general analytic function of J_0. The above algebraic relations are constructed in order that, in the representation theory, the n-th eigenvalue of operator J_0 is given by the n-th iteration, through the function f, of an initial value α_0. The operator

$$C = J_+\,J_- - J_0 = J_-\,J_+ - f(J_0), \quad (4)$$

is a Casimir operator of the algebra. The representation theory of the algebra can be analyzed assuming that we have an irreducible representation of the algebra given by eqs. (1-3). Consider the state $|0\rangle$ with the lowest eigenvalue of the Hermitian operator J_0,

$$J_0\,|0\rangle = \alpha_0\,|0\rangle\,. \quad (5)$$

For each value of α_0 and the parameters of the algebra we have a different vacuum that for simplicity will be denoted by $|0\rangle$. As $|0\rangle$ is the vacuum, we have,

$$J_-\,|0\rangle = 0. \quad (6)$$

As consequence of the algebraic relations (1-3, 5, 6) we obtain for a general functional f

$$J_0\,|m\rangle = f^m(\alpha_0)\,|m\rangle, \quad m = 0, 1, 2, \cdots\,, \quad (7)$$
$$J_+\,|m\rangle = N_m\,|m+1\rangle, \quad (8)$$
$$J_-\,|m\rangle = N_{m-1}\,|m-1\rangle, \quad (9)$$

where $N_m^2 = f^{m+1}(\alpha_0) - \alpha_0$ and we have used $f^0(\alpha_0) = \alpha_0$. Note that $f^m(\alpha_0)$ denotes the m-th iterate of f,

$$\alpha_m \equiv f^m(\alpha_0) = f(\alpha_{m-1})\,. \quad (10)$$

Eqs. (7-9) define a general n-dimensional representation for the algebra. In order to solve it, i.e., to construct the conditions under which we have finite- and infinite-dimensional representations we have to specify the functional $f(J_0)$. Heisenberg algebra is the simplest particular case of algebra (1-3) and we can see that if we choose $f(J_0) = J_0 + 1$ the algebra given by eqs. (1-3) becomes the Heisenberg algebra. In [6] we used linear and quadratic functionals, leading to multi parametric deformations of the Heisenberg algebra. Also, we showed in [6] that it is the iteration aspect of the algebra that allow us to find their representations through the analysis of the stability of the fixed points of the function f and their composed functions [5, 6].

Here, in this paper, we shall use the inverse approach utilized in [5, 6], where it was studied general functional forms of f. Now, we look for a simple physical problem with a known spectrum and try to obtain the generalized algebra related to it. To implement this program we will need the formalism of the non commutative differential calculus mainly studied by Dimakis et al [7, 8, 9].

In [7] a formalism was developed for a one-dimensional spacial lattice with finite spacing, i.e., a discrete space. We will sketch here an analogous formalism for a momentum-space instead of the position-space. The reason is that in many physical problems the momentum space is already discretized, with only some allowed values. In the one-dimensional infinite square-well potential for example, that will be analyzed below, the allowed values for the (adimensional) momenta are only the positive integers, as it is well-known. Thus, the non commutative differential calculus approach seems to be specially appropriated to be used in the momentum space. The formulae used here are analogous to the formulae used in [7], and the reader should see this paper for a more detailed exposition and explanation of the non commutative calculus (remembering again that their formulae were derived for a discrete position-space). Therefore, let us consider an one dimensional lattice in a momentum space where the momenta are allowed only to take discrete values, say p_0, $p_0 + a$, $p_0 + 2a$, $p_0 + 3a$ etc, with $a > 0$. The non commutative differential calculus is based on the expression

$$[p, dp] = dp\,a\,, \quad (11)$$

implying that

$$f(p)\,dg(p) = dg(p)\,f(p+a)\,, \quad (12)$$

for all functions f and g. Let us introduce partial derivatives by

$$d\,f(p) = dp\,(\partial_p\,f)\,(p) = (\bar{\partial}_p\,f)\,(p)\,dp\,, \quad (13)$$

where the left and right discrete derivatives are given by:

$$(\partial_p\,f)\,(p) = \frac{1}{a}\,[f(p+a) - f(p)]\,, \quad (14)$$
$$(\bar{\partial}_p\,f)\,(p) = \frac{1}{a}\,[f(p) - f(p-a)]\,, \quad (15)$$

and satisfies

$$(\bar{\partial}_p\,f)\,(p) = (\partial_p\,f)\,(p-a)\,. \quad (16)$$

The Leibniz rule for the right discrete derivative can be written as:

$$(\partial_p\,fg)\,(p) = (\partial_p f)\,(p) g\,(p) + f(p+a)(\partial_p g)\,(p)\,, \quad (17)$$

with a similar formula for the left derivative [7].

