Supersymmetry, Supercurrent, and Scale Invariance

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CBPF–NF–072/96
UGVA–DPT 1996/08–938
hep-th/9611003

Lectures given at the Catholic University of Petrópolis (RJ) and at the CBPF (Rio de Janeiro), September 1995 and July-August 1996
(Notes written in collaboration with Oswaldo Monteiro Del Cima)

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[1]On leave of absence from: *Département de Physique Théorique – Université de Genève, 24 quai E. Ansermet – CH-1211 Genève 4, Switzerland.*

[2]Supported in part by the Swiss National Science Foundation and the Brazilian National Research Council (CNPq)

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Preface

These notes are an expanded version of a set of lectures I have given at the Catholic University of Petrópolis (UCP) and at the Centro Brasileiro de Pesquisas Físicas (CBPF).

I thank all the members of the theoretical physics departments of both institutions for their very warm hospitality, and the students who assisted to these lectures, with the hope that they have taken profit from them.

I feel me particularly indebted towards my friends Renato M. Doria (UCP), José A. Helayel Neto, Marco Antonio de Andrade, Oswaldo M. Del Cima (CBPF) and Silvio P. Sorella (UERJ) for their continuous assistance during my stays in Brazil and for the many interesting scientific discussions we had together.

I must specially thank Oswaldo M. Del Cima for his very valuable help for writing these notes.

Finally I thank the Brazilian National Research Council (CNPq) and the CBPF for their generous financial assistance.

1 Introduction

The aim of the present lectures is to give an introduction to the renormalization of supersymmetric gauge theories in 4-dimensional space-time. This will include the analysis of the ultraviolet divergences, and much emphasis will be put on the so-called "ultraviolet finite" models. Exemples of the latters might be relevant as realistic "grand unified theories" of the particle interactions.

Some "textbook knowledge" of renormalization theory is expected from the listeners. The approach I shall follow is that of "algebraic renormalization", see e.g. [1]. On the other hand, the supersymmetry formalism, in particular the superspace formalism developed in these lectures, is not supposed to be known in advance. One may however consult the classical textbooks on the subject [2, 3, 4, 5], as well as reviews such as the ones collected in [6]. The book [7] also presents this formalism, with more emphasis on the problem of renormalization. I shall follow the notations and conventions of [7].

Usual symmetries, either of the space-time type – e.g. Poincaré – or of the internal type – e.g. U(1)×SU(2)×SU(3) or SU(5) – are described by Lie groups [8]. Is it possible to unify both types of supersymmetry? The "no-go theorem" of Coleman and Mandula [9] answers by the negative. More precisely, it states that any Lie group containing the Poincaré group and an internal symmetry group as maximal subgroups is the trivial product of both. In other words, internal symmetry transformations always commute with the Poincaré transformations.

The hypotheses of this theorem are quite general. They consist in the axioms of relativistic quantum field theory [10], in the existence of a unitary S-matrix and in the assumption that all symmetries are realised in terms of Lie groups. A way to circumvent it was however found by Haag, Lopuschanski and Sohnius [11]. These authors simply relaxed one of the hypotheses of the no-go theorem, namely the one which concerns the groups of symmetry. They assumed that the infinitesimal generators of the symmetry obey a *superalgebra*. A superalgebra is a generalization of the notion of a Lie algebra, where some of the infinitesimal generators are fermionic, which means that some of the commutation rules are replaced by anticommutation rules. The result of [11] is still very restrictive: the only superalgebras compatible with the general axioms of relativistic quantum field theory and with the unitarity of the S-matrix are the supersymmetries of the Wess-Zumino type, i.e. those where the fermionic generators carry a spin 1/2.

Another theoretical motivation for studying supersymmetry is offered by *string theory* [12]. Indeed, the presence of fermionic string states together with bosonic ones, imposes a supersymmetric structure to the theory. In the effective field theories which approximate string theory in the energy domain below the Planck mass, equal to 10^{19} GeV, this structure manifests itself as a Wess-Zumino supersymmetry.

A further motivation for supersymmetry is found in the solution of the *hierarchy problem* [13, 14] of the grand unified theories. In these theories [15], which tend to unify all the particles and forces described by the standard model of particle interactions [16], two energy scales must be introduced, typically of the order of 10^3 - 10^4 GeV – the electro-weak scale – and 10^{15} - 10^{16} GeV – the grand unification scale. This means that one has to "fine tune" a mass difference expressed by a number with more than 12 significative digits! This fine tuning would be perfectly utopic in the framework of conventional gauge theories, since the presence of quadratic divergences of the mass corrections induces a strong instability of the difference of the renormalized masses, which must be fine tuned at each order of the perturbative calculus. The interest in considering supersymmetric theories is that ultraviolet divergences are milder, in particular the mass corrections depend only on the logarithm of the ultraviolet cut-off, instead of its square. The huge mass differences in grand unified theories are then much more stable[3].

Supersymmetry having thus a tendency to cancel some of the ultraviolet divergences, a natural question to ask is: could supersymmetry eventually lead to a complete cancellation of these divergences? Let us mention that searches for general ultraviolet finite models have been done – up to the order of the two-loop graphs. They have lead to the conclusion that supersymmetry is most likely required [17, 18].

Some ultraviolet finite supersymmetric models have been known since a long time. All these models had an extended supersymmetry: N=4 [19] or N=2 [20, 21], where N counts the fermionic generators. However, gauge models with extended supersymmetry are not physically appealing since they don't accomodate chiral fermions – in contrast with the N=1 models. More recently, finite models with N=1 supersymmetry were proposed. A complete list of such models, finite at least up to the two-loop order [22], was first obtained in [23, 24]. Then some proposals for all order finiteness were done [25, 26]. A common feature of these finite N=1 supersymmetric models is that they are based on a simple gauge group – hence they possess a single gauge coupling constant – and also that their Yukawa coupling constants must be functions of the gauge coupling constant. This indicates them as valuable candidates for grand unified theories, which moreover possess the power to predict the fermion masses since the Yukawa couplings are no more arbitrary parameters, in contrast to the usual, i.e. nonfinite, grand unifications.

Finally, a general criterion for the all order finiteness was given [27]-[32]. This criterion states a set of necessary and sufficient conditions for a theory to have of all its Callan-Symanzik "β-functions" vanishing to all orders of perturbation theory. Only the knowledge of the general expression for the one loop β-functions [33] is required. The physical meaning of vanishing β is the absence of scale anomalies, hence the scale invariance of the theory – at least asymptotically if massive particles are present. This does not mean complete ultraviolet finiteness, since infinite renormalizations of the field amplitudes are still allowed. The nonphysical character of the latter [1] however justifies the terminology of "ultraviolet finiteness".

Applications of the criterion of ultraviolet finiteness to realistic models based on the grand unification group SU(5) with three fermion generations have been performed recently [34, 35] (see also [25] for a different approach.)

[3] Such a picture is more understandable, in physical terms, within a framework where one considers the field theoretical model as an effective field theory, the ultraviolet cut-off being a physical parameter of an hypothetical exact theory – e.g. string theory – describing the phenomena at very high energies. This parameter might be the Planck mass.

2 Generalities

2.1 Extended Supersymmetry Algebra

The basis of the extended N-supersymmetry algebra consists of [11]:

- Bosonic (*even*) hermitean generators T_a, $a = 1 \cdots \dim(G)$, of some Lie group G,
- The generators P_μ and $M_{[\mu\nu]}$ of the 4-dimensional Poincaré group.
- Fermionic (*odd*) generators Q^i_α, $\alpha = 1,2$; $i = 1, \cdots, N$ belonging to a dimension N representation of G, and their conjugates $\bar{Q}^{\dot\alpha}_i$.
- Central charges $Z^{[ij]}$, i.e. bosonic operators commuting with all the T_a's and all the Q_α's and $\bar{Q}^{\dot\alpha}$'s, as well as with the Poincaré generators.

The T_a's and $Z^{[ij]}$'s are scalars, whereas the Q^i_α's belong to the representation $(1/2, 0)$ of the Lorentz group and the $\bar{Q}^{\dot\alpha}_i$'s to the conjugate representation $(0, 1/2)$. The latters are written as Weyl spinors, with two complex components[4].

The general superalgebra of N-extended supersymmetry, also called the N-super-Poincaré algebra, reads (we write only the nonvanishing (anti)commutators):

$$
\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma}) \ , \\
&[M_{\mu\nu}, P_\lambda] = i(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda}) \ ,
\end{aligned}
\tag{2.1}
$$

$$
[T_a, T_b] = i f_{ab}{}^c T_c \ , \tag{2.2}
$$

$$
\begin{aligned}
&\left\{ Q^i_\alpha, Q^j_\beta \right\} = \varepsilon_{\alpha\beta} Z^{[ij]} \ , \\
&\left\{ Q^i_\alpha, \bar{Q}^j_{\dot\alpha} \right\} = 2\delta^{ij} \sigma^\mu_{\alpha\dot\alpha} P_\mu \ ,
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
&[Q^i_\alpha, M_{\mu\nu}] = \frac{1}{2} (\sigma_{\mu\nu})_\alpha{}^\beta Q^i_\beta \ , \\
&[Q^i_\alpha, T_a] = (R_a)^i{}_j Q^j_\alpha \ .
\end{aligned}
\tag{2.4}
$$

This result is the most general one for a massive theory. In a massless theory, another set of fermionic charges, S^i_α (and their conjugates), may be present. Then, the Lie group G is U(N) for $N \neq 4$, and either U(4) or SU(4) for N=4. The superalgebra moreover contains all the generators of the conformal group – which contains the Poincaré group as a subgroup: one calls it the N-superconformal algebra.

In these lectures we will restrict ourselves to the case N=1.

2.2 N=1 Superfields

In the N=1 case, the part of the superalgebra (2.1)-(2.4) which involves the spinor charges reduces to the original Wess-Zumino algebra

$$
\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \ , \quad \{Q_\alpha, Q_\beta\} = 0 \ , \quad \left\{ \bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta} \right\} = 0 \ , \tag{2.5}
$$

[4] The notations and conventions are detailed in Appendix A.

to

$$
\begin{aligned}
&[Q_\alpha, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \ , \quad [\bar{Q}^{\dot\alpha}, M_{\mu\nu}] = -\frac{1}{2}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta} \ , \\
&[Q_\alpha, P_\mu] = 0 \ , \quad [\bar{Q}_{\dot\alpha}, P_\mu] = 0 \ ,
\end{aligned}
\tag{2.6}
$$

and to

$$
[Q_\alpha, R] = -Q_\alpha \ , \quad [\bar{Q}_{\dot\alpha}, R] = \bar{Q}_{\dot\alpha} \ . \tag{2.7}
$$

Here, R is the infinitesimal generator of an Abelian group into which the internal symmetry group G has shrunk.

The objects which transform covariantly under the supersymmetry transformations are the superfields[5], either of the general type, or of the chiral type. As explained in Appendix A, a superfield is a superspace function $\phi(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha})$, where θ_α, $\alpha = 1, 2$, are complex Grassmann variables, and $\bar\theta_{\dot\alpha}$ their complex conjugates. A chiral superfield $A(x, \theta, \bar\theta)$, resp. antichiral superfield $\bar{A}(x, \theta, \bar\theta)$, is a superfield obeying the constraint

$$
\bar{D}_{\dot\alpha} A = 0 \ , \quad \text{resp.} \quad D_\alpha \bar{A} = 0 \ , \tag{2.8}
$$

where D_α, $\bar{D}_{\dot\alpha}$ are the covariant superspace derivatives (A.4). The component fields of a superfield span a supermultiplet, i.e. an irreducible representation of the supersymmetry algebra. The translation, supersymmetry and R transformation laws of a superfield ϕ are defined by the superspace differential operators

$$
\begin{aligned}
&\delta^P_\mu \phi = \partial_\mu \phi \ , \\
&\delta^Q_\alpha \phi = \left(\frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \right) \phi, \\
&\delta^{\bar Q}_{\dot\alpha} \phi = \left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \right) \phi \ ,
\end{aligned}
\tag{2.9}
$$

and

$$
\delta^R \phi = i \left(n + \theta^\alpha \frac{\partial}{\partial\theta^\alpha} - \bar\theta^{\dot\alpha} \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \right) \phi \ . \tag{2.10}
$$

In the last equation the real number n is the "R-weight" of the superfield ϕ. The R-weigths of a pair of complex conjugates superfields are opposite to each other. These differential operators fulfil the algebra

$$
\begin{aligned}
&\left\{ \delta^Q_\alpha, \delta^{\bar Q}_{\dot\alpha} \right\} = -2i\sigma^\mu_{\alpha\dot\alpha}\delta^P_\mu \ , \\
&[\delta^Q_\alpha, \delta^R] = i\delta^Q_\alpha \ , \quad [\delta^{\bar Q}_{\dot\alpha}, \delta^R] = -i\delta^{\bar Q}_{\dot\alpha} \ , \\
&\text{(the other (anti)commutators vanishing)} \ .
\end{aligned}
\tag{2.11}
$$

2.3 Invariant Actions and Ward Identity Operators

A supersymmetric classical action Σ is given by the superspace integral – as defined by (A.11) – of some local functional of the superfields entering the considered theory, and of their covariant derivatives. Such integrals are indeed invariant under supersymmetry transformations.

The actions which will be considered in these lectures will be invariant as well under other symmetry transformations. These invariances will be expressed in a functional way. Let denote by $\delta_X \varphi$ the infinitesimal transformation of the superfield φ along the generator X of the (super)group

[5] See Appendix A for the definitions, notations and conventions.

of symmetries, e.g. one of the transformations (2.9), (2.10). Let us define the associated functional Ward identity (WI) operator as the differential operator

$$W_X := -i\sum_{\varphi}\int \delta_X\varphi\frac{\delta}{\delta\varphi} \ . \tag{2.12}$$

The summation runs over all superfields φ. The superspace functional derivatives are defined by (A.16). We don't specify the integration measure, which is dV, dS or $d\bar{S}$ according to the type of φ.

The invariance of the classical action Σ is then expressed by the *Ward identity* (WI)

$$W_X\Sigma = 0 \ . \tag{2.13}$$

An important property of the the WI operators is that they obey the superalgebra

$$[W_{X_a}, W_{X_b}] = if_{abc}W_{X_c} \ , \tag{2.14}$$

if the differential operators or matrices δ_X obey the (anti)commutation rules

$$[\delta_{X_a}, \delta_{X_b}] = f_{abc}\delta_{X_c} \ . \tag{2.15}$$

In the equations above the brackets are "graded commutators", i.e. anticommutators $\{\,,\,\}$ if both arguments are odd, and commutators $[\,,\,]$ otherwise.

As a rule, the WI operators obey the same (super)algebra as the abstract (super)algebra of the generators, the superalgebra (2.5) - (2.7) for instance.

3 The Baby Model

3.1 The Action

The simplest N=1 supersymmetric model in four dimensions is the model of Wess and Zumino[36], which consists of a chiral superfield A in self-interaction. Its action reads

$$\Sigma = \frac{1}{16}\int dV\ A\bar{A} + \int dS\ W(A) + \int d\bar{S}\ \bar{W}(\bar{A})\ , \tag{3.1}$$

with the *superpotential*

$$W(A) = \frac{1}{4}\left(\frac{m}{2}A^2 + \frac{\lambda}{12}A^3\right)\ , \tag{3.2}$$

the mass m and coupling constant λ being real. In components (A.6), we have

$$\begin{aligned}\Sigma = \int d^4x\ \Big(& F\bar{F} + \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + \partial^\mu A\partial_\mu\bar{A} \\ & -\frac{m}{4}(4AF - \psi^2 + \text{conj.}) - \frac{\lambda}{8}(2A^2F - A\psi^2 + \text{conj.})\Big)\end{aligned} \tag{3.3}$$

One sees that the complex scalar field F is auxiliary, i.e. its equation of motion can be solved algebraically:

$$F = F(\bar{A}) = 4\bar{W}'(\bar{A}) = mA + \frac{\lambda}{4}A^2\ . \tag{3.4}$$

We may, if we want, insert this into the action, obtaining

$$\Sigma = \int d^4x\ \left(\frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + \partial^\mu A\partial_\mu\bar{A} + \frac{m}{4}\psi^2 + \frac{m}{4}\bar{\psi}^2 + \frac{\lambda}{8}A\psi^2 + \frac{\lambda}{8}\bar{A}\bar{\psi}^2 - V(A,\bar{A})\right)\ ,$$

with a potential given by

$$V(A,\bar{A}) = F(\bar{A})\bar{F}(A) = \left|mA + \frac{\lambda}{4}A^2\right|^2\ ,$$

which turns out to be positive.

