Light traps in nonlinear dielectric media

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Abstract

We analyze the propagation of light in a medium that responds in a nonlinear way to an electric field. We report on the case of a dielectric medium having a fourth order electric susceptibility that presents a new trapping effect.

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I. INTRODUCTION

Maxwell theory implies that discontinuities of the electromagnetic field propagates through characteristics null surfaces in Minkowski spacetime. This property is no longer true for nonlinear electrodynamics [1]. In the present paper we will concentrate our analysis to wave propagation in a dielectric medium that responds nonlinearly to an external stimulus. There seems to be no better way to examine such wave propagation than Hadamard's method. We shall see that the net result of such investigation shows that the waves propagate no more as null cones of Minkowski background, but instead as null cones of another effective geometry. Let us point out that one could describe this propagation without any use of such effective modification of the metric properties of the spacetime. Nevertheless, such procedure is by far simpler than dealing with a complicated non null surface in the actual Minkowski geometry [2]. We would like to emphasize that such an effective metric is nothing but a very useful way to describe the characteristic surface of propagation of the nonlinear field.

In section 2 we review briefly the Hadamard method and show how this effective geometry naturally appears in that context. We sketch some kinematical properties induced by such description, thus providing its consistency.

Section 3 is devoted to reveal some important new properties of the waves that this interpretation leads to, namely the existence of light traps inside some negatively charged nonlinear dielectric matter. We describe the formal requirements for such traps to exist and also show how they read in terms of the physical properties of the material these traps should be built of.

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II. WAVE PROPAGATION IN NONLINEAR MEDIA

The electromagnetic field in an arbitrary medium is relativistically represented by two skew-symmetric tensors $F_{\mu\nu}$ and $P_{\mu\nu}$. It is useful to decompose these tensors into its corresponding electric and magnetic components as measured by an arbitrary observer endowed with a four-velocity v^μ. We call electric component E_μ and magnetic component H_σ the vectors associated to $F_{\mu\nu}$. Correspondingly, D_μ and B_σ are the vectors associated to $P_{\mu\nu}$ through the standard expressions:

$$F_{\mu\nu} = E_\mu\, v_\nu - E_\nu\, v_\mu + \eta^{\rho\sigma}{}_{\mu\nu}\, v_\rho\, H_\sigma \tag{1}$$

and

$$P_{\mu\nu} = D_\mu\, v_\nu - D_\nu\, v_\mu + \eta^{\rho\sigma}{}_{\mu\nu}\, v_\rho\, B_\sigma. \tag{2}$$

We choose a standard observer by setting

$$v^\mu = \delta^\mu_0. \tag{3}$$

Maxwell equations are given by:

$$F^{*\mu\nu}{}_{,\nu} = 0, \tag{4}$$

$$P^{\mu\nu}{}_{,\nu} = 0. \tag{5}$$

We limit our analysis to electrostatic fields inside isotropic dielectrics, for which $P^{\mu\nu}$ and $F^{\mu\nu}$ are related by

$$P_{\mu\nu} = \epsilon(E) F_{\mu\nu}. \tag{6}$$

In a nonlinear dielectric medium the response of the polarization to an electric field is described by expressing the scalar function ϵ as a power series in terms of the field strength E:

$$\epsilon = 1 + \chi_2\, E + \chi_3\, E^2 + \chi_4\, E^3 + \ldots \tag{7}$$

where the constants χ_n are known as the n-order nonlinear optical susceptibility. Note that we are using the standard convention [3] which relates χ_n with the expansion of the polarization vector.

We are looking here for the velocities of propagation of the waves. In other words our aim is to examine the evolution of the discontinuities of the field. The simplest and direct way to do this is to follow the method originally set out by Hadamard [4]. Let us make a brief survey of its main steps.

Let Σ be the surface of discontinuity defined by the equation

$$\Sigma(x^\mu) = constant. \tag{8}$$

The Hadamard condition concerning the value of the field passing through this surface makes use of the jump value of the quantities associated to the field. We set for the definition of the discontinuity through the surface Σ:

$$[F_{\mu\nu}]_\Sigma \equiv F^{(+)}_{\mu\nu} - F^{(-)}_{\mu\nu}, \tag{9}$$

in which the symbol $[J]_\Sigma$ represents the discontinuity of the function J through the surface Σ, where $F^{(+)}_{\mu\nu}$ and $F^{(-)}_{\mu\nu}$ represent the limit value of $F_{\mu\nu}$ when approaching Σ by either of its sides. We set for the values of the discontinuity of the field and its first derivative the corresponding quantities:

$$[F_{\mu\nu}]_\Sigma = 0, \tag{10}$$

and

$$[F_{\mu\nu,\lambda}]_\Sigma = f_{\mu\nu} k_\lambda. \tag{11}$$

in which the vector k_λ is the normal to the surface Σ, that is $k_\lambda = \Sigma_{,\lambda}$, and $f_{\mu\nu}$ is called the tensor of discontinuities.