Let us now introduce the momentum shift operators

$$A = 1 + a\,\partial_p \quad (18)$$
$$\bar{A} = 1 - a\,\bar{\partial}_p\,, \quad (19)$$

which increases (decreases) the value of the momentum by a

$$(Af)(p) = f(p+a) \qquad (20)$$
$$(\bar{A}f)(p) = f(p-a) \qquad (21)$$

and satisfies

$$A\bar{A} = \bar{A}A = 1\,, \qquad (22)$$

where 1 means the identity on the algebra of functions of p. Let us now introduce the momentum operator [7]

$$(Pf)(p) = p\,f(p)\,, \qquad (23)$$

($P^\dagger = P$), which returns the value of the variable of the function f. Clearly,

$$AP = (P+a)A \qquad (24)$$
$$\bar{A}P = (P-a)\bar{A}\,. \qquad (25)$$

Integrals can also be defined in this formalism but it is rather a technical point and the interested reader can look the paper [7] for a detailed explanation. Here we will only use the definition of a definite integral of a function f from p_d to p_u (p_u being equal to $p_d + Ma$, where M is a positive integer) as

$$\int_{p_d}^{p_u} dp\,f(p) = a\sum_{k=0}^{M} f(p_d + k\,a)\,. \qquad (26)$$

Using eq. (26), an inner product of two (complex) functions f and g can be defined as

$$\langle f\,,\,g\rangle = \int_{p_d}^{p_u} dp\,f(p)^*\,g(p)\,, \qquad (27)$$

where * indicates the complex conjugation of the function f. Clearly, the norm $\langle f\,,\,f\rangle \geq 0$ is zero only when f is identically null. The set of equivalent classes of normalizable functions f ($\langle f\,,\,f\rangle$ is finite) is a Hilbert space and it can be shown that the operators A and $\bar{A}$ are well defined in this space [7]. We have

$$\langle f, Ag\rangle = \langle \bar{A}f, g\rangle\,, \qquad (28)$$

where

$$\bar{A} = A^\dagger\,, \qquad (29)$$

being $A^\dagger$ the adjoint operator of A. Eqs. (22) and (29) show that A is a unitary operator. It is also possible to define a position operator X given as $X = (\partial_p + \bar{\partial}_p)/2i$ [7]. With this very short adapted review of the non commutative differential calculus we can go further and, together with the generalization of the Heisenberg algebra, analyze the physical example of the quantum mechanical infinite one dimensional square-well potential.

Thus, let us assume a one dimensional system with zero potential between zero and L and infinite elsewhere. As it is well-known, the spectrum of the Hamiltonian ($H = cP^2$, $c = 1/2m$, $\hbar = 1$) with the above boundary conditions is proportional to n^2, where $n = 1, 2, 3, \ldots$. The momentum is quantized and proportional to n. Therefore, we can see the momentum space as an one dimensional periodic lattice with constant spacing $a = \pi/L$, clearly a candidate to apply the non commutative differential calculus sketched before. We then take the momentum operator in the Hamiltonian $H = cP^2$, with the above boundary conditions, as defined in eq. (23).

The Hamiltonian's eigenvalue associated with the (n+1)-th level is proportional to $(n+1)^2$ and we can write

$$e_{n+1} = b(n+1)^2 = (\sqrt{e_n} + \sqrt{b})^2\,, \qquad (30)$$

where e_n is the eigenvalue of the Hamiltonian associated with the n-th level and $b = \pi^2/2mL^2$. As J_0 is related to the Hamiltonian [5, 6] and their eigenvalues should satisfy the iterations given by a function f in eqs. (1 - 3), we see that if we choose this function as

$$f(x) = (\sqrt{x} + \sqrt{b})^2\,, \qquad (31)$$

the J_0 in eqs. (1-3) has eigenvalues equal to the energy eigenvalues of the square-well potential. Eqs. (1-3) can then be rewritten for this case as

$$[J_0, J_+] = 2\sqrt{b}\,J_+\sqrt{J_0} + b\,J_+\,, \qquad (32)$$
$$[J_0, J_-] = -2\sqrt{b}\sqrt{J_0}\,J_- - b\,J_-\,, \qquad (33)$$
$$[J_+, J_-] = -2\sqrt{b}\sqrt{J_0} - b\,. \qquad (34)$$

The square root of the generator J_0 is well defined since this is a Hermitian operator and can be diagonalized.