3.2 Field Equations

The field equations read

$$\begin{aligned}\frac{\delta\Sigma}{\delta A} &= \frac{1}{16}\bar{D}^2\bar{A} + \frac{m}{4}A + \frac{\lambda}{16}A^2 = 0 \\ \frac{\delta\Sigma}{\delta\bar{A}} &= \frac{1}{16}D^2A + \frac{m}{4}\bar{A} + \frac{\lambda}{16}\bar{A}^2 = 0\ .\end{aligned} \tag{3.5}$$

One may combine them in order to find

$$\begin{aligned}4m\frac{\delta\Sigma}{\delta A} - \bar{D}^2\frac{\delta\Sigma}{\delta\bar{A}} &= (\partial^2 + m^2)A + \text{interaction} = 0 \\ 4m\frac{\delta\Sigma}{\delta\bar{A}} - D^2\frac{\delta\Sigma}{\delta A} &= (\partial^2 + m^2)\bar{A} + \text{interaction} = 0\ .\end{aligned} \tag{3.6}$$

3.3 Free Propagators

The computation of the free propagators amounts to compute the Green functions of the theory without self-interaction – i.e. with $\lambda = 0$ – but in presence of an external chiral superfield source J coupled to A. This is described by the action

$$\Sigma_J = \frac{1}{16}\int dV\ A\bar{A} + \frac{m}{8}\int dS\ A^2 + \frac{m}{8}\int d\bar{S}\ \bar{A}^2 + \int dS\ JA + \int d\bar{S}\ \bar{J}\bar{A}\ , \tag{3.7}$$

leading to the field equations

$$\begin{aligned} &\frac{1}{16}\bar{D}^2\bar{A} + \frac{m}{4}A = -J \\ &\frac{1}{16}D^2 A + \frac{m}{4}\bar{A} = -\bar{J}\ . \end{aligned} \tag{3.8}$$

Combining them as above, we find

$$\begin{aligned} (\partial^2 + m^2)A &= \bar{D}^2\bar{J} - 4mJ \\ (\partial^2 + m^2)\bar{A} &= D^2 J - 4m\bar{J}\ , \end{aligned} \tag{3.9}$$

The equations for the propagators are obtained by differentiating with respect to the sources:

$$\begin{aligned} (\partial^2 + m^2)\Delta_{AA}(1,2) &= (\partial^2 + m^2)\frac{\delta A(1)}{i\delta J(2)} = 4im\delta_S(1,2)\ , \\ (\partial^2 + m^2)\Delta_{A\bar{A}}(1,2) &= (\partial^2 + m^2)\frac{\delta A(1)}{i\delta \bar{J}(2)} = -i\bar{D}^2\delta_{\bar{S}}(1,2)\ , \\ (\partial^2 + m^2)\Delta_{\bar{A}\bar{A}}(1,2) &= (\partial^2 + m^2)\frac{\delta \bar{A}(1)}{i\delta \bar{J}(2)} = 4im\delta_{\bar{S}}(1,2)\ , \end{aligned} \tag{3.10}$$

where δ_S and $\delta_{\bar{S}}$ are the chiral and antichiral superspace Dirac distributions given by (A.14). The notation $f(1)$ means $f(x_1, \theta_1, \bar{\theta}_1)$, etc.

In order to solve the latter system, one introduces the causal scalar propagator $\Delta_c(x)$ defined as a particular inverse of the Klein-Gordon operator:

$$\begin{aligned} &i(\partial^2 + m^2)\Delta_c(x) = \delta^4(x)\ \ : \\ &\Delta_c(x) = \frac{1}{(2\pi)^4}\int d^4p\ e^{ipx}\tilde{\Delta}_c(p)\ ,\quad \tilde{\Delta}_c(p) = \frac{i}{p^2 - m^2 + i0}\ , \end{aligned} \tag{3.11}$$

with the notation

$$\frac{1}{z + i0} := \lim_{\varepsilon\to 0,\ \varepsilon>0} \frac{1}{z + i\varepsilon}\ .$$

Then

$$\begin{aligned} \Delta_{AA}(1,2) &= m\theta_{12}^2 e^{i(\theta_1\sigma\bar{\theta}_2 - \theta_2\sigma\bar{\theta}_1)\partial}\Delta_c(x_1 - x_2)\ , \\ \Delta_{A\bar{A}}(1,2) &= e^{i(\theta_1\sigma\bar{\theta}_2 + \theta_2\sigma\bar{\theta}_1 - \theta_{12}\sigma\bar{\theta}_{12})\partial}\Delta_c(x_1 - x_2)\ , \\ \Delta_{\bar{A}\bar{A}}(1,2) &= m\bar{\theta}_{12}^2 e^{i(\theta_1\sigma\bar{\theta}_2 - \theta_2\sigma\bar{\theta}_1)\partial}\Delta_c(x_1 - x_2)\ . \end{aligned} \tag{3.12}$$

Taking the Fourier transform with respect to $x_1 - x_2$ yields the momentum space propagators

$$\begin{aligned} \hat{\Delta}_{AA}(1,2) &= m\theta_{12}^2 e^{-(\theta_1\sigma\bar{\theta}_2 - \theta_2\sigma\bar{\theta}_1)p}\hat{\Delta}_c(p)\ , \\ \hat{\Delta}_{A\bar{A}}(1,2) &= e^{-(\theta_1\sigma\bar{\theta}_2 + \theta_2\sigma\bar{\theta}_1 - \theta_{12}\sigma\bar{\theta}_{12})p}\hat{\Delta}_c(p)\ , \\ \hat{\Delta}_{\bar{A}\bar{A}}(1,2) &= m\bar{\theta}_{12}^2 e^{-(\theta_1\sigma\bar{\theta}_2 - \theta_2\sigma\bar{\theta}_1)p}\hat{\Delta}_c(p)\ . \end{aligned} \tag{3.13}$$

4 Super Yang-Mills Theory

This section contains a general description of the N=1 supersymmetric gauge theories and of their gauge fixing procedure at the classical level. I follow [7], up to small changes in the notation.

4.1 Pure Super Yang-Mills Action

The supermultiplet of gauge fields is given by the components of the superfield (see (A.3))

$$\begin{aligned}\phi(x,\theta,\bar\theta) = C(x) + \theta\chi(x) + \bar\theta\bar\chi(x) + \frac{1}{2}\theta^2 M(x) + \frac{1}{2}\bar\theta^2 \bar M(x) \\ + \theta\sigma^\mu\bar\theta v_\mu(x) + \frac{1}{2}\bar\theta^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar\theta\bar\lambda(x) + \tfrac{1}{4}\theta^2\bar\theta^2 D(x) \ ,\end{aligned} \tag{4.1}$$

ϕ as well as each of its components belong to the adjoint representation of the gauge group G. We use a matricial notation:

$$\varphi = \varphi^a\tau_a \ , \quad \varphi = \phi,\ C, \chi, \cdots \tag{4.2}$$

where the matrices τ_a form the basis of the Lie group G in the defining representation of G – e.g. the Pauli matrices in the case $G = \mathrm{SU}(2)$ – normalized in such a way that

$$[\tau_a, \tau_b] = i f_{abc}\tau_c \ , \quad \mathrm{Tr}\ \tau_a\tau_b = \delta_{ab} \ . \tag{4.3}$$

The gauge transformations are implicitly defined by

$$e^{\phi'} = e^{-i\bar\Lambda}e^{\phi}e^{i\Lambda} \ , \quad \text{with} \quad \bar D_{\dot\alpha}\Lambda = 0 \ , \tag{4.4}$$

where $\Lambda = \Lambda^a\tau_a$, which explicitly yields, for the infinitesimal transformations,

$$\begin{aligned}\delta_{\rm gauge}\phi &= \frac{i}{2}L_\phi(\Lambda+\bar\Lambda) + \frac{i}{2}\left(L_\phi\coth(L_\phi/2)\right)(\Lambda-\bar\Lambda) \\ &= i(\Lambda-\bar\Lambda) + \frac{i}{2}[\phi,\Lambda+\bar\Lambda] + \frac{i}{12}[\phi,[\phi,\Lambda-\bar\Lambda]] + O(\phi^3) \ ,\end{aligned} \tag{4.5}$$

with $L_\phi X = [\phi, X]$.

Remark. Later we shall see that this transformation law is only a particular case of a general transformation law defined by

$$e^{\mathcal{F}(\phi')} = e^{-i\bar\Lambda}e^{\mathcal{F}(\phi)}e^{i\Lambda} \ , \tag{4.6}$$

where $\mathcal{F}(\phi)$ is an arbitrary function of ϕ, only restricted by the requirement to be in the adjoint representation like ϕ.

The group G will be supposed to be a simple Lie group. Generalization to a general compact Lie group is straightforward. The pure super Yang-Mills (SYM) action reads [37] (the conventions are those of [7])

$$\begin{aligned}\Sigma_{\rm SYM} = -\frac{1}{128g^2}\mathrm{Tr}\int dS\ F^\alpha F_\alpha \ , \\ \text{with} \quad F_\alpha = \bar D^2\left(e^{-\phi}D_\alpha e^{\phi}\right) \ .\end{aligned} \tag{4.7}$$

4.2 The Wess-Zumino Gauge:

Expanding in components the chiral superfield Λ according to (A.6):

$$\Lambda(x,\theta,\bar\theta) = e^{-i\theta\sigma^\mu\bar\theta\partial_\mu}\left(a(x)+\theta\eta(x)+\theta^2 f(x)\right) \ , \tag{4.8}$$

one can write the gauge transformations for the components of the gauge superfield ϕ as

$$\begin{aligned}
&C' = C + i(a-\bar a) + \cdots \ , \quad \chi' = \chi + i\eta + \cdots \ , \quad M' = M + 2if + \cdots \ , \\
&v'_\mu = v_\mu + \partial_m(a+\bar a) + \cdots \ , \quad \lambda' = \lambda + \bar\sigma^\mu\partial_\mu\eta + \cdots \ , \\
&D' = D - i\partial^2(a-\bar a) + \cdots \ .
\end{aligned} \tag{4.9}$$

where the dots stand for the non-Abelian part of the transformations. One can see that the transformations of the lower components C, χ and M do not involve any derivative of the components of Λ. It follows that one can solve algebraically for $\mathrm{Im}\, a$, η and f the equations $C' = \chi' = M' = 0$. Thus there always exist a gauge transformation which allows to fix to zero these lower components of ϕ. This defines the Wess-Zumino gauge [38]. In this gauge only the higher components, i.e. the gauge field v_μ, the "gaugino" λ and the D-field, are non-zero. From the components of Λ, only $\mathrm{Re}\, a$ remains free. It corresponds to the usual gauge transfornations:

$$A'_\mu = e^{-i\omega}(\partial_\mu + i[A_\mu,\omega])e^{i\omega} \ , \quad \lambda' = e^{-i\omega}\lambda e^{i\omega} \ , \quad D' = e^{-i\omega}De^{i\omega} \ , \tag{4.10}$$

where one has set

$$\omega := \mathrm{Re}\, a \ , \quad A_\mu := \frac{1}{2}v_\mu \ . \tag{4.11}$$

The SYM action (4.7) now reduces to the more familiar one

$$\Sigma_{\mathrm{SYM\ (WZgauge)}} = \frac{1}{g^2}\mathrm{Tr}\int d^4x\ \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\lambda\sigma^\mu D_\mu\bar\lambda + \frac{1}{2}D^2\right) \ , \tag{4.12}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \ , \quad D_\mu\cdot = \partial_\mu\cdot + i[A_\mu,\cdot].$$

Of course, the Wess-Zumino gauge is not preserved by the supersymmetry transformations (A.17). However, the action (4.12) is still invariant under the following combination of infinitesimal supersymmetry and gauge transformations :

$$\begin{aligned}
\delta_\alpha A_\mu &= \frac{1}{4}(\sigma_\mu\bar\lambda)_\alpha, \\
\delta_\alpha\lambda^\beta &= \delta_\alpha^\beta D + 2\sigma^{\mu\nu}F_{\mu\nu} \ , \\
\delta_\alpha\bar\lambda_{\dot\alpha} &= 0 \ , \\
\delta_\alpha D &= -i(\sigma^\mu D_\mu\bar\lambda)_\alpha \ .
\end{aligned} \tag{4.13}$$

These transformations are nonlinear, which is a source of complications for the renormalization [39]. Moreover, the supersymmetry algebra closes on the "covariant translations", instead of the simple translations as in (A.2): one has to replace the derivative ∂_μ in the translation operator by the covariant derivative D_μ, when acting on λ and D, and replace $\partial_\mu A_\nu$ by $F_{\mu\nu}$. The reader may consult [40] for recent progress in this direction.

4.3 Gauge Fixing and BRS Invariance

For the rest of these lectures, we shall choose a supersymmetric gauge fixing, instead of the Wess-Zumino one described in the preceding subsection. This gauge fixing will be a supersymmetric extension of the Lorentz gauge $\partial^\mu v_\mu = 0$. Observing that $\partial^\mu v_\mu$ is a component of the chiral superfield

$$\bar{D}^2 D^2\phi = e^{-i\theta\sigma^\mu\bar{\theta}\partial_\mu}\left(4(D - \partial^2 C - 2i\partial v) - 8i\theta\sigma\partial(\bar{\lambda} + i\partial\chi\sigma) - 8\theta^2\partial^2 M\right) , \tag{4.14}$$

we shall implement the condition $\bar{D}^2 D^2\phi = 0$, with the help of a Lagrange multiplier chiral superfield $B = B^a\tau_a$. We thus add to the action the piece

$$\frac{1}{8}\mathrm{Tr}\int dS\; B\bar{D}^2 D^2\phi + \text{ c.c.} = \frac{1}{8}\mathrm{Tr}\int dV\;(BD^2\phi + \bar{B}\bar{D}^2\phi) .$$

Since the gauge group is non-Abelian one has still to add Faddeev-Popov ghost fields. The gauge condition and the gauge parameter Λ being chiral, these ghost fields will be chiral as well. We note that $c_- = c_-^a\tau_a$ and $c_+ = c_+^a\tau_a$. They are the antighost and the ghost, respectively. Their components $c_\pm$ and $\bar{c}_\pm$ are *anticommuting* or Grassmann chiral superfields.

Before introducing them in the action, let us define the BRS transformations, under which the total action will have to be invariant:

$$\begin{aligned}
s\phi &= \frac{1}{2}L_\phi(c_+ + \bar{c}_+) + \frac{1}{2}\left(L_\phi\coth(L_\phi/2)\right)(c_+ - \bar{c}_+) \\
&= c_+ - \bar{c}_+ + \frac{1}{2}[\phi, c_+ + \bar{c}_+] + \cdots , \\
sc_+ &= -c_+^2 , \quad s\bar{c}_+ = -\bar{c}_+^2 , \quad \left(sc_+^a = -\frac{i}{2}f_{abc}c_+^b c_+^c \right) \\
sc_- &= B , \quad s\bar{c}_- = \bar{B} , \\
sB &= 0 , \quad s\bar{B} = 0 .
\end{aligned} \tag{4.15}$$

One checks that the BRS operator s is an *antiderivation* which is *nilpotent*:

$$s^2 = 0 . \tag{4.16}$$

One sees that the BRS transformation of the gauge superfield ϕ is just the gauge transformation (4.5) – up to a factor i. The gauge invariant action (4.7) thus is already BRS invariant. The gauge fixing piece of the action will be defined as

$$\begin{aligned}
\Sigma_{\rm gf} &= \frac{1}{8}s\,\mathrm{Tr}\int dV\;\left(c_- D^2\phi + \bar{c}_-\bar{D}^2\phi\right) \\
&= \frac{1}{8}\mathrm{Tr}\int dV\;\left(BD^2\phi + \bar{B}\bar{D}^2\phi - c_- D^2 s\phi - \bar{c}_-\bar{D}^2 s\phi\right) .
\end{aligned} \tag{4.17}$$

Its BRS invariance follows from the nilpotency of s. The last term, which involves the ghosts and the antighosts, is the supersymmetric extension of the usual Faddeev-Popov action.