From equations (5) and (6) we obtain

$$(\epsilon F^{\mu\nu})_{,\nu} = 0. \tag{12}$$

The discontinuity of this relation yields

$$[\epsilon_{,\nu} F^{\mu\nu} + \epsilon F^{\mu\nu}{}_{,\nu}]_\Sigma = 0. \tag{13}$$

Making use of

$$[\epsilon_{,\nu}]_\Sigma = -\frac{\epsilon'}{2E} F^{\alpha\beta} \, [F_{\alpha\beta,\nu}]_\Sigma \, , \tag{14}$$

where $\epsilon' \equiv \partial\epsilon/\partial E$, and (11) we achieve the propagation relation

$$\frac{\epsilon'}{2E} F^{\alpha\beta} F^{\mu\nu} f_{\alpha\beta} k_\nu - \epsilon f^{\mu\nu} k_\nu = 0. \tag{15}$$

In order to get rid of the information of the field discontinuities, $f_{\mu\nu}$, we recall (4) in the form

$$F_{\alpha\beta,\lambda} + F_{\lambda\alpha,\beta} + F_{\beta\lambda,\alpha} = 0. \tag{16}$$

The discontinuity of (16) through Σ converts $F_{\alpha\beta}$ into $f_{\alpha\beta}$ and the partial derivatives with respect to the coordinate variable x^ν into the propagation vector k_ν, that is

$$f_{\alpha\beta} \, k_\lambda + f_{\lambda\alpha} \, k_\beta + f_{\beta\lambda} \, k_\alpha = 0. \tag{17}$$

Contracting this expression with $F^{\alpha\beta} k_\sigma \eta^{\lambda\sigma}$ one reaches

$$F^{\alpha\beta} f_{\alpha\beta} k_\sigma k_\rho \eta^{\rho\sigma} = -2 F^{\alpha\beta} f_{\beta\lambda} k_\alpha k_\sigma \eta^{\lambda\sigma}. \tag{18}$$

Using this result into equations (15) we finally get

$$F^{\rho\sigma} f_{\rho\sigma} \left\{ \epsilon \, \eta^{\alpha\beta} + \frac{\epsilon'}{E} F^{\alpha\nu} F_\nu{}^\beta \right\} k_\alpha k_\beta = 0. \tag{19}$$

Whenever the scalar $F^{\rho\sigma} f_{\rho\sigma}$ does not vanish[1] one arrives at a fundamental (algebraic) equation

$$k_\mu k_\nu g^{\mu\nu} = 0, \tag{20}$$

in which k_ν is the wave propagation vector and $g^{\mu\nu}$ is an effective geometry defined by

$$g^{\mu\nu} = \epsilon \, \eta^{\mu\nu} + \frac{\epsilon'}{E} F^\mu{}_\alpha F^{\alpha\nu}. \tag{21}$$

Using (1) we can re-write this under the form:

$$g^{\mu\nu} = \epsilon \, \eta^{\mu\nu} - \frac{\epsilon'}{E} \left(E^\mu \, E^\nu - E^2 \, \delta^\mu_0 \, \delta^\nu_0 \right) , \tag{22}$$

where $E^2 \equiv - E_\alpha \, E^\alpha > 0$. In other words,

$$g^{00} = \epsilon + \epsilon' E \tag{23}$$

$$g^{ij} = - \epsilon \, \delta^{ij} - \frac{\epsilon'}{E} \, E^i \, E^j \tag{24}$$

We point out that such effective geometry determines the path of the electromagnetic waves.

This shows that the discontinuities of the electromagnetic field inside a nonlinear dielectric medium propagates through null cones of an effective geometry which depends on the characteristics of the medium by equation (21). We shall prove now that this path is nothing but a geodesic in this associated geometry.