We then have an algebra eqs. (32-34) where, by construction, the eigenvalues of J_0 , e_n, are the energy eigenvalues of the quantum mechanical one dimensional infinite square-well potential and $J_\pm$ act as ladder operators. In order to have a complete description, similar to the case of the one-dimensional harmonic oscillator, we must realize the operators $J_{(\pm\,,\,0)}$ in terms of physical operators. We propose for this problem the following realization:

$$J_+ = \sqrt{c}\,P\,(1 + a\,\partial_p) \qquad (35)$$
$$J_- = \sqrt{c}\,(1 - a\,\bar{\partial}_p)\,P \qquad (36)$$
$$J_0 = c\,P^2\,. \qquad (37)$$

Clearly, J_0 is the Hamiltonian and can be written, analogously to the harmonic oscillator case, as an ordered product of ladder operators

$$J_0 = J_+\,J_- = c\,P^2\,, \qquad (38)$$

as $J_+ = \sqrt{c}\,PA$, $J_- = \sqrt{c}\,\bar{A}P$ and, according to eq. (22), $A\bar{A} = 1$. Using eqs. (24-25) it is straightforward to check that these operators indeed satisfy the commutation relations given by eqs. (32-34), applying them to a function (state) of p. We stress that, the operators

P and X are the momentum and position operators in the momentum space for the one-dimensional infinite square-well potential. Moreover, it is possible to write the operators P and X in terms of the ladder operators $J_\pm$ and the operator J_0.

Fock space representations of the algebra generated by J_0 and $J_\pm$, eqs. (32-34), are obtained considering eigenstates of J_0, with fixed values of the momentum. Let us call $|n\rangle$ the eigenstate of J_0 whose momentum is associated with the quantum number n, $n = 0, 1, 2, 3, \ldots$. The eigenvalue α_n that appears in eqs. (5-10) can be put as $\alpha_n = b\, n^2$ and the eqs. (7-9) can be rewritten as

$$J_0\,|n\rangle = b\, n^2\,|n\rangle, \quad n = 0, 1, 2, \cdots, \tag{39}$$

$$J_+\,|n\rangle = \sqrt{b}\,(n+1)\,|n+1\rangle\,, \tag{40}$$

$$J_-\,|n\rangle = \sqrt{b}\, n\,|n-1\rangle\,, \tag{41}$$

$$P|n\rangle = a\, n\,|n\rangle\,, \tag{42}$$

where $N_n^2 = b\,(n+1)^2$.

Hence, we see that an algebraic formalism similar to the harmonic oscillator algebra is constructed for another physical problem: the one dimensional infinite square-well potential in quantum mechanics. The sequence of energy eigenvalues of the infinite square-well potential hides an algebra, whose symmetries were not suspected up to now. This Heisenberg-like algebra, that we call square-well algebra, is an example of a large class of generalized Heisenberg algebras, class that contains q-oscillators as a particular case, recently constructed [5, 6]. It is interesting to stress that the number-like and ladder operators are realized in terms of physical operators of the system as in the harmonic oscillator. Also, a number interpretation is possible, allowing us to consider a system with higher momenta or several systems with lower momenta. It would be tempting to find applications of this second quantization type approach in condensed matter and quantum field theory.

Our results indicate that the procedure adopted in this paper, i.e., to find a hidden algebraic structure of a physical system with the corresponding physical realization of the algebra generators, can be applied to other quantum systems. The introduction of generalized Heisenberg algebras for Hamiltonian systems constitutes thus a very powerful tool for extracting symmetries that conventional treatments are unable to disclose. We have exemplified the use of this algebra for a simple quantum mechanical problem, namely the infinite square-well potential. However, usefulness of the method also lies in its application to more complex physical systems.

The most difficult task in this method is to realize $J_{(\pm,\,0)}$ in terms of physical operators of the system, such that the algebra is still satisfied, with the product $J_+\,J_-$ being proportional to the Hamiltonian of the problem studied, as was done in this paper. Work in this direction is under way.

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