Remark. One may add to the gauge fixing action a supplementary term

$$\Sigma_{(\alpha)} = 2\alpha\,\mathrm{Tr}\int dV\; B\bar{B} , \tag{4.18}$$

where α is a dimensionless *gauge parameter*. One can show [7] that the physical content of the theory does not depend on it. This makes of the B field an auxiliary field which can be eliminated by using its equation of motion

$$\bar{D}^2\bar{B} = -\frac{1}{16\alpha}\bar{D}^2 D^2\phi , \tag{4.19}$$

thus yielding, for the B-dependent terms of the action the expression

$$-\frac{1}{256\alpha}\mathrm{Tr}\int dV\;\phi\left(D^2\bar{D}^2 + \bar{D}^2 D^2\right)\phi , \tag{4.20}$$

which is the supersymmetrization of the Stueckelberg gauge fixing

$$-\frac{1}{2\alpha}\mathrm{Tr}\int d^4x\ (\partial^\mu v_\mu)^2\ .$$

But, using the fact that the physical quantities are independent from α [41], we shall keep $\alpha = 0$ through the rest of these lectures. This corresponds to a supersymmetrization of the Landau gauge fixing

$$\mathrm{Tr}\int d^4x\ (\mathrm{Re}\,F_B)\,\partial^\mu v_\mu,$$

where F_B is the θ^2-component of the chiral superfield B.

4.4 Matter Fields

Having written all the pieces building the classical gauge fixed action of the pure super Yang-Mills action, let us introduce matter. The latter is described by a set of chiral superfields

$$A^i(x,\theta,\bar\theta) = e^{-i\theta\sigma^\mu\bar\theta\partial_\mu}\left(A^i(x) + \theta\psi^i(x) + \theta^2 F^i(x)\right)\ , \tag{4.21}$$

which belong to some representation R of the gauge group. Their BRS transformations – identical to their infinitesimal gauge transformations up to a factor i – read

$$sA^i = -c_+^a {T_a}^i{}_j A^j \equiv -(c_+A)^i\ , \quad s\bar A_i = \bar A_j {T_a}^j{}_i \bar c_+^a \equiv (\bar A\bar c_+)_i\ , \tag{4.22}$$

where the hermitean matrices T_a are the generators of the gauge group in the representation R.

The BRS-invariant action for the matter fields reads

$$\Sigma_{\mathrm{matter}} = \frac{1}{16}\int dV\ \bar A e^{\phi^a T_a} A + \int dS\ W(A) + \int d\bar S\ \bar W(\bar A)\ , \tag{4.23}$$

with the superpotential W given by

$$W(A) = \frac{1}{8}m_{(ij)}A^iA^j + \lambda_{(ijk)}A^iA^jA^k\ , \tag{4.24}$$

the mass matrix m_{ij} and the Yukawa coupling constants λ_{ijk} being invariant symmetric tensors in the representation R.

4.5 R Invariance

It is easy to check that, in the massless case ($m_{ij} = 0$), ′the classical action given by (4.7), (4.17) and (4.23) is invariant under the R-transformations generically defined by (2.10), the R-weights n of the various superfields of the present theory being given in Table 1.

This symmetry will play a very important role in the sequel.

4.6 Slavnov-Taylor identity

In order to express the BRS invariance of the theory through a Ward identity, we have to take care of the nonlinearity of the BRS transformations. We couple the latters with external superfields ϕ^*, A^{*i} and c_+^* by adding to the action the piece

$$\Sigma_{\mathrm{ext}} = \int dV\ \mathrm{Tr}\,\phi^* s\phi + \left[\int dS\ \left(A^{*i}sA_i + \mathrm{Tr}\,c_+^* sc_+\right) + \mathrm{c.c.}\right] \equiv \int\sum_\varphi \varphi * s\varphi\ . \tag{4.25}$$

	θ^α	D_α	ϕ	A	c_+	c_-	B	ϕ^*	A^*	c_+^*
d	$-\frac{1}{2}$	$\frac{1}{2}$	0	1	0	1	1	2	2	3
n	-1	1	0	$-\frac{2}{3}$	0	-2	-2	0	$-\frac{4}{3}$	-2
$\Phi\Pi$	0	0	0	0	1	-1	0	-1	-1	-2

Table 1: Dimensions d, R-weights n and ghost numbers $\Phi\Pi$.

The BRS invariance of the total action

$$\Sigma := \Sigma_{\rm SYM} + \Sigma_{\rm matter} + \Sigma_{\rm gf} + \Sigma_{\rm ext} \tag{4.26}$$

is now expressed by the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0 \ , \tag{4.27}$$

with

$$\begin{aligned} \mathcal{S}(\gamma) &= {\rm Tr}\int dV \frac{\delta\gamma}{\delta\phi^*}\frac{\delta\gamma}{\delta\phi} + \left(\int dS \left\{ \frac{\delta\gamma}{\delta A^{*i}}\frac{\delta\gamma}{\delta A_i} + {\rm Tr}\,\frac{\delta\gamma}{\delta c_+^*}\frac{\delta\gamma}{\delta c_+} + {\rm Tr}\, B\frac{\delta\gamma}{\delta c_-}\right\} + {\rm c.c.}\right) \\ &\equiv \int \sum_\varphi \frac{\delta\gamma}{\delta\varphi^*}\frac{\delta\gamma}{\delta\varphi} + B\frac{\delta\gamma}{\delta c_-} \ . \end{aligned} \tag{4.28}$$

The gauge fixing is defined in a functional way by the condition (supersymmetric Landau gauge)

$$\frac{\delta\Sigma}{\delta B} = \frac{1}{8}\bar{D}^2 D^2\phi \ , \tag{4.29}$$

and its complex conjugate.

Differentiating the Slavnov-Taylor identity with respect to B or $\bar{B}$ and using (4.29) yield the ghost equations

$$\mathcal{G}_+\Sigma = 0 \ , \quad \bar{\mathcal{G}}_+\Sigma = 0 \ , \tag{4.30}$$

with

$$\mathcal{G}_+ = \frac{\delta}{\delta c_-} + \frac{1}{8}\bar{D}^2 D^2 \frac{\delta}{\delta\phi^*} \ . \tag{4.31}$$

They imply that the theory depends on c_-, $\bar{c}_-$ and ϕ^* only through the combination

$$\hat{\phi^*} = \phi^* - \frac{1}{8}(D^2 c_- + \bar{D}^2\bar{c}_-) \ . \tag{4.32}$$

Hence

$$\begin{aligned} \Sigma &= \Sigma(\phi, A, c_+, B, \hat{\phi^*}, A^*, c_+^*) \\ &= \Sigma_{\rm SYM}(\phi) + \Sigma_{\rm matter}(\phi, A) + \Sigma_{\rm ext}(\phi, A, c_+, \hat{\phi^*}, A^*, c_+^*) \\ &\quad + \frac{1}{8}{\rm Tr}\int dV (B D^2\phi + \bar{B}\bar{D}^2\phi) \ , \end{aligned} \tag{4.33}$$

4.7 Rigid Invariance

The total action is invariant under the rigid transformations

$$\begin{aligned} &\delta_{\rm rig}\varphi = i[\omega, \varphi] \ , \quad \varphi = \phi,\, c_\pm,\, B,\, \phi^*,\, c_+^* \ , \\ &\delta_{\rm rig}A^i = i\omega^a T_a{}^i{}_j A^j \ , \quad \delta_{\rm rig}A_i^* = -i\omega^a A_j^* T_a{}^j{}_i \ , \end{aligned} \tag{4.34}$$

which correspond to gauge transformations with constant parameters ω^a.

Rigid invariance does not necessarily hold in general. It holds here because the gauge fixing condition respects it. This would not be the case with a more general gauge fixing condition, such as a 't Hooft-like gauge, for example.

4.8 Ward Identities and Algebra

Beyond BRS invariance, the theory posesses invariances under supersymmetry, translations, R-transformations and rigid transformations. The four latter symmetries being linear are expressed by the Ward identities

$$W_X \Sigma := -i \sum_\varphi \int \delta_X \varphi \frac{\delta}{\delta\varphi} \Sigma = 0 \ , \quad X = Q_\alpha,\ P_\mu,\ R, \text{ and rigid transf. }, \tag{4.35}$$

where $\delta^P_\mu \varphi = \partial_\mu \varphi$, and δ^Q_α, δ^R, $\delta_{\rm rig}$ are defined by (2.9), (2.10), (4.34), respectively.

The Ward identity operators together with the Slavnov-Taylor operator and the gauge fixing and ghost equation operators, obey the algebra (null (anti-)commutators are not written)

$$\begin{array}{l}
\left[W^Q_\alpha, W^{\bar Q}_{\dot\alpha}\right] = 2\sigma^\mu_{\alpha\dot\alpha} W^P_\mu \ , \quad \left[W^Q_\alpha, W^R\right] = -W^Q_\alpha \ , \quad \left[W^Q_{\dot\alpha}, W^R\right] = +W^Q_{\dot\alpha} \ , \\
W^Q_\alpha \mathcal{S}(\gamma) - \mathcal{S}_\gamma W^Q_\alpha \gamma = 0 \ , \quad W^R \mathcal{S}(\gamma) - \mathcal{S}_\gamma W^R \gamma = 0 \ , \quad W_{\rm rig} \mathcal{S}(\gamma) - \mathcal{S}_\gamma W_{\rm rig} \gamma = 0 \ , \quad \forall \gamma \ , \\
\frac{\delta}{\delta B} \mathcal{S}(\gamma) - \mathcal{S}_\gamma \left(\frac{\delta}{\delta B}\gamma - \frac{1}{8}\bar D^2 D^2 \phi \right) = \mathcal{G}_+ \gamma \ , \quad \forall \gamma \ , \\
\mathcal{G}_+ \mathcal{S}(\gamma) + \mathcal{S}_\gamma \mathcal{G}_+ \gamma = 0 \ , \quad \forall \gamma \ , \\
\mathcal{S}_\gamma \mathcal{S}(\gamma) = 0 \ , \quad \forall \gamma \ , \\
{\mathcal{S}_\gamma}^2 = 0 \quad \text{if} \quad \mathcal{S}(\gamma) = 0 \ ,
\end{array} \tag{4.36}$$

γ denoting a functional of the superfields and $\mathcal{S}_\gamma$ the "linearized" Slavnov-Taylor operator at the "point" γ:

$$\begin{aligned}
\mathcal{S}_\gamma = \ & {\rm Tr} \int dV \left(\frac{\delta\gamma}{\delta\phi^*} \frac{\delta}{\delta\phi} + \frac{\delta\gamma}{\delta\phi} \frac{\delta}{\delta\phi^*} \right) \\
& + \left(\int dS \left(\frac{\delta\gamma}{\delta A^*} \frac{\delta}{\delta A} + \frac{\delta\gamma}{\delta A} \frac{\delta}{\delta A^*} + {\rm Tr}\, \frac{\delta\gamma}{\delta c^*_+} \frac{\delta}{\delta c_+} + {\rm Tr}\, \frac{\delta\gamma}{\delta c_+} \frac{\delta}{\delta c^*_+} + {\rm Tr}\, B \frac{\delta}{\delta c_-} \right) + \text{c.c.} \right)
\end{aligned} \tag{4.37}$$

Note that, since the classical action Σ obeys the Slavnov-Taylor identity, $\mathcal{S}_\Sigma$ is nilpotent:

$$ {\mathcal{S}_\Sigma}^2 = 0 \ . \tag{4.38}$$

4.9 General Classical Action

The general solution of the classical problem, i.e. of solving the Slavnov-Taylor identity for the classical action, taking into account the gauge condition (4.29) and the Ward identities (4.35) for supersymmetry, R-invariance and rigid invariance, is given by

$$\begin{aligned}
\Sigma(\phi, A, c_+, B, \hat\phi^*, A^*, c^*_+) = \Sigma_{\rm SYM}(\phi') + \Sigma_{\rm matter}(\phi', A') + \Sigma_{\rm ext}(\phi', A', c'_+, \hat\phi^{*\prime}, A^{*\prime}, {c^*_+}') \\
+ \frac{1}{8} {\rm Tr} \int dV (B D^2 \phi + \bar B \bar D^2 \phi) \ ,
\end{aligned} \tag{4.39}$$

with

$$
\begin{aligned}
&\phi' = \mathcal{F}(\phi) \ , \qquad \hat{\phi^*}'(z) = \frac{\delta}{\delta\phi'(z)}\int dV(z')\hat{\phi^*}(z')\mathcal{F}^{-1}(\phi')(z')\bigg|_{\phi'=\mathcal{F}(\phi)} \ , \\
&A' = z_1 A \ , \qquad A^{*\prime} = \frac{1}{z_1}A^* \ , \\
&c'_+ = z_2 c_+ \ , \qquad c_+^{*\prime} = \frac{1}{z_2}c_+^* \ .
\end{aligned}
\tag{4.40}
$$

Eqs. (4.40) represent field renormalizations. Due to the dimensionlessness of ϕ its renormalization is non-linear: the function $\mathcal{F}(\phi)$ is an arbitrary formal power series in ϕ:

$$
\mathcal{F}_a(\phi) = \sum a_k \; t^{(k)}_{aa_1\cdots a_n}\phi^{a_1}\cdots\phi^{a_n} \tag{4.41}
$$

Due to the rigid invariance (4.34), the numbers $t^{(k)}_{aa_1\cdots a_n}$ are components of invariant tensors of the group. (In the same way the masses m_{ij} and the couplings λ_{ijk} in (4.23) are invariant tensors in the matter field representation). We shall restrict ourselves in the following to the massless case $m_{ij} = 0$.

One can check that the dependence on the renormalization parameters a_k is non-physical. This is expressed by the fact that the derivative of the action with respect to each a_k is a BRS-variation:

$$
\frac{\partial}{\partial a_k}\Sigma = \mathcal{S}_\Sigma \Delta_k \ , \tag{4.42}
$$

where Δ_k is some local functional. This means that the a_k are gauge parameters [41].

Remark. Let us open a "parenthesis": The gauge condition and the ghost equation lead to the decomposition

$$
\Sigma = \hat{\Sigma}(\phi,\ A,\ c_+,\ \hat{\phi^*},\ A^*, c_+^*) + \frac{1}{8}\mathrm{Tr}\int dV(BD^2\phi + \bar{B}\bar{D}^2\phi) \ . \tag{4.43}
$$

The Slavnov-Taylor identity then reads

$$
\mathcal{S}(\Sigma) = \frac{1}{2}\mathcal{B}_{\hat{\Sigma}}\hat{\Sigma} = 0 \ , \tag{4.44}
$$

with

$$
\begin{aligned}
\mathcal{B}_\gamma = \ & \mathrm{Tr}\int dV\left(\frac{\delta\gamma}{\delta\phi^*}\frac{\delta}{\delta\phi} + \frac{\delta\gamma}{\delta\phi}\frac{\delta}{\delta\phi^*}\right) \\
& + \left(\int dS\left(\frac{\delta\gamma}{\delta A^*}\frac{\delta}{\delta A} + \frac{\delta\gamma}{\delta A}\frac{\delta}{\delta A^*} + \mathrm{Tr}\,\frac{\delta\gamma}{\delta c_+^*}\frac{\delta}{\delta c_+} + \mathrm{Tr}\,\frac{\delta\gamma}{\delta c_+}\frac{\delta}{\delta c_+^*}\right) + \text{c.c.}\right)
\end{aligned}
\tag{4.45}
$$

obeying the identities

$$
\begin{aligned}
&\mathcal{B}_\gamma\mathcal{B}_\gamma\gamma = 0 \ , \qquad \forall\,\gamma \ , \\
&\mathcal{B}_\gamma\mathcal{B}_\gamma = 0 \qquad \text{if} \qquad \mathcal{B}_\gamma\gamma = 0 \ .
\end{aligned}
\tag{4.46}
$$

4.10 Soft Breakings of Supersymmetry

If supersymmetry has some relevance, it must be broken at "low" energy (typically below $\approx$ 1Tev). A spontaneous breakdown is conceivable at the level of supergravity, i.e. of local supersymmetry [42], or at the level of superstring theory [43].

But, in the low energy domain, where gravitational interaction appears to be negligible, i.e. in the domain of *rigid* supersymmetry, the breakdown arises in the form of an explicit breakdown by nonsupersymmetric mass terms. Such a breakdown is soft[6], which means that it does not affect the behaviour of the theory in the high energy domain, where supersymmetry thus remains valid.