A. The effective null geodesics

The geometrical relevance of the effective geometry (21) goes beyond its immediate definition. Indeed, in the following it will be shown that the integral curves of

[1] All specific cases covered by this work satisfy the inequality $f^{\alpha\beta} F_{\alpha\beta}\big|_\Sigma \neq 0$.

the vector k_ν (*i.e.*, the trajectories of such nonlinear *photons*) are in fact geodesics. In order to achieve this result it will be required an underlying Riemannian structure for the manifold associated with the effective geometry. In other words, this means a set of Levi-Civita connection coefficients $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}$, by means of which there exists a covariant differential operator ∇_λ (the *covariant derivative*) such that

$$\nabla_\lambda g^{\mu\nu} \equiv g^{\mu\nu}{}_{;\,\lambda} \equiv g^{\mu\nu}{}_{,\,\lambda} + \Gamma^\mu{}_{\sigma\lambda} g^{\sigma\nu} + \Gamma^\nu{}_{\sigma\lambda} g^{\sigma\mu} = 0. \quad (25)$$

From (25) it follows that the effective connection coefficients are completely determined from the effective geometry by the usual Christoffel formula.

Contracting (25) with $k_\mu k_\nu$ results

$$k_\mu k_\nu g^{\mu\nu}{}_{,\,\lambda} = -2 k_\mu k_\nu \Gamma^\mu{}_{\sigma\lambda} g^{\sigma\nu}. \quad (26)$$

Differentiating (20) and remembering $g^{\mu\nu} = g^{\nu\mu}$ one gets

$$2 k_{\mu,\,\lambda} k_\nu g^{\mu\nu} + k_\mu k_\nu g^{\mu\nu}{}_{,\,\lambda} = 0. \quad (27)$$

Inserting (26) for the last term on the left hand side of (27) we obtain

$$g^{\mu\nu} k_{\mu,\,\lambda} k_\nu - g^{\sigma\nu} \Gamma^\mu{}_{\sigma\lambda} k_\mu k_\nu = 0. \quad (28)$$

Relabeling contracted indices we can rewrite (28) as

$$g^{\mu\nu} k_{\mu;\,\lambda} k_\nu \equiv g^{\mu\nu} \left[k_{\mu,\,\lambda} - \Gamma^\sigma{}_{\mu\lambda} k_\sigma \right] k_\nu = 0. \quad (29)$$

Now, as the propagation vector $k_\mu = \Sigma_{,\,\mu}$ is an exact gradient one can write $k_{\mu;\,\lambda} = k_{\lambda;\,\mu}$. With this identity and defining $k^\nu \equiv g^{\mu\nu} k_\mu$ equation (29) reads

$$k_{\mu;\,\lambda} k^\lambda = 0, \quad (30)$$

which states that k_μ is a geodesic vector. Remembering it is also a null vector (with respect to the effective geometry $g^{\mu\nu}$) its integral curves are thus null geodesics. We can restate our previous demonstration as:

> The discontinuities of the electromagnetic field inside a nonlinear dielectric medium propagates through **null geodesics** of an effective geometry which depends on the characteristics of the medium.

III. GEOMETRICAL ASPECTS OF LIGHT TRAPS

Let us restrict the analysis of this section to a particular solution of Maxwell's equations inside a dielectric medium. Assuming a spherically symmetric configuration with a radial electric field $F^{tr} = E(r)$ (all other components vanish), and rewriting field equations (5) in spherical coordinates one obtains the well known solution

$$D = \epsilon\, E = \frac{Q}{r^2}, \quad (31)$$

where Q is a constant.

The propagation of waves into such dielectric medium is ruled by the effective geometry (21), whose non trivial components reduce to

$$g^{tt} = -g^{rr} = \epsilon + \epsilon'\, E. \quad (32)$$

From (32) it follows that there may exist some critical radius r_c for which g^{rr} may vanish which implicitly defines a distinguished surface Ω through

$$\Omega : \qquad (\epsilon + \epsilon' E)|_{r_c} = 0. \quad (33)$$

Let us concern ourselves to the exam of the physical consequences of such peculiar region. First of all, we remember that a sign reversion[2] of g^{tt} and g^{rr} physically means that r becomes a *time-like* coordinate inside the region bounded by Ω. Note that Ω is a null surface (*i.e.*, $g^{\mu\nu} r_{,\mu} r_{,\nu}|_\Omega = 0$) of the effective geometry. Continuity of in-falling null geodesics implies that the future light cones are directed inwards; therefore Ω is an one way membrane, as no light can exit from the finite region of space bounded by the surface Ω. Differentiating (31) we get