Such soft breakings are conveniently described in the Symanzik formalism [44], which consists in adding to the action couplings with "shifted" fields (here: shifted superfields) in order to keep record of the transformation properties of the breaking terms, in such a way that the Ward identities still hold. Let us see an example of breaking by a gauge invariant gluino mass term $\mathrm{Tr}\,(\lambda^\alpha\lambda_\alpha)$. One observes that

$$\mathrm{Tr}\,\lambda^\alpha\lambda_\alpha = \mathrm{Tr}\,F^\alpha F_\alpha|_{\theta=0}\ , \tag{4.47}$$

i.e., this term is equal to the first component of the chiral superfield $\mathrm{Tr}\,F^\alpha F_\alpha$ (where F_α is given by (4.7)).

One then introduces the shifted chiral superfield

$$E' = E + m\theta^2 \tag{4.48}$$

where E is a gauge invariant classical chiral superfield, and m is a parameter with the dimension of a mass. The term

$$\int dS E' \mathrm{Tr}\,F^\alpha F_\alpha \tag{4.49}$$

just gives the gluino mass term (4.47) at $E = 0$. Moreover the action containing this term still obeys the supersymmetry Ward identity

$$W^Q_\alpha \Sigma = 0\ , \tag{4.50}$$

where

$$W^Q_\alpha = W^{Q\,(\mathrm{old})}_\alpha + \left[\int dS\ \delta^Q_\alpha E' \frac{\delta}{\delta E'} + c.c.\right]\ , \tag{4.51}$$

where $\delta^Q_\alpha E'$ resuts from the application of the supersymmetry differential generator given by eq. (A.1) of Appendix A to E':

$$\delta^Q_\alpha E' = \delta^Q_\alpha E + 2m\theta_\alpha\ . \tag{4.52}$$

One easily sees that, at $E = 0$, the Ward identity (4.50) reads

$$W^{Q\,(\mathrm{old})}_\alpha \Sigma = -\int d^4x\ m\mathrm{Tr}\,(\lambda_\alpha D + \ldots)\ . \tag{4.53}$$

The right-hand side is, as one expects, the variation of the spacetime integral of the gluino masss term (4.47), computed using the transformation rules (A.17) of Appendix A. The advantage of the supersymmetric formalism is that the algebra (2.14) is preserved by the Ward operator involving such shifted superfields.

[6] "soft" is taken here in the sense of power-counting [44]. This definition is more general that the one of [14], which only keeps breakings which do not give rise to UV divergences more severe than logarithmic.

5 Superspace Feynman Graphs

A short account of the supergraph formalism and of its consequences will be given. A more detailed account is given in [7]. See [2] for a somewhat alternative presentation.

5.1 The Free Propagators

The free propagators are the Green functions of the theory defined by the quadratic part of the classical action, the fields $\varphi_i(x)$ being coupled to external sources $J^i(x)$, $(i = 1, \cdots, n)$. Let us first illustrate the procedure for obtaining them in the case the φ^i's are scalar fields. The field equations corresponding to the action

$$\Sigma_{\rm free}(\varphi) = \int dx \left(\frac{1}{2}\varphi_i K^{ij}\varphi_j + J^i\varphi_i \right) \ , \tag{5.1}$$

where K^{ij} is a matrix of partial derivative operators, read

$$K^{ij}\varphi_j + J^i = 0 \ . \tag{5.2}$$

The Green functions are defined as the solutions of the equations

$$K^{ij}{\Delta^{\rm c}}_{jk}(x) = i\delta^i_k \delta(x-y) \ , \tag{5.3}$$

with the Feynman-Stueckelberg causal prescription as a boundary condition. Then, the solution of the field equations (5.2) will read

$$\begin{array}{l} \varphi_i(x) = i\displaystyle\int dx {\Delta^{\rm c}}_{ij}(x-y)J^j(y) \ , \\ \text{or, formally:} \qquad \varphi = i\Delta^{\rm c} * J \ , \quad \text{with} \quad \Delta^{\rm c} = \dfrac{i}{K} \ . \end{array} \tag{5.4}$$

E.g., in the case of one scalar field, $K = -\partial^2 - m^2$,

$$\begin{array}{l} \Delta^{\rm c} = \dfrac{-i}{\partial^2 + m^2} = \dfrac{1}{(2\pi)^4}\displaystyle\int dp \, e^{ip(x-y)}\tilde{\Delta}^{\rm c}(p) \ , \\ \text{with} \quad \Delta^{\rm c}(p) = \dfrac{i}{p^2 - m^2 + i0} =: \lim_{\varepsilon\to+0} \dfrac{i}{p^2 - m^2 + i\varepsilon} \ . \end{array} \tag{5.5}$$

Let us go to the super Yang-Mills theory[7] beginning with the matter fields. The free action being diagonal we don't write the summation indices:

$$\Sigma_{\rm free}(A) = \frac{1}{16}\int dV \, \bar{A}A + \int dS \, AJ + \int d\bar{S} \, \bar{A}\bar{J} \ . \tag{5.6}$$

The corresponding field equations read

$$\frac{1}{16}\bar{D}^2\bar{A} = -J \ , \quad \frac{1}{16}D^2 A = -\bar{J} \ .$$

Applying the operator $\bar{D}^2$ to the second equation, and using the commutation relations (A.28) of Appendix A, we find the equation

$$\partial^2 A = \bar{D}^2\bar{J} \ ,$$

[7] We consider the massless theory, leaving the massive case as an exercise to the reader.

whose solution reads[8]

$$A = i\Delta^c{}_{AA} * J + i\Delta^c{}_{A\bar{A}} * \bar{J} = \frac{1}{\partial^2}\bar{D}^2\bar{J} \ .$$

This yields

$$\begin{aligned}
&\Delta^c{}_{AA}(x_1,\theta_1,\bar\theta_1;x_2,\theta_2,\bar\theta_2) \equiv \Delta^c{}_{AA}(1,2) = 0 \ , \\
&\Delta^c{}_{A\bar{A}}(x_1,\theta_1,\bar\theta_1;x_2,\theta_2,\bar\theta_2) \equiv \Delta^c{}_{A\bar{A}}(1,2) = \frac{-i}{\partial^2}\bar{D}^2\delta_{\bar{S}}(1,2) \\
&\quad = \frac{1}{(2\pi)^4}\int d^4p \ \tilde\Delta^c_{A\bar{A}}(p;1,2) \ , \quad \text{with} \quad \tilde\Delta^c_{A\bar{A}}(p;1,2) = e^{-(\theta_1\sigma\bar\theta_2 - \theta_2\sigma\bar\theta_1 - \theta_{12}\sigma\bar\theta_{12})p}\frac{i}{p^2+i0} \ ,
\end{aligned} \tag{5.7}$$

where we have used the expressions (A.14), (A.15) of Appendix A for the superspace Dirac distributions and their derivatives.

The free action for the ghost fields reads

$$\begin{aligned}
\Sigma_{\rm free}(c_+,c_-) &= -\frac{1}{8}\int dV \ c_-D^2c_+ + \int dS \ \left(J_{c_+}c_+ + J_{c_-}c_-\right) \quad + \ \text{c.c.} \\
&= \int dS \ \left(2c_-D^2c_+ + J_{c_+}c_+ + J_{c_-}c_-\right) \quad + \ \text{c.c.}
\end{aligned}$$

From the field equations

$$2\partial^2c_+ = J_{c_-} \ , \quad -2\partial^2c_- = J_{c_+} \ ,$$

we immediately deduce the ghost propagators (directly written in momentum space):

$$\begin{aligned}
&\tilde\Delta^c_{c_+c_-}(p;1,2) = e^{-(\theta_1\sigma\bar\theta_2-\theta_2\sigma\bar\theta_1)p}(\theta_{12})^2\frac{i}{8(p^2+i0)} \\
&\tilde\Delta^c_{c_+\bar{c}_-}(p;1,2) = 0 \ .
\end{aligned} \tag{5.8}$$

In the sector of the gauge and Lagrange-multiplier fields the free action is again diagonal in the Yang-Mills indices:

$$\begin{aligned}
\Sigma_{\rm free}(\phi,B) = &\int dV \ \left(-\frac{1}{128g^2}D\phi\bar{D}^2D\phi + \frac{1}{8}(BD^2\phi + \bar{B}\bar{D}\phi)\right) \\
&+ \int dV \ J_\phi\phi + \int dS \ J_BB + \int d\bar{S} \ J_{\bar{B}}\bar{B} \ .
\end{aligned}$$

It yields the field equations

$$\begin{aligned}
&\frac{1}{64g^2}D\bar{D}^2D\phi + \frac{1}{8}D^2B + \frac{1}{8}\bar{D}^2\bar{B} = -J_\phi \ , \\
&\frac{1}{8}\bar{D}^2D^2\phi = -J_B \ , \\
&\frac{1}{8}D^2\bar{D}^2\phi = -J_{\bar{B}} \ .
\end{aligned} \tag{5.9}$$

Applying $\bar{D}^2$ on the first equation we obtain

$$\frac{1}{8}\bar{D}^2D^2B = -\bar{D}^2J_\varphi \ ,$$

which solves into

$$B = \frac{1}{2}\frac{1}{\partial^2}\bar{D}^2J_\phi \ .$$

[8] The superspace integration measure used in the convolution product $*$ is the one appropriate to the type of the superfields involved in each case.

From the latter equation one reads out the propagators (in momentum space):

$$\begin{aligned} &\tilde{\Delta}^{c}_{BB} = \tilde{\Delta}^{c}_{B\bar{B}} = 0 \ , \\ &\tilde{\Delta}^{c}_{B\phi}(p;1,2) = e^{-(\theta_1\sigma\bar{\theta}_2 - \theta_2\sigma\bar{\theta}_1)p}(\theta_{12})^2 \frac{i}{8(p^2+i0)} \ . \end{aligned} \tag{5.10}$$

In order to find the remaining propagators, we act on the first of the equations (5.9) with the "transverse" projector P^{T} (A.29), and sum up the the second and the third of these equations. Combining both equations thus obtained, making use of the completeness property $P^{\mathrm{T}} + P^{\mathrm{L}} = 1$ (see (A.29)), we obtain

$$\frac{1}{8g^2}\partial^2\phi = -P^{\mathrm{T}}J_\phi + \frac{1}{16g^2}(J_B + J_{\bar{B}}) \ ,$$

which yields

$$\Delta^{c}{}_{\phi\phi}(1,2) = \frac{8ig^2}{\partial^2} P^{\mathrm{T}}\delta_V(1,2) \ . \tag{5.11}$$

Remark. Note that this yields, in momentum space, a double pole at $p^2 = 0$:

$$\frac{1}{(p^2+i0)^2} \ , \tag{5.12}$$

which constitutes an infrared singularity. We shall comment on this point at the beginning of Section 6.

5.2 Feynman rules and Power-Counting

The contribution of an L-loop superspace Feynman graph γ consists in a product of superpropagators or of covariant derivatives thereof. It has the form of an L-loop integral

$$J_\gamma(p,\tilde{\theta}) = \int d^{4L}k \ I_\gamma(p,k,\tilde{\theta}) \ , \tag{5.13}$$

where $\tilde{\theta} = \theta$ or $\bar{\theta}$, and where the p's and the k's denote the internal and external momenta, respectively. The precise structure of the integrand I_γ follows from the following momentum space Feynman rules (for the 1PI amputated diagrams):

1. For each internal line, write the corresponding superpropagator, with appropriate derivatives if the vertices coupled by the line involve superfield derivatives.
2. For each external (amputated) leg, write a superspace Dirac distribution: δ_V, δ_S or $\delta_{\bar{S}}$ (see (A.14)) according to the nature of the field (real, chiral or antichiral) associated to the leg.
3. At each vertex integrate over its $\tilde{\theta}$ variables, with the integration measure which corresponds to the nature of the vertex.

As we have seen in the preceding subsection, the propagators as well as their covariant derivatives have the structure

$$\Delta^{c}(p;1,2) = e^{-(\theta_1\sigma\bar{\theta}_2 - \theta_2\sigma\bar{\theta}_1)p} f(p,\tilde{\theta}_{12}) \ , \tag{5.14}$$

where $\tilde\theta_{ij} = \tilde\theta_i - \tilde\theta_j$. The superspace Dirac distributions have the same structure. It follows that the integrand I_γ has the general form

$$
\begin{aligned}
&I_\gamma(p,k,\tilde\theta) = e^{E(p,\tilde\theta)} \bar I_\gamma(p,k,\tilde\theta_{..}) \ , \\
&\text{with} \quad E(p,\tilde\theta) = -\sum_{i=1}^{n-1} (\theta_i \sigma \bar\theta_n - \theta_n \sigma \bar\theta_i) p_i \ ,
\end{aligned}
\tag{5.15}
$$

where $\bar I_\gamma$ only depends on the differences $\tilde\theta_{..}$.

Remark. This corresponds to the general structure of a superfield Green function following from the supersymmetry Ward identities:

$$
\tilde G(p,\tilde\theta) = e^{E(p,\tilde\theta)} \bar G(p,\tilde\theta_{..}) \ .
$$

Expanding (5.15) in powers of the $\tilde\theta$'s, we obtain

$$
I_\gamma(p,k,\tilde\theta) = e^{E(p,\tilde\theta)} \sum_{\omega=0}^{\Omega} (\tilde\theta_{..})^\omega I_\omega(p,k) \ , \tag{5.16}
$$

where $(\tilde\theta_{..})^\omega$ stands for a generic monomial in the variables $\tilde\theta_{ij}$ of degree ω – a subsummation over all the independent monomials with the same degree being implicitly understood. The maximum degree is given by

$$
\Omega = \begin{cases} 4N_V + 2N_S + 2N_{\bar S} - 4 & \text{in the generic case} \ , \\ 2N_S - 2 & \text{if } N_V = N_{\bar S} = 0 \ , \end{cases} \tag{5.17}
$$

where N_V, N_S, resp. $N_{\bar S}$ are the numbers of vector, chiral, resp. antichiral external legs of the 1PI graph under consideration. A simple argument based on dimensional analysis shows that the degrees of divergence d_ω of the integrands I_ω are related to each other by the formula[9]

$$
d_\omega \le d_0 + \frac{1}{2}\omega \ . \tag{5.18}
$$

A detailed analysis[10] leads for d_0 to the upperbound

$$
d_0 \le 4 - \sum_V (d_V + 2) - \sum_S (d_S + 1) - \sum_{\bar S} (d_{\bar S} + 1) \ , \tag{5.19}
$$

where d_V, d_S, resp. $d_{\bar S}$ are the dimensions (see Table 1 of in Section 4) of the superfields corresponding to the vector, chiral, resp. antichiral external legs of the diagram. The maximum degree of divergence of a supergraph is given by (use (5.17)):

$$
d_\Omega \le \begin{cases} 2 - \sum_V d_V - \sum_S d_S - \sum_{\bar S} d_{\bar S} & \text{in the generic case} \ , \\ 3 - \sum_S d_S & \text{if } N_V = N_{\bar S} = 0 \ . \end{cases} \tag{5.20}
$$

[9]Due to the presence of possible mass terms, such a powercounting based on dimensional analysis yields only upperbounds.

[10]d_0 in fact is the degree of divergence which one would obtain through usual power-counting for the component diagram whose external legs correspond to the highest θ-components of the superfields coresponding to the legs of the superdiagram.

5.3 Nonrenormalization Theorem for the Superpotential

Applying this result to the super Yang-Mills theory described in the preceding section, we find that the potentially divergent diagrams (we don't consider here the diagrams with ghost external legs) are those contributing to the following vertex functions [11]:

$$\begin{aligned}
&\Gamma_{\phi_1\cdots\phi_{N_V}}\ , \quad \forall\, N_V \ : && d_\Omega \leq 2\ , \\
&\Gamma_{\phi_1\cdots\phi_{N_V}A\bar{A}}\ , \quad \forall\, N_V \ : && d_\Omega \leq 0\ , \\
&\Gamma_{A_1A_2}\ : && d_\Omega \leq 1\ , \\
&\Gamma_{A_1A_2A_3}\ : && d_\Omega \leq 0\ .
\end{aligned} \tag{5.21}$$

But it turns out that the actual degrees of divergence are lower. In particular, those for the purely chiral vertex functions Γ_{AA} and Γ_{AAA} are negative: the corresponding diagrams are convergent. This is the content of *the nonrenormalization theorem for the chiral vertices.* This theorem follows from the vanishing of the radiative (i.e. loop graph) corrections to the purely chiral vertex functions at zero external momenta:

$$\Gamma^{(\text{rad. corr.})}_{A_1\cdots A_N}(p)\Big|_{p=0} = 0\ . \tag{5.22}$$

The latter result indeed implies that these radiative corrections must have external momentum factors, of degree 2 at least due to Lorentz invariance: hence their effective degree of divergence is lowered by 2 at least, which makes it negative in view of (5.21). Before proving (5.22), let us note that this means that the effective superpotential defined (in momentum space) as

$$W_{\text{eff}}(A) = \Gamma(\varphi)\big|_{\text{ at zero momenta, } \varphi=0\ \forall\varphi \text{ except } \varphi=A}\ , \tag{5.23}$$

does not get any quantum correction:

$$W_{\text{eff}}(A) = W_{\text{class}}(A)\ , \tag{5.24}$$

where $W_{\text{class}}(A)$ is the classical superpotential, given by (3.2) or (4.24), describing the self-interaction of the matter fields. Eq. (5.24) is the content of the *nonrenormalization theorem for the superpotential.*

Proof of (5.22): Let us consider the contribution to the vertex function $\Gamma^{(\text{rad. corr.})}_{A_1\cdots A_N}$, taken at zero momentum, of a 1PI diagram containing n_V, n_S, resp. $n_{\bar{S}}$ vertices for the vector, chiral, resp. antichiral type. Since all the external legs are chiral, all the variables $\bar{\theta}$ are integrated. Before these integrations, the integrand is a function of the differences $\bar{\theta}_{..}$ of the $\bar{\theta}$ associated to each of the vertices since the external momenta are set to zero (c.f. (5.15)). There are at most $2(n_V + n_{\bar{S}} - 1)$ such independent variables. But the total number of $\bar{\theta}$-integrations is equal to $2(n_V + n_{\bar{S}})$, which implies a vanishing integral.

6 Renormalization

The material which follows is only a summary. A more complete exposition may be found in [7] and in the original paper [45].

The renormalization program consists in showing that there exists a quantum theory, constructed as a perturbative expansion in $\hbar$, whose Green functions obey all the conditions defining

[11] A vertex functions is the sum of the contributions of the one-particle irreducible graphs only to a given Green functions, amputated from its external legs (c.f. Appendix B).

a given classical theory. If this programs succeeds, and if the resulting theory depends on a *finite* number of free physical parameters, the theory is called renormalizable. The theory is called anomalous if the fulfilment of some of the conditions turns out to be impossible (see [1]).

These conditions have been expressed in Section 4 for the super Yang-Mills theory as a set of identities (gauge condition, Ward identities, Slavnov-Taylor identity, etc.) which the classical action Σ has to fulfil. These functionals identities have to generalize for the vertex functional $\Gamma(\varphi)$. The latter is indeed the natural object to consider in the quantum theory. It generates the vertex functions, i.e. the contributions of the 1-particle irreducible Feynman graphs to the Green functions, amputated from their external legs. Let us note[12] that, in the classical limit $\hbar = 0$, the vertex functional coincides with the classical action:

$$\Gamma(\varphi) = \Sigma(\varphi) + O(\hbar) \ , \tag{6.1}$$

and, for future use, that the vertex functional $\Delta \cdot \Gamma(\varphi)$ corresponding to a composite field insertion coincides, in the classical limit, with the local functional (classical field polynomial) Δ:

$$\Delta \cdot \Gamma(\varphi) = \Delta(\varphi) + O(\hbar\Delta) \ . \tag{6.2}$$

6.1 The Infrared Problem

A difficulty, genuine to supersymmetric gauge theories in four-dimensional space-time, is the appearance of a pseudoscalar field $C(x)$ (the $\theta = 0$ component of the gauge superfield (4.1) which is both massless (due non-Abelian gauge invariance) and dimensionless. Its propagator in momentum space is of the form $1/(k^2 + i0)^2$. It therefore presents an infrared singularity since it is non-integrable at $k = 0$. There are two known ways out of this difficulty. The first, better known, way is to work in the Wess-Zumino gauge [38] (see Subsection 4.2), where the field C is absent.

The second procedure [7, 46] for circumventing the infrared problem consists in the introduction of a mass μ^2 for the field C, in such a way that the physical quantities do not depend on μ^2. This is achieved by using the possibility of performing a non-linear field redefinition of the gauge superfield as in (4.40), (4.41), but in a supersymmetry breaking way:

$$\phi' = (1 + \frac{\mu^2}{2}\theta^2\bar\theta^2)\phi \tag{6.3}$$

The propagator of C becomes proportional to $1/(k^2 - \mu^2 + i0)^2$: μ^2 plays the role of an infrared regulator. On the other hand μ^2, like the parameters a_k in (4.41), is a gauge parameter: the physical quantities are independent of it. In particular the breakdown of supersymmetry, parametrized by μ^2, does not affect the physical quantities.

For details we refer the reader to the original literature [46, 7]. In these notes we shall simply assume that all fields are made massive by adding suitable supersymmetric mass terms in the action. Since these masses in general will break BRS invariance and R-invariance, we assume the corresponding Slavnov-Taylor and Ward identities to hold in the asymptotic region of momentum space only , where the effect of the masses is negligeable. All equalities in the following have to be understood in this sense.

6.2 Renormalization of the Linear Identities

As we have outlined at the beginning of the present section, our aim is to establish the validity, to all orders, of the functional identities used to define the zeroth order theory given by the classical action. These identities, now written for the vertex functional Γ (see (6.1)), are:

[12] See Appendix B for more details on the generating functionals.

– the Ward identities for R-invariance, supersymmetry and rigid invariance, (4.35) (translation invariance being obvious)

$$W_X\Gamma := -i\sum_{\varphi}\int \delta_X\varphi\frac{\delta}{\delta\varphi}\Gamma = 0\ ,\quad X = R,\, Q_\alpha \text{ and rigid transf.}\ , \tag{6.4}$$

– the gauge condition (4.29) and the ghost equation[13] (4.30)

$$\begin{aligned}
&\frac{\delta\Gamma}{\delta B} = \frac{1}{8}\bar{D}^2D^2\phi\ ,\\
&\mathcal{G}_+\Gamma := \left(\frac{\delta}{\delta c_-} + \frac{1}{8}\bar{D}^2D^2\frac{\delta}{\delta\phi^*}\right)\Gamma = 0\ ,
\end{aligned} \tag{6.5}$$

– the Slavnov-Taylor identity (4.27)

$$\mathcal{S}(\Gamma) := \mathrm{Tr}\int dV\ \frac{\delta\Gamma}{\delta\phi^*}\frac{\delta\Gamma}{\delta\phi} + \left(\int dS\ \left\{\frac{\delta\Gamma}{\delta A^{*i}}\frac{\delta\Gamma}{\delta A_i} + \mathrm{Tr}\,\frac{\delta\Gamma}{\delta c_+^*}\frac{\delta\Gamma}{\delta c_+} + \mathrm{Tr}\,B\frac{\delta\Gamma}{\delta c_-}\right\} + \text{c.c.}\right) = 0\ . \tag{6.6}$$

We begin by giving a very short description of the way the linear identities (6.4)-(6.5) may be proven[14], leaving the Slavnov-Taylor identity (6.6) for the next subsection. Let us rewrite the identities (6.4)-(6.5), to be proven, as

$$\mathcal{F}_A\Gamma = 0\ , \tag{6.7}$$

where the index A enumerates all the components of each of them. We also include the translation operators. The operators $\mathcal{F}_A$ form a superalgebra

$$[\mathcal{F}_A, \mathcal{F}_B] = c_{ABC}\mathcal{F}_C\ , \tag{6.8}$$

– the brackets [,] being commutators or anticommutators – which is a subalgebra of the complete (including BRS) algebra (4.36).

The proof of the functional identities (6.7) is inductive and begins with the assumption that they hold up to the loop order $n-1$:

$$\mathcal{F}_A\Gamma = \hbar^n\Delta_A + O(\hbar^{n+1})\ . \tag{6.9}$$

Due to the quantum action principle [47], the possible breaking in the right-hand side is a local field insertion, integrated or not according to the nature of the left-hand side, and of dimension bounded from above by the dimension of the left-hand side[15]. At its lowest nonvanishing order, i.e. at the order n, it is a classical local functional Δ_A of the fields.

The algebraic relations (6.8) applied to the vertex functional Γ yield, at the order n, the consistency conditions

$$\mathcal{F}_A\Delta_B \mp \mathcal{F}_B\Delta_A = c_{ABC}\Delta_C\ . \tag{6.10}$$

It can be checked, in our case, that the general solution of the consistency conditions has the "trivial" form

$$\Delta_A = \mathcal{F}_A\hat{\Delta}\ , \tag{6.11}$$

[13] As we have already, said the ghost equation follows in fact from the gauge condition and from the Slavnov-Taylor identity. But it is useful to begin by showing its validity, prior to the proof of the Slavnov-Taylor identity, because it will give a further constraint on the possible breakings of the latter.

[14] The proof actually given in the literature (see [7]) does not take all the identities at once together, but treats them in sequence, each one after the other. The present description (c.f. [1]) is more concise, but equivalent.

[15] In fact we only take under consideration the terms of maximum dimension, since the lower dimension ones mix with the breakings due to the possible noninvariant mass terms, whose effects we have decided not to worry with.

where $\hat{\Delta}$ is an integrated local functional of dimension 4, i.e. of the dimension of the action. We then redefine the action as

$$\Sigma' = \Sigma - \hbar^n \hat{\Delta} \ , \tag{6.12}$$

which amounts to redefine the vertex functional as

$$\Gamma' = \Gamma - \hbar^n \hat{\Delta} + O(\hbar^{n+1}) \ . \tag{6.13}$$

Clearly, the new vertex functional obeys the functional identities (6.7) to the next order:

$$\mathcal{F}_A \Gamma = O(\hbar^{n+1}) \ . \tag{6.14}$$

This ends the inductive proof of their validity to all orders.

Remark. The proof we have sketched includes in particular that of the absence of anomaly for supersymmetry, which can be found in detail in [48]. The proof in the latter reference holds for supersymmetric theories with a field content corresponding to the class of super Yang-Mills theories considered here. More general cases, where supersymmetry anomalies could occur – although no concrete example of this is known – were considered in Refs. [49, 50].

6.3 Renormalization of BRS Invariance

The treatment of the renormalization problem for BRS invariance, namely the proof of the Slavnov-Taylor identity (6.6) – with possible anomalies – is closely parallel to the one for the nonsupersymmetric gauge theories discussed e.g. in [1]. There is also here one single possible anomaly, which is a supersymmetric extension of the usual Adler-Bardeen anomaly. It has the form of an infinite power series in the gauge superfield ϕ:

$$\mathcal{A} = \mathrm{Tr} \int dV \left(c_+ D^\alpha \phi \bar{D}^2 D_\alpha \phi - \bar{c}_+ \bar{D}_{\dot\alpha} \phi D^2 \bar{D}^{\dot\alpha} \phi + O(\phi^3) \right) \tag{6.15}$$

There is no simple closed expression for $\mathcal{A}$ (see [51]). The references [45, 7] state its existence and uniqueness. Explicit constructions may be found in [52].

Let us sketch the demonstration, which makes use of the same inductive procedure as for the linear functional identities in the last subsection. First, through the quantum action principle and from the assumption that the Slavnov-Taylor identity (6.6) has been proven up to order $n-1$ in $\hbar$, we can write

$$\mathcal{S}(\Gamma) = \hbar^n \Delta \cdot \Gamma = \hbar^n \Delta + O(\hbar^{n+1}) \ , \tag{6.16}$$

where Δ is an integrated local functional of the fields, of dimension[16] 4 and ghost number 1 (see Table 1 in Section 4 for dimensions and quantum numbers). From the algebra (4.36) and the fulfilment of the linear functional identities (6.4)-(6.5), we deduce that the most general form for the breaking D is restricted by the constraints

$$\begin{aligned} &W^R \Delta = 0 \ , \quad W^Q_\alpha \Delta = 0 \ , \quad W_{\rm rig} \Delta = 0 \ , \\ &\frac{\delta}{\delta B} \Delta = 0 \ , \quad \mathcal{G}_+ \Delta = 0 \ , \end{aligned} \tag{6.17}$$

and

$$\mathcal{S}_\Sigma \Delta = 0 \ , \tag{6.18}$$

[16] Again we neglect lower dimension terms.

where $\mathcal{S}_\Sigma$ is the linearized Slavnov-Taylor operator (4.37), Σ being the classical action. Due to the last of algebra relations (4.36) and to the fulfilment of the Slavnov-Taylor identity by the classical action, $\mathcal{S}_\Sigma$ is nilpotent:

$$\mathcal{S}_\Sigma^2 = 0 \ .$$

Solving (6.18) is thus a problem of cohomology in the space of the local functional of dimension 4, ghost number 1 and subjected to the constraints (6.17). A detailed analysis shows that the BRS constraint (6.18) has the general solution

$$\Delta = \mathcal{S}_\Sigma \tilde{\Delta} + r\mathcal{A} \ . \tag{6.19}$$

$\tilde{\Delta}$ is an integrated local functional of dimension 4 and ghost number 0: its absorption as a conterterm $-\hbar^n\tilde{\Delta}$ in the action eliminates it from the breaking Δ, in the same way as the possible breakings of the linear functional identities were eliminated (c.f. Eqs. (6.11) to (6.14)). We are left with the term $r\mathcal{A}$, with $\mathcal{A}$ given by (6.15) and r a calculable function of the parameters of the theory. Since it cannot be written as a $\mathcal{S}_\Sigma$-variation and it represents the cohomology of the nipotent operator $\mathcal{S}_\Sigma$ in the space of functionals under consideration. From the physical point of view, $\mathcal{A}$ represents the gauge anomaly, i.e. an obstruction to the implementation of BRS invariance beyond the classical approximation.

Remarks.

1. At the one-loop order, the anomaly coefficient r appears as an algebraic expression which is the same as in the usual gauge theories [1]. It follows that the absence of the anomaly in the one-loop order is assured by the usual conditions on the choice of the group representations for the matter fields. Its absence to all higher orders is then assured by a supersymmetric generalization of the nonrenormalization theorem of Bardeen (see [1], e.g.). Although such a generalization has not been explicitly checked, one may expect its validity, the supersymmetric adaptation of the proof looking obvious.

2. The anomaly (6.15) obeys the constraint

$$\left(\int dS \, \frac{\delta}{\delta c_+} + \int d\bar{S} \, \frac{\delta}{\delta \bar{c}_+}\right) \mathcal{A} = 0 \ ,$$

which follows from the validity of the "antighost equation" (6.23) – to be shown in Subsection 6.4 – and from the algebraic identity (6.26) together with rigid invariance, the independence from the Lagrange multiplyer field B being taken into account (see (6.17)).

6.4 The Antighost Equation

It is known [53] that in the Landau gauge – and in some noncovariant linear gauges as well [54] – the coupling of the Faddeev-Popov ghost c_+ is severely constrained by a functional identity, the "antighost equation". Its main consequence is the nonrenormalization of the ghost field, a property which turns out to be very useful in the proof of various nonrenormalization theorems [55, 56]. Let us show that such an identity also holds [57] in SYM theories in the supersymmetric Landau gauge (4.17).