$$\frac{\partial r}{\partial E} = -\frac{r^3}{2\,Q} (\epsilon + \epsilon' E), \quad (34)$$

which vanishes at the surface Ω. In order to ensure that $E(r)$ is well defined on the neighborhood of Ω (that is, $r = r(E)$ does not have an extremum at Ω) we impose that $\frac{\partial^2 r}{\partial E^2}$ must also vanish at the critical surface Ω. Besides, one requires that the first non zero derivative of g^{rr} with respect to E, at Ω, should be even. The most simple case when all the above hold is encompassed by the following

> **Lemma**: The region $r < r_c$ is a *light trap* if there exists a spherically symmetric domain inside some compact dielectric medium whose associated effective geometry (21), induced by a radial electrostatic field (31), satisfies the following properties:
>
> i) $g^{rr}|_\Omega = 0$;
>
> ii) $\left.\frac{\partial g^{rr}}{\partial E}\right|_\Omega = 0$;
>
> iii) $\left.\frac{\partial^2 g^{rr}}{\partial E^2}\right|_\Omega \neq 0$;
>
> iv) the maximum integer n such that $\frac{\partial^n g^{rr}}{\partial E^n} \neq 0$ is odd;
>
> v) $Q < 0$.

[2] Notice that coordinates r and t interchange their role when crossing the distinguished surface Ω. Such behavior is similar in Schwarzschild geometry.

The usefulness of such a lemma is to guide one in modeling a medium in order to present light trap properties. Specifically, let us consider a negatively charged dielectric medium whose susceptibility ϵ depends on the electric field up to its third power[3]

$$\epsilon = 1 + \chi_2\, E + \chi_3\, E^2 + \chi_4\, E^3. \tag{35}$$

Assuming that the electric field approaches a finite value E_o at Ω the conditions of the lemma for a dielectric medium with $\epsilon(E)$ given by equation (35) simplify to

$$1 + 2\chi_2 E_o + 3\chi_3 E_o^2 + 4\chi_4 E_o^3 = 0, \tag{36}$$

$$\chi_2 + 3\chi_3 E_o + 6\chi_4 E_o^2 = 0, \tag{37}$$

$$\chi_3 + 4\chi_4 E_o \neq 0. \tag{38}$$

The solution of this algebraic system is

$$E_o = -\frac{2}{3}\frac{\chi_2}{\chi_3}\left(1 + \beta\sqrt{1 - \frac{9\chi_3}{4{\chi_2}^2}}\right), \tag{39}$$

where

$$\beta = \mathrm{sgn}\left[1 - \frac{3{\chi_3}^2}{16{\chi_4}^2}\left(\frac{3{\chi_3}^2 - 8\chi_2\chi_4}{4{\chi_2}^2 - 9\chi_3}\right)\right]. \tag{40}$$

The inequality (38) implies that the parameters χ_2, χ_3 and χ_4 must be such that $\frac{\chi_3}{{\chi_2}^2} < \frac{4}{9}$ and $\frac{\chi_2\chi_4}{{\chi_3}^2} < \frac{3}{8}$.

In the asymptotic regimes we have $E(r \to \infty) \to 0$ and $E(r \to 0) \to -\mathrm{sgn}\,(\chi_4)\infty$. Thus, we conclude that $g^{rr}|_{r\to\infty} < 0$ and $g^{rr}|_{r\to 0} > 0$. This completes the proof that the above lemma holds for such medium.

IV. CONCLUSION

We have shown in this work how the refractive index of matter can be translated to the geometrical language, in the context of geometrical optics, thus allowing room for the concept of a light trap. It is shown that the discontinuities propagate through *null geodesics* of the effective geometry (21). The development of the geometrical approach, properly restricted to the particular case of a spherically symmetric electrostatic background, leads to enough requirements for such trapping to exist, which are fulfilled for an isotropic negatively charged dielectric whose susceptibility is of the form (35) and such that constraint (39) holds. This trapping is a new effect of nonlinear dielectric media which deserves further investigation.

[3]This is the lowest order for which this isotropic dependence leads to geometrical trap effects. Generalization to higher order terms in E is immediate and induces minor changes in the model, equations (36)-(38).

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