Differentiating the classical action (4.26) with respect to the ghost field c_+ we obtain

$$\begin{aligned}\frac{\delta\Sigma}{\delta c_+} &= \frac{1}{16}\bar{D}^2[D^2c_-,\phi] + \frac{1}{16}\bar{D}^2[\bar{D}^2\bar{c}_-,\phi] \\ &- \frac{1}{2}\bar{D}^2[\phi^*,\phi] - \bar{D}^2\left(\hat{\phi}^*M(\phi)\right) + [c_+^*,c_+] + A^*T_aA\tau_a \ ,\end{aligned} \tag{6.20}$$

where $M(\phi)$, which appears in the nonlinear part of the BRS transformation of the gauge superfield (see (4.15)), is defined by

$$
\begin{aligned}
s\phi &= \frac{1}{2}[\phi, c_+ + \bar{c}_+] + M(\phi)(c_+ - \bar{c}_+) \ , \\
(M(\phi))^a_b \, (c_+ - \bar{c}_+)^b &:= \frac{1}{2}\Big((L_\phi \mathrm{coth}(L_\phi/2))(c_+ - \bar{c}_+)\Big)^a
\end{aligned}
$$

At this point one should observe that the right-hand side of (6.20) thus contains terms nonlinear in the quantum fields. These composite terms, being subject to renormalization, spoil the usefulness of this equation. However, considering the corresponding equation for $\bar{c}_+$:

$$
\begin{aligned}
\frac{\delta\Sigma}{\delta\bar{c}_+} = &\frac{1}{16}D^2[\bar{D}^2\bar{c}_-, \phi] + \frac{1}{16}D^2[D^2 c_-, \phi] \\
&- \frac{1}{2}D^2[\phi^*, \phi] + D^2\left(\hat{\phi}^* M(\phi)\right) + [\bar{c}^*_+, \bar{c}_+] - \bar{A}^* T_a \bar{A}\tau_a \ ,
\end{aligned}
\tag{6.21}
$$

adding together the superspace integrals of the equations (6.20), (6.21) and using[17] the Landau gauge condition (4.29), one obtains the antighost equation we are looking for:

$$
\mathcal{G}_- \Sigma = \Delta_{\text{class}} \ , \tag{6.22}
$$

with

$$
\mathcal{G}_- := \int dS \, \left(\frac{\delta}{\delta c_+} - \left[c_-, \frac{\delta}{\delta B}\right]\right) + \int d\bar{S} \, \left(\frac{\delta}{\delta \bar{c}_+} - \left[\bar{c}_-, \frac{\delta}{\delta \bar{B}}\right]\right) \tag{6.23}
$$

and

$$
\Delta_{\text{class}} := -\int dV \, [\phi^*, \phi] + \int dS \, \left([c^*_+, c_+] + (A^* T_a A)\tau_a\right) + \int d\bar{S} \, \left([\bar{c}^*_+, \bar{c}_+] - (\bar{A} T_a \bar{A}^*)\tau_a\right) \ . \tag{6.24}
$$

We remark that the undesired nonlinear terms present in each of the equations (6.20) and (6.21) have been cancelled. We are thus left with the breaking (6.24) which, being now linear in the quantum fields, will not be renormalized, i.e., it will remain a classical breaking.

Equation (6.22) has now a form which allows one to consider its validity to all orders of perturbation theory. That it indeed holds as it stands at the quantum level:

$$
\mathcal{G}_- \Gamma = \Delta_{\text{class}} \ , \tag{6.25}
$$

may be shown without any difficulty by repeating exactly the argument given in [53, 1] for the nonsupersymmetric case.

Let us finally remark that the sum of the superspace-integrated functional derivatives with respect to c_+ and $\bar{c}_+$ in (6.23) is in fact the space-time integral of the functional derivative with respect to the real part of the $\theta = 0$ component of c_+. It coincides with the functional operator appearing in the nonsupersymmetric version of the antighost equation.

We shall now derive an interesting consequence of the antighost equation. Using the "anticommutation relation"

$$
\mathcal{G}_- \mathcal{S}(\Gamma) + \mathcal{S}_\Gamma \left(\mathcal{G}_- \Gamma - \Delta_{\text{class}}\right) = W_{\text{rig}} \Gamma \ , \tag{6.26}
$$

[17] Use has to be made of the identity

$$
\int dS \; \bar{D}^2[D^2 c_-, \phi] = \int dS \; [c_-, \bar{D}^2 D^2 \phi] \ ,
$$

and of its complex conjugate.

where $\mathcal{S}_\Gamma$ is the linearized Slavnov-Taylor operator defined accordingly to (4.37), and

$$
\begin{aligned}
&W_{\rm rig}\Gamma := \int dV\ \left(\left[\phi, \frac{\delta\Gamma}{\delta\phi}\right] + \left\{\phi^*, \frac{\delta\Gamma}{\delta\phi^*}\right\} \right) \\
&+\int dS\ \left(\left\{c_+, \frac{\delta\Gamma}{\delta c_+}\right\} + \left[c_+^*, \frac{\delta\Gamma}{\delta c_+^*}\right] + \left[B, \frac{\delta\Gamma}{\delta B}\right] + \left\{c_-, \frac{\delta\Gamma}{\delta c_-}\right\} + \left(\frac{\delta\Gamma}{\delta A}T_a A\right)\tau_a - \left(A^* T_a \frac{\delta\Gamma}{\delta A^*}\right)\tau_a \right) \\
&+\int d\bar{S}\ \left(\left\{\bar{c}_+, \frac{\delta\Gamma}{\delta \bar{c}_+}\right\} + \left[\bar{c}_+^*, \frac{\delta\Gamma}{\delta \bar{c}_+^*}\right] + \left[\bar{B}, \frac{\delta\Gamma}{\delta \bar{B}}\right] + \left\{\bar{c}_-, \frac{\delta\Gamma}{\delta \bar{c}_-}\right\} - \left(\bar{A} T_a \frac{\delta\Gamma}{\delta \bar{A}}\right)\tau_a - \left(\frac{\delta\Gamma}{\delta \bar{A}^*}T_a \bar{A}^*\right)\tau_a \right) \\
&= 0\ .
\end{aligned}
\tag{6.27}
$$

One thus sees that, in the Landau gauge, the identity

$$
W_{\rm rig}\Gamma = 0 \tag{6.28}
$$

follows from the Slavnov-Taylor identity and the antighost equation. This is the Ward identity expressing the invariance of the theory under the rigid transformations (4.34), corresponding to the transformations of the gauge group with constant parameters.

6.5 Invariant Counterterms

Once the gauge fixing condition (4.29), the ghost equation(4.30), the Slavnov-Taylor identity (4.27), the Ward identity for R-invariance (third of equs. (4.35)) and the antighost equation (6.25) have been established[18] at a given arbitrary order $\hbar^n$ as shown in Section 6, we are still free to introduce in the action, at the same order, counterterms which do not spoil these identities. A generic "invariant counterterms" Δ has thus to obey to the constraints

$$
\begin{aligned}
&\frac{\delta\Delta}{\delta B} = 0\ ,\quad \mathcal{G}_+\Delta = 0\ ,\quad W_R\Delta = 0\ ,\quad W_{\rm rig}\Gamma = 0\ ,\quad \mathcal{G}_-\Delta = 0\ , \\
&\mathcal{S}_\Sigma\Delta = 0\ ,
\end{aligned}
\tag{6.29}
$$

where $\mathcal{S}_\Sigma$ is the linearized Slavnov-Taylor operator (4.37). The ghost number of Δ is 0 and its dimension 4 [19].

The general solution of these constraints is a linear superposition of the following terms:

$$
\begin{aligned}
&{\rm Tr}\int dS\ F^\alpha F_\alpha\ ,\quad \int dS\ \lambda_{(ijk)}A^iA^jA^k + {\rm c.c.}\ ,\quad \mathcal{S}_\Sigma{\rm Tr}\int dS\ A_i^*A^j + {\rm c.c.}\ , \\
&\mathcal{S}_\Sigma{\rm Tr}\int dV\ \hat{\phi}^*\phi\ ,\quad \mathcal{S}_\Sigma{\rm Tr}\int dV\ \hat{\phi}^*(\phi)^k\ ,
\end{aligned}
\tag{6.30}
$$

where Σ is the most general classical action as given by (4.39), and $\hat{\phi}^*$ is the shifted external superfield (4.32).

Another, more convenient, basis for the invariant counterterms is given by the expressions

$$
\nabla_I\Sigma\ , \tag{6.31}
$$

where the operators ∇_I respectively are

$$
\begin{aligned}
&\partial_g\ ,\ \partial_{\lambda_{ijk}}\ ,\ \mathcal{N}_\phi = N_\phi - N_{\phi^*} - N_{c_-} - N_{\bar{c}_-} - N_B - N_{\bar{B}}\ , \\
&\mathcal{N}_i^j = \int dS\left(A^j\frac{\delta}{\delta A^i} - A_i^*\frac{\delta}{\delta A_j^*}\right) + {\rm conj.}\ , \\
&\partial_{a_k}\ ,
\end{aligned}
\tag{6.32}
$$

[18] Poincaré invariance and supersymmetry are obvious since the renormalization scheme preserves them explicitly.
[19] We don't consider counterterms of lower dimension, as they would affect the mass terms which anyhow break these symmetries.

where we have introduced the "counting operators"

$$N_\varphi = \int \varphi \frac{\delta}{\delta\varphi} \ , \quad \varphi = \phi,\, \phi^*,\, c_\pm,\, B \ . \tag{6.33}$$

The invariance of the expressions (6.31) follows from the operators (6.32) being "symmetric", i.e. from their "(anti)commutativity"[20] with the operators appearing in the constraints (6.29):

$$\begin{aligned} &\nabla_I \mathcal{S}(\gamma) - \mathcal{S}_\gamma \nabla_I \gamma = 0 \ , \quad \forall\gamma \ , \\ &\left[\nabla_I, \frac{\delta}{\delta B}\right] = 0 \ , \quad [\nabla_I, \mathcal{G}_\pm] = 0 \ , \quad [\nabla_I, W_R] = 0 \ . \end{aligned} \tag{6.34}$$

This is clear for ∂_g, ∂_λ and ∂_{a_k}. For the operators $\mathcal{N}$ this follows from

$$\mathcal{N}_\phi \gamma = \mathcal{S}_\gamma \mathrm{Tr} \int dV\ \hat{\phi}^* \phi \ , \quad \mathcal{N}_i^j \gamma = \mathcal{S}_\gamma \mathrm{Tr} \int dS\ A_i^* A^j + \text{conj.} \ .$$

One thus sees that the counterterms, in the form (6.31), correspond to a renormalization of the parameters of the action and of the field amplitudes. This shows the stability of the theory under the perturbative quantum fluctuations.

Remarks.

1. The latter property, which is equivalent to the stability of the classical action under the effect of small perturbations, characterizes the renormalizability of the theory.
2. A renormalization of the ghost field c_+ would be implied by a counterterm

$$\begin{aligned} -\mathcal{S}_\Sigma \mathrm{Tr} \int dS\ c_+^* c_+ + \text{conj.} &= \left(N_{c_+} - N_{c_+^*} + \text{conj.}\right) \Sigma \\ &= \mathrm{Tr} \int dS\ c_+^* s c_+ + \text{conj.} + \int dS\ A^* sA + \text{conj.} + \int dV\ \hat{\phi}^* s\phi \ . \end{aligned}$$

But such a term, depending on the superfield c_+ without derivative, is forbidden by the last of the constraints (6.29), which corresponds to the antighost equation (6.25).

The coefficients of the invariant counterterms (6.30) or (6.31) are still free parameters. They are usually fixed by imposing normalization conditions which define the field amplitudes and the physical parameters of the theory [1]. In our case we can choose the conditions given in Eqs. (5.180-182) of Ref. [7]. We only mention here that they involve vertex functions taken at some fixed 4-impulsions characterized by a normalization mass κ, and that, in the tree approximation, they reproduce the parametrization of the classical action given in section 4.

6.6 Callan-Symanzik Equation

The classical theory is scale invariant if all the fields are massless, or at least asymptotically scale invariant if there are masses. This is no longer true for the quantum theory. This "scale anomaly"

[20] There are in fact two nonvanishing anticommutators, namely

$$\left[\mathcal{N}_\phi, \frac{\delta}{\delta B}\right] = \frac{\delta}{\delta B} \ , \quad \left[\mathcal{N}_\phi, \mathcal{G}_+\right] = \mathcal{G}_+ \ ,$$

but this has no consequence since they are applied to the action which obeys the gauge condition and the ghost equation.

is best expressed by the Callan-Symanzik equation. In order to derive it, let us introduce the dilatation generator

$$D := \sum_{\text{all mass parameters } m} m\frac{\partial}{\partial m} \ , \tag{6.35}$$

where the summation is taken over all the mass parameters, including the normalization mass κ introduced through the normalization conditions. Application of this operator to the classical action yields the equation

$$D\Sigma = 0 \ , \tag{6.36}$$

which expresses the asymptotic scale invariance of the classical theory[21].

Let us now apply the dilatation generator to the full vertex functional Γ. Through the quantum action principle, we obtain

$$D\Gamma = \Delta \cdot \Gamma \ , \tag{6.37}$$

where Δ is an insertion of dimension 4, of order $\hbar$ and whose effect is to break asymptotic scale invariance.

Noting that D is a symmetric operator according to the definition given in (6.34), we conclude that Δ is an invariant insertion, which we can expand in a basis of invariant dimension 4 insertions. Such a basis may be provided by the set of insertions

$$\{\nabla_I \Gamma\} \ , \tag{6.38}$$

where the ∇_I's are the symmetric operators (6.32). This is a quantum extension of the classical basis of counterterms (6.31). The expansion of the right-hand side of (6.37) yields the Callan-Symanzik equation

$$C\Gamma := \left(D + \beta_g\partial_g + \beta_{ijk}\partial_{\lambda_{ijk}} - \gamma_\phi\mathcal{N}_\phi - \gamma^i_j\mathcal{N}^j_i - \gamma_k\partial_{a_k}\right)\Gamma = 0 \ . \tag{6.39}$$

The coefficients β and γ are of order $\hbar$. The former correspond to the renormalization of the coupling constants, the latter – the "anomalous dimensions" – to the renormalization of the field amplitudes and of the unphysical parameters a_k (see (4.41)-(4.42)).

Remark. There is no anomalous dimension for the ghost field c_+. This is a consequence of the antighost equation (6.25) (see the second remark at the end of Subsection 6.5).

[21] Recall that (non-gauge invariant) mass terms being implicitly present, such an equality is valid up to terms of dimension less than the dimension of the left-hand-side, i.e. less than 4. These terms are negligible at high momenta.

7 Supercurrent

The matter presented in this section is extracted from the original papers [58], the book [7], with slight modifications introduced later in the papers [28, 30].

7.1 Classical Theory

We have seen that supersymmetry and BRS invariance together with power counting fix the action (4.39). We have also seen (Subsection 4.5) that the latter turns out to be also invariant[22] under the following chiral phase transformation, called R-invariance [59] (we ommit the infinitesimal parameter):

$$\delta^R\varphi = i\left(n_\varphi + \theta^\alpha\frac{\partial}{\partial\theta^\alpha} - \bar\theta^{\dot\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\right)\varphi\ . \tag{7.1}$$

The "R-weights" n_φ are given in table 1 (Subsection 4.5)

Let us recall that the R-transformation commutes with BRS, but not with supersymmetry:

$$[\delta^Q_\alpha, \delta^R] = i\delta^Q_\alpha\ , \qquad [\delta^{\bar Q}_{\dot\alpha}, \delta^R] = -i\delta^{\bar Q}_{\dot\alpha}\ . \tag{7.2}$$

Taking into account the Wess-Zumino algebra (2.11) we see that the generators of R-transformations, supersymmetry and translations form a supermultiplet, supersymmetry acting on them by (anti)commutation. The supercurrent [60] is then the supermultiplet which contains the conserved Noether currents R_μ, $Q_{\mu\alpha}$ and $T_{\mu\nu}$ associated respectively to the invariances under R, supersymmetry and translations. The supercurrent is represented by a vector superfield

$$V_\mu(x,\theta,\bar\theta) = R_\mu(x) - i\theta^\alpha Q_{\mu\alpha}(x) + i\bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}_\mu(x) - 2(\theta\sigma^\nu\bar\theta)T_{\mu\nu}(x) + \cdots \tag{7.3}$$

where we have written only the most relevant terms. $T_{\mu\nu}$ is the "improved" (i.e. symmetric, traceless in the classical approximation) energy-momentum tensor. $Q_{\mu\alpha}$ is also traceless in the classical approximation – in the sense: $\sigma^\mu_{\alpha\dot\alpha}\bar Q^{\dot\alpha}_\mu = 0$.

We are going to show that the precise identification of these currents and of their properties follow from the *supertrace identities*, written here in the classical approximation:

$$\begin{aligned} &\bar D^{\dot\alpha} V_{\alpha\dot\alpha} = -2w_\alpha\Sigma - \frac{4}{3}D_\alpha S^0\ , \quad D^\alpha V_{\alpha\dot\alpha} = -2\bar w_{\dot\alpha}\Sigma - \frac{4}{3}\bar D_{\dot\alpha}\bar S^0\ , \\ &\text{with}\quad V_\mu = \sigma^{\alpha\dot\alpha}_\mu V_{\alpha\dot\alpha}\ , \quad V_{\alpha\dot\alpha} = \frac{1}{2}\sigma^\mu_{\alpha\dot\alpha}V_\mu\ , \end{aligned} \tag{7.4}$$

where Σ is the classical action (4.26), and w_α is the functional differential operator (we use the

[22] I recall that due to the fields being all massive in order to avoid infrared problems, BRS as well as R-invariance hold only asymptotically in momentum space. Hence all the following equations are meant to hold asymptotically only.

definitions of Refs. [28, 30], slightly different from those of [58, 7])[23]

$$
\begin{aligned}
w_\alpha = 2\mathrm{Tr}\,\Big(&-\bar{D}^2 D_\alpha\phi\frac{\delta}{\delta\phi} + D_\alpha\phi\bar{D}^2\frac{\delta}{\delta\phi} + \phi^*\bar{D}^2 D_\alpha\frac{\delta}{\delta\phi^*} - \bar{D}^2\phi^* D_\alpha\frac{\delta}{\delta\phi^*} \\
&+ D_\alpha c_+\frac{\delta}{\delta c_+} - c_+^* D_\alpha\frac{\delta}{\delta c_+^*} - c_- D_\alpha\frac{\delta}{\delta c_-} - B D_\alpha\frac{\delta}{\delta B}\Big) \\
&+\frac{4}{3}D_\alpha A\frac{\delta}{\delta A} - \frac{2}{3}A D_\alpha\frac{\delta}{\delta A} + \frac{2}{3}D_\alpha A^*\frac{\delta}{\delta A^*} - \frac{4}{3}A^* D_\alpha\frac{\delta}{\delta A^*}\ .
\end{aligned}
\tag{7.5}
$$

This operator is BRS-symmetric, i.e.:

$$
w_\alpha\mathcal{S}(\gamma) - \mathcal{S}_\gamma w_\alpha = 0\ , \quad \forall\gamma\ .
$$

The chiral superfield[24] S^0 is a polynomial in the superfields ϕ, c_- and B:

$$
S^0 = \frac{1}{8}s\,\mathrm{Tr}\left(\bar{D}^2(2c_- D^2\phi - D^2 c_-\phi)\right) + \mathrm{conj.}\ , \tag{7.6}
$$

which is not gauge invariant. It is however BRS invariant, but nonphysical since it is a s-variation. It possesses the properties

$$
\begin{aligned}
&\int dS S^0 - \int d\bar{S}\bar{S}^0 = 0 \\
&\int dS S^0 + \int d\bar{S}\bar{S}^0 = 2(N_B + N_{c_-} + \mathrm{conj.})\Sigma\ ,
\end{aligned}
\tag{7.7}
$$

where N_φ is the φ-field counting operator (6.33).

Without going into details, (see [7, 58]) let us write the BRS invariant supercurrent, solution of the supertrace identities, for the classical theory:

$$
V_{\alpha\dot\alpha} = \frac{1}{6}[D_\alpha, \bar{D}_{\dot\alpha}](\bar{A}e^{T^a\phi_a}A) + \frac{1}{2}\bar{D}_{\dot\alpha}(\bar{A}e^{T^a\phi_a})e^{-T^a\phi_a}D_\alpha(e^{T^a\phi_a}A) + \frac{1}{16}\mathrm{Tr}\,\left(F^\alpha e^{-\phi}F_\alpha e^{\phi}\right) + \cdots\ , \tag{7.8}
$$

with F_α given in (4.7), and where the dots represent non-gauge-invariant terms produced by the gauge fixing, ghost terms and external field contributions.

In order to see that the supertrace identities (7.4) yield the conservation of the currents associated to R, supersymmetry and translation invariances, let us define the superfield currents

$$
\begin{aligned}
&\hat{R}_\mu := V_\mu =: R_\mu + O(\theta)\ , \\
&\hat{Q}_{\mu\alpha} := i\,(D_\alpha V_\mu - (\sigma_\mu\bar\sigma^\nu D)_\alpha V_\nu) =: Q_{\mu\alpha} + O(\theta)\ , \\
&\hat{T}_{\mu\nu} := -\frac{1}{16}\left(V_{\mu\nu} + V_{\nu\mu} - 2g_{\mu\nu}V_\lambda{}^\lambda\right) =: T_{\mu\nu} + O(\theta)\ , \\
&\quad\text{with}\quad V_{\mu\nu} := \sigma_{\mu\beta\dot\beta}[D^\beta, \bar{D}^{\dot\beta}]V_\nu\ .
\end{aligned}
\tag{7.9}
$$

We first check the conservation law of $\hat{R}_\mu$, i.e. of the supercurrent V_μ. It is obtained by applying D^α on the first of the supertrace identities (7.4), $\bar{D}^{\dot\alpha}$ on the second one, then adding together

---

[23] For a chiral field φ of weight n: $\quad w_\alpha = nD_\alpha\left(\varphi\frac{\delta}{\delta\varphi}\right) + 2D_\alpha\varphi\frac{\delta}{\delta\varphi}\ .$
For a vector superfield V (of weight 0), on may have:

$$
\text{either:}\quad w_\alpha = -2\left(\bar{D}^2 D_\alpha V\frac{\delta}{\delta V} - D_\alpha V\bar{D}^2\frac{\delta}{\delta V}\right)\ ,\quad \text{or:}\quad w_\alpha = 2\left(V\bar{D}^2 D_\alpha\frac{\delta}{\delta V} - \bar{D}^2 V D_\alpha\frac{\delta}{\delta V}\right)\ .
$$

[24] This form of S^0 is the one found in [28, 30]. It differs of the form given in [58] or in [7] due to a different choice for the gauge condition.

the identities thus obtained , and finally using the anticommutation rule (A.5) for the covariant derivatives. We get

$$\begin{aligned}&\partial^\mu V_\mu = i\hat{w}^R\Sigma + \frac{2}{3}i(D^2S^0 - \bar{D}^2\bar{S}^0) ,\\ &\text{with} \quad \hat{w}^R = D^\alpha w_\alpha - \bar{D}_{\dot\alpha}\bar{w}^{\dot\alpha} .\end{aligned} \tag{7.10}$$

The other conservation laws follow from the latter identity and from the supertrace identity. One finds, altogether:

$$\begin{aligned}&\partial^\mu \hat{R}_\mu = i\hat{w}^R\Sigma + \frac{2}{3}i(D^2S^0 - \bar{D}^2\bar{S}^0) ,\\ &\partial^\mu \hat{Q}_{\mu\alpha} = i\hat{w}^Q_\alpha\Sigma ,\\ &\partial^\mu \hat{T}_{m\nu} = i\hat{w}^P_\nu\Sigma ,\end{aligned} \tag{7.11}$$

with

$$\begin{aligned}&\hat{w}^R = D^\alpha w_\alpha - \bar{D}_{\dot\alpha}\bar{w}^{\dot\alpha} ,\\ &\hat{w}^Q_\alpha = iD_\alpha(D^\beta w_\beta - \bar{D}_{\dot\beta}\bar{w}^{\dot\beta}) - 4i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar{w}^{\dot\alpha} ,\\ &\hat{w}^P_\nu = -\frac{1}{16}\bar{\sigma}^{\dot\alpha\alpha}_\nu\left(D^2\bar{D}_{\dot\alpha}w_\alpha + \bar{D}^2D_\alpha\bar{w}_{\dot\alpha} + [D_\alpha, \bar{D}_{\dot\alpha}](D^\beta w_\beta - \bar{D}_{\dot\beta}\bar{w}^{\dot\beta})\right) + \frac{i}{2}\partial_\nu(D^\beta w_\beta + \bar{D}_{\dot\beta}\bar{w}^{\dot\beta}) .\end{aligned} \tag{7.12}$$

One checks that the space-time integration of the latter functional operators at $\theta = 0$ yields the Ward operators of the corresponding symmetries:

$$\int d^4x\ \hat{w}^R = W^R + O(\theta) , \quad \int d^4x\ \hat{w}^Q_\alpha = W^Q_\alpha + O(\theta) , \quad \int d^4x\ \hat{w}^P_\nu = W^P_\nu + O(\theta) , \tag{7.13}$$

It follows that, taken at $\theta = 0$, the equations (7.11) express the conservation of the Noether currents R_μ, $Q_{\mu\alpha}$, $T_{\mu\nu}$ associated to R-invariance, supersymmetry and translation invariance, respectively, identified as the $\theta = 0$ components of the superfield currents (7.9). (The conservation of R_μ holding up to the nonphysical breaking in S^0).

One also finds that, beyond the conservation laws (7.11), the identities (7.4) also contain the *trace identity*

$$\hat{T}^\lambda_\lambda = -\frac{3}{2}\left(D^\alpha w_\alpha + \bar{D}_{\dot\alpha}\bar{w}^{\dot\alpha}\right)\Sigma - \left(D^2S^0 + \bar{D}^2\bar{S}^0\right) . \tag{7.14}$$

Thus the conserved energy-momentum tensor contained in the supercurrent is symmetric and traceless (up to the nonphysical S^0-terms): this identifies it as the improved energy-momentum tensor. Defining thus the current

$$\hat{D}_\mu = x^\nu\hat{T}_{\mu\nu} =: D_\mu + O(\theta) , \tag{7.15}$$

we check that the latter is conserved (modulo the nonphysical breaking terms in S^0):

$$\partial^\mu\hat{D}_\mu = \left(ix^\nu\hat{w}^P_\nu - \frac{3}{2}(D^\alpha w_\alpha + \bar{D}_{\dot\alpha}\bar{w}^{\dot\alpha})\right)\Sigma - \left(D^2S^0 + \bar{D}^2\bar{S}^0\right) . \tag{7.16}$$

Its $\theta = 0$ component D_μ is nothing else than the dilatation current, conserved in the classical approximation[25]. Indeed, the space-time integration of the right-hand side of (7.16) yields, with (7.7) taken into account, the dilatation Ward identity:

$$\begin{aligned}&\int d^4x\ \left(ix^\nu\hat{w}^P_\nu - \frac{3}{2}(D^\alpha w_\alpha + \bar{D}_{\dot\alpha}\bar{w}^{\dot\alpha})\right)\Sigma - \left(\int dS\ S^0 + \int d\bar{S}\ \bar{S}^0\right) = iW^D\Sigma + O(\theta) = 0 ,\\ &\text{with} \quad W^D := -i\sum_\varphi\int\delta_D\varphi\frac{\delta}{\delta\varphi} , \quad \delta_D\varphi = \left(d_\varphi + x^\mu\partial_\mu + \frac{1}{2}\theta^\alpha\frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\bar{\theta}^{\dot\alpha}\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}}\right)\varphi ,\end{aligned} \tag{7.17}$$

[25] We recall that we neglect every breaking due to the masses.

d_φ being the dimension of the superfield φ.

Remarks.

1. The scale dimensions d_φ of the superfields contained in this theory are the canonical ones. They are given in Table 1 (in Subsection 4.5). Looking to (7.7), one sees that the S^0-term in the right-hand side of (7.14) does contribute to the dimensions of B and c_-. If it were absent, the wrong dimension 3 would have been obtained for these two fields.

2. The supertrace identities (7.4) also imply the "spinor trace identity"

$$\bar\sigma_\mu^{\dot\alpha\alpha}\hat Q_\alpha^\mu = -12i\bar w^{\dot\alpha}\Sigma - 8i\bar D^{\dot\alpha}\bar S^0 \ .$$

This allows to define (at $\theta = 09$)

$$S_{\mu\alpha} = ix_\nu\sigma^\nu_{\alpha\dot\alpha}\bar Q^{\dot\alpha}_\mu \ ,$$

the Noether current asssociated to conformal supersymmetry. This, together with the special conformal current

$$K_{\mu\nu} = \left(2x_\nu x^\lambda - \delta_\nu^\lambda x^2\right) T_{\mu\lambda} \ ,$$

completes the list of the Noether currents associated to the superconformal group [61, 60, 7].

7.2 Renormalization of the Supercurrent

Statement of the Result

One has to show that the supercurrent identities (7.4) are renormalizable [7, 58] in the sense that there exists a BRS-invariant quantum extension of the supercurrent (7.8) and a chiral insertion S of dimension 3 and R-weight -2, such that the identities

$$\begin{aligned} \bar D^{\dot\alpha}V_{\alpha\dot\alpha}\Gamma &= -2w_\alpha\Gamma - \frac{4}{3}D_\alpha(S+S^0)\cdot\Gamma \ , \\ D^\alpha V_{\alpha\dot\alpha}\Gamma &= -2\bar w_{\dot\alpha}\Gamma - \frac{4}{3}\bar D_{\dot\alpha}(\bar S+\bar S^0)\cdot\Gamma \ , \end{aligned} \tag{7.18}$$

hold to all orders. S^0 is now a quantum extension of (7.6), which will be defined later on in such a way that it remains a BRS variation – i.e., now, a variation under the linearized Slavnov-Taylor operator $\mathcal{S}_\Gamma$ – and that it still obeys the identities (7.7). The new chiral insertion $S\cdot\Gamma$ in (7.18) is an anomaly, which does not spoil the conservation of the currents $Q_{\mu\alpha}$ and $T_{\mu\nu}$ defined from V_μ by (7.9), but breaks[26] the conservation of $\hat R_\mu$ (c.f. (7.11)):

$$\partial_\mu\hat R^\mu\cdot\Gamma = i\ \hat w^R\Gamma\big|_{\theta=0} + \frac{2}{3}i\left(D^2S^0 - \bar D^2\bar S^0\right)\cdot\Gamma + \frac{2}{3}i\left(D^2S - \bar D^2\bar S\right)\cdot\Gamma \ , \tag{7.19}$$

and similarly gives an anomaly to the trace identity (7.14):

$$\hat T^\lambda_\lambda\cdot\Gamma = -\frac{3}{2}\left(D^\alpha w_\alpha + \bar D_{\dot\alpha}\bar w^{\dot\alpha}\right)\Gamma - \left(D^2S^0 + \bar D^2\bar S^0\right)\cdot\Gamma - \left(D^2S + \bar D^2\Sigma\right)\cdot\Gamma \ . \tag{7.20}$$

[26] Recall that the term in : S^0 : and conj. does not represent a breaking, since it is a total derivative (see the first of Eqs. (7.7)), and moreover is nonphysical, being a BRS varition.

The space-time integral of the sum of the first two terms in the right-hand-side being equal, at $\theta = 0$, to the action on Γ of the dilatation Ward identity operator (7.17), this yields, as in the classical limit, the dilatation Ward identity, but now broken by the scale anomaly S:

$$W_D \Gamma = -i \left(\int dS S + \int d\bar{S} \bar{S} \right) \cdot \Gamma \ . \tag{7.21}$$

The scale anomaly in the r.h.s. of (7.21) can be written in a suggestive way by expanding the dimension 3 chiral insertion S in an appropriate BRS invariant basis:

$$S = \beta_g L_g + \sum \beta_{ijk} L_{ijk} - \gamma_\phi L_\phi - \sum \gamma^i_j L^j_i - \sum_k \gamma_k L_k \ . \tag{7.22}$$

The basic elements L_I are defined (up to total derivatives) through the action principle by

$$\left(\int dS L_I + \int d\bar{S} \bar{L}_I \right) \cdot \Gamma = \nabla_I \Gamma \ , \tag{7.23}$$

whith the "symmetric operators" ∇_I defined by (6.32). Using now the dimension analysis identity

$$i W^D \Gamma + D\Gamma = 0 \ , \tag{7.24}$$

where D is the dilatation generator (6.35), we see that the broken Ward identity (7.21) is nothing else than the Callan-Symanzik equation (6.39).

Sketch of the Proof of the Renormalized Supertrace Identities

The quantum action principle yields

$$w_\alpha \Gamma = \Delta_\alpha \cdot \Gamma \ , \tag{7.25}$$

where Δ_α is an insertion of dimension 7/2. Due to the algebraic identities

$$\begin{aligned}
& w_\alpha \mathcal{S}(\gamma) - \mathcal{S}_\gamma w_\alpha \gamma = 0 \ , \quad \forall \gamma \ , \\
& [W^R, w_\alpha] = \left(-1 + \theta \frac{\partial}{\partial \theta} + \bar\theta \frac{\partial}{\partial \bar\theta} \right) w_\alpha \ , \\
& \left[\frac{\delta}{\delta B(1)}, w_\alpha(2) \right] = -2\delta_S(1,2) D_\alpha \frac{\delta}{\delta B(2)} \ , \quad \left[\frac{\delta}{\delta \bar{B}(1)}, w_\alpha(2) \right] = 0 \ , \\
& (\mathcal{G}_+(1), w_\alpha(2)) = \\
& \quad -2\delta_S(1,2) D_\alpha \frac{\delta}{\delta c_-(2)} + \frac{1}{4} \bar{D}^2 D^2 \delta_V(1,2) \bar{D}^2 D_\alpha \frac{\delta}{\delta \phi^*(2)} - \frac{1}{4} \bar{D}^2 D^2 \bar{D}^2 \delta_V(1,2) D_\alpha \frac{\delta}{\delta \phi^*(2)} \ , \\
& (\bar{\mathcal{G}}_+(1), w_\alpha(2)) = \frac{1}{4} D^2 \bar{D}^2 \delta_V(1,2) \bar{D}^2 D_\alpha \frac{\delta}{\delta \phi^*(2)} \ ,
\end{aligned} \tag{7.26}$$

the insertion Δ_α is submitted to the constraints

$$
\begin{aligned}
&\mathcal{S}_\Gamma \Delta_\alpha \cdot \Gamma = 0 \ , \\
&W^R \Delta_\alpha \cdot \Gamma = \left(-1 + \theta\frac{\partial}{\partial\theta} + \bar\theta\frac{\partial}{\partial\bar\theta}\right) \Delta_\alpha \cdot \Gamma \ , \\
&\frac{\delta}{\delta B(1)}\Delta_\alpha(2)\cdot\Gamma = \frac{1}{4}\left(\bar D^2 D^2 \delta_S(2,1) D_\alpha\phi(2) - D^2\delta_S(2,1)\bar D^2 D_\alpha\phi(2) - \delta_S(2,1) D_\alpha \bar D^2 D^2\phi(2)\right) \ , \\
&\frac{\delta}{\delta \bar B_(1)}\Delta_\alpha(2)\cdot\Gamma = -\frac{1}{4}\bar D^2\delta_{\bar S}(2,1)\bar D^2 D_\alpha\phi(2) \ , \\
&\mathcal{G}_+(1)\Delta_\alpha(2)\cdot\Gamma \\
&\quad = -\frac{1}{4}\left(\bar D^2 D^2\delta_S(2,1) D_\alpha\, s\phi(2) - D^2\delta_S(2,1)\bar D^2 D_\alpha\, s\phi(2) - \delta_S(2,1) D_\alpha \bar D^2 D^2\, s\phi(2)\right) \ , \\
&\bar{\mathcal{G}}_+(2)\Delta_\alpha(2)\cdot\Gamma = \frac{1}{4}\bar D^2\delta_{\bar S}(2,1)\bar D^2 D_\alpha\, s\phi(2) \ .
\end{aligned}
\tag{7.27}
$$

We have to solve these constraints. Let us first look for a special solution. Such a solution is given, at the classical level, by the expression[27]

$$
\Delta^0_\alpha = -\frac{1}{2}\mathcal{D}^{\dot\alpha} V^0_{\alpha\dot\alpha} - \frac{2}{3} D_\alpha S^0 \ , \tag{7.28}
$$

with

$$
\begin{aligned}
&\Delta^0_\alpha = s\,\hat\Delta_\alpha \ , \quad V^0_{\alpha\dot\alpha} = s\,\hat V_{\alpha\dot\alpha} \ , \quad S^0 = s\,\hat S \ , \\
&\quad \hat\Delta_\alpha = \frac{1}{4}\mathrm{Tr}\,\left(\bar D^2 D^2 c_- D_\alpha\phi - D^2 c_- \bar D^2 D_\alpha\phi - c_- D_\alpha \bar D^2 D^2\phi - \bar D^2 \bar c_- \bar D^2 D_\alpha\phi\right) \ , \\
&\quad \hat V_{\alpha\dot\alpha} = \frac{1}{3}\mathrm{Tr}\,\Big(D_\alpha \bar D_{\dot\alpha} D^2 c_-\phi + \bar D_{\dot\alpha} D^2 c_- D_\alpha\phi - \bar D_{\dot\alpha} D_\alpha c_- D^2\phi \\
&\qquad\qquad + D^2 c_-[D_\alpha, DB_{\dot\alpha}]\phi - D_\alpha c_- \bar D_{\dot\alpha} D^2\phi + c_- D_\alpha \bar D_{\dot\alpha} D^2\phi \Big) \ + \ \text{conj.} \ , \\
&\quad \hat S = \frac{1}{8}\mathrm{Tr}\,\left(\bar D^2(2c_- D^2\phi - D^2 c_-\phi)\right) + \text{conj.} \ .
\end{aligned}
\tag{7.29}
$$

(The expression for S^0 is the same as given in (7.6).) We notice that the "hat" quantities $\hat\Delta_\alpha$, etc., obey the same identity (7.28) as their BRS variations Δ^0_α, etc.

We shall define the quantum extensions of the expressions Δ^0_α, etc., as the quantum BRS variations – i.e. the variations under $\mathcal{S}_\Gamma$ – of the "Wick products" of the bilinear expressions $\hat\Delta_\alpha$, etc.:

$$
\Delta^0_\alpha = \mathcal{S}_\Gamma : \hat\Delta_\alpha : \ , \quad V^0_{\alpha\dot\alpha} = \mathcal{S}_\Gamma : \hat V_{\alpha\dot\alpha} : \ , \quad S^0 = \mathcal{S}_\Gamma : \hat S : \ . \tag{7.30}
$$

We define the Wick product of a bilinear expression AB at a superspace point (x,θ) as the insertion $: AB :$ obtained by subtracting off the infinite part of the Wilson expansion [62] of the bilocal T-product $T(A(x+\varepsilon,\theta)B(x-\varepsilon,\theta))$ at $\varepsilon^\mu = 0$:

$$
\text{Fin. part}\ \lim_{\varepsilon\to 0} T\left(A(x+\varepsilon,\theta)B(x-\varepsilon,\theta)\right) =\, : AB : (x,\theta) \ .
$$

Since these renormalized quantities obey the same equation as the classical ones, namely

$$
: \hat\Delta_\alpha : \cdot\Gamma = -\frac{1}{2}\mathcal{D}^{\dot\alpha} : \hat V_{\alpha\dot\alpha} : \cdot\Gamma - \frac{2}{3} D_\alpha : \hat S : \cdot\Gamma \ ,
$$

[27](7.28) being in fact obtained through applying the operator w_α (7.5) on the gauge fixing term (4.17) of the classical action, the fulfilment of the constraints (7.27) is obvious.

it immediately follows that the same holds for the renormalized Δ^0, etc. defined by (7.30). It is also evident that such a renormalization by "point splitting regularization" preserves all the symmetry properties of the corresponding classical expression. Δ^0 in particular obeys the constraints (7.27). V^0 and S^0 are explicitly $\mathcal{S}_\Gamma$-variations, the former being a real superfield and the latter a chiral superfield obeying the constraints (7.7).

We can thus write

$$\Delta_\alpha \cdot \Gamma = \Delta^0_\alpha \cdot \Gamma + \Delta'_\alpha \cdot \Gamma \ , \tag{7.31}$$

where Δ'_α obeys homogeneous constraints, namely the constraints (7.27) with the right-hand sides replaced by zero. The general solution for Δ'_α has the desired form:

$$\Delta'_\alpha \cdot \Gamma = -\frac{1}{2}\bar{\mathcal{D}}^{\dot\alpha} V'_{\alpha\dot\alpha} \cdot \Gamma - \frac{2}{3} D_\alpha S \cdot \Gamma \ , \quad \mathcal{S}_\Gamma V'_{\alpha\dot\alpha} \cdot \Gamma = 0 \ , \quad \mathcal{S}_\Gamma S \cdot \Gamma = 0 \ , \tag{7.32}$$

where $V'_{\alpha\dot\alpha}$ and S are $\mathcal{S}_\Gamma$-invariant, the former being a real superfield and the latter a chiral superfield[28]. This establishes the existence of a BRS invariant supercurrent

$$V_{\alpha\dot\alpha} \cdot \Gamma =: V^0_{\alpha\dot\alpha} : \cdot \Gamma + V'_{\alpha\dot\alpha} \cdot \Gamma \ , \qquad \mathcal{S}_\Gamma V_{\alpha\dot\alpha} \cdot \Gamma = 0 \ , \tag{7.33}$$

and of a BRS invariant supertrace anomaly S obeying the supertrace identities (7.18).

[28] The proof, which is rather lengthy, may be found in pages 255-259 of [7].

8 Finite Theories

Our aim is now to show that, in some circumstances, a supersymmetric gauge theory may be finite. "Finiteness" means here the vanishing of the Callan-Symanzik β-functions, the anomalous dimensions γ possibly remaining nonzero. In other words it means the scale invariance of the physical quantities (e.g. Green functions of BRS-invariant operators), since the anomalous dimensions do not touch them.

We shall only give a rough sketch of the construction, and advise the reader to consult the original literature [28, 30] for complete proofs. The starting point is the relation between the scale anomaly and the anomaly of the axial R-current, which follows from the supertrace identities (7.18). This anomaly is given essentially by the β-functions (see (7.22)), which get contributions from all orders of perturbation theory.

We want now to use the nonrenormalization theorem of the axial anomaly, which holds for the supercurrent constructed in the preceding section (see App. A of ref. [30]). In order to state this theorem we need to expand the supercurrent anomaly S in a basis different from the one used in (7.22). The new expansion reads

$$S \cdot \Gamma = \bar{D}^2 \left[r K^0 + J^{\rm inv} \right] \cdot \Gamma \ , \tag{8.1}$$

where $J^{\rm inv}$ is a BRS invariant real insertion and the insertion K^0 is alike a Chern-Simons form. The latter indeed is a quantum extension of the expression

$$e^{-\phi} D^\alpha e^\phi \bar{D}^2 \left(e^{-\phi} D_\alpha e^\phi \right) \ , \tag{8.2}$$

whose real part builds up the super-Yang-Mills action, and whose imaginary part is a supersymmetric generalization of the Bardeen current K^μ – defined by $\partial_\mu K^\mu = F_{\mu\nu} \tilde{F}^{\mu\nu}$. More precisely, K^0 belongs to a set of insertions K^q obeying "descent equations"

$$\begin{aligned}
&\mathcal{S}_\Gamma [K^0 \cdot \Gamma] = \bar{D}_{\dot\alpha} [K^{1\dot\alpha} \cdot \Gamma] \ , \\
&\mathcal{S}_\Gamma [K^{1\dot\alpha} \cdot \Gamma] = \left(\bar{D}^{\dot\alpha} D^\alpha + 2 D^\alpha \bar{D}^{\dot\alpha} \right) [K^2_\alpha . \Gamma] \ , \\
&\mathcal{S}_\Gamma [K^2_\alpha \cdot \Gamma] = D_\alpha [K^3 \cdot \Gamma] \ , \\
&\mathcal{S}_\Gamma [K^3 \cdot \Gamma] = 0, \qquad \bar{D}_{\dot\alpha} [K^3 \cdot \Gamma] = 0 \ ,
\end{aligned} \tag{8.3}$$

analogous to the usual ones (see [1], e.g.), where $\mathcal{S}_\Gamma$ is the linearized Slavnov-Taylor operator defined by (4.37). It follows from the first of Eqs. (8.3) that $\bar{D}^2 [K^0 \cdot \Gamma]$ is a BRS invariant insertion.

We don't give here the construction of the insertions K^q, and only state that the last one, K^3, is a quantum extension of

$$K^3_{\rm class} = \frac{1}{3} {\rm Tr}\, c_+^3 \ . \tag{8.4}$$

It can be shown[29] [57], that the antighost equation (6.25) forbids any counterterm for the insertion ${\rm Tr}\, c_+^3$. Hence the latter is UV-finite, and the quantum insertion K^3 is thus unambiguosly fixed. It follows then, by solving the descent equations up from the bottom, that K^0 is uniquely defined modulo a BRS invariant insertion – absorbed in $J^{\rm inv}$, in Eq. (8.1), and modulo a total derivative $\bar{D}$ – which we shall neglect since we are interested in $\bar{D}^2 K^0$. These remarks are at the basis of the following statement:

Supersymmetric Nonrenormalization Theorem. The coefficient r in (8.1) gets contributions only from one-loop graphs [30].

[29] In [28, 30], this result was obtained as a consequence of the nonrenormalization theorem of chiral insertions. But the latter holds only in the case of exact supersymmetry. The argument mentioned presently is more general [57].

The next step consists in comparing both expansions (7.22) and (8.1) for the supercurrent anomaly. But we need to modify slightly the former expansion. Choosing

$$L_i^j \Gamma = \left(A^j \frac{\delta}{\delta A^i} - A_i^* \frac{\delta}{\delta A_j^*} \right) \Gamma \ , \tag{8.5}$$

in accordance with (7.23) and (6.32), we split the set $\{L_i^j\}$ in two subsets

$$\{L_{0a} = e_{aj}^i L_i^j \ , \ a = 1,\ 2,\ \cdots\},\ \{\mathrm{L}_{1K} = f_{Kj}^i L_i^j \ , \ K = 1,\ 2,\ \cdots\} \tag{8.6}$$

defined as following. The associated counting operators

$$\mathcal{N}_a = e_{aj}^i \mathcal{N}_i^j \tag{8.7}$$

form a basis for the counting operators which annihilate the matter field self-interactions, i.e. the superpotential (4.24):

$$\mathcal{N}_a W(A) = 0 \ . \tag{8.8}$$

The latter conditions correspond in fact to the set of renormalizable chiral symmetry Ward identities

$$W_a \Gamma = e_{aj}^i \left(\int dS L_i^j - \int d\bar{S} \bar{L}_i^j \right) \Gamma = 0 \ , \tag{8.9}$$

which constrain these self-interactions. The L_{1K} complete the basis of the linear space spansed by the L_i^j. Moreover the L_{1K} form a basis for the insertion which are *genuinely chiral*, i.e. which are chiral but are not of the form $\bar{D}^2(\cdots)$[30].

With this, (7.22) becomes

$$S = \beta_g L_g + \sum_{ijk} \beta_{ijk} L^{ijk} - \gamma_\phi L_\phi - \sum_a \gamma_{0a} L_{0a} - \sum_K \gamma_{1K} L_{1K} - \sum_k \gamma_k L_k \ . \tag{8.10}$$

Each of the chiral insertions L_I also possesses an expansion of the type (8.1):

$$\begin{aligned} L_g &= \bar{D}^2 \left[\left(\frac{1}{128 g^3} + r_g \right) K^0 + J_g^{\mathrm{inv}} \right] + L_g^c \ , \\ L_A &= \bar{D}^2 \left[r_A K^0 + J_A^{\mathrm{inv}} \right] + L_A^c \ , \quad A = (ijk),\ \phi,\ 0a,\ 1K,\ k \ , \end{aligned} \tag{8.11}$$

where r_g and the r_A are of order $\hbar$ at least, and L_g^c, L_A^c are genuinely chiral insertions.

One notes that

$$r_\phi = 0 \ , \quad r_{1K} = 0 \ . \tag{8.12}$$

The first equality is due to the possibility of preserving to all orders [28] the property

$$L_\phi = \bar{D}^2 \mathcal{L}_\phi \ , \tag{8.13}$$

where $\mathcal{L}_\phi$ is BRS-invariant and real. The second one is obvious from the very definition of L_{1K} as a genuinely chiral insertion (which also implies $J_{1K}^{\mathrm{inv}} = 0$).

We now substitute S and the basis elements L_I in (8.10) by their expressions (8.1) and (8.11), respectively. Identifying the coefficient of the resulting $\bar{D}^2 K^0$ term yields the equation

$$r = \beta_g \left(\frac{1}{128 g^2} + r_g \right) + \beta_{ijk} r^{ijk} - \gamma_{0a} r_{0a} - \gamma_k r_k \ . \tag{8.14}$$

[30] This is obvious, in the classical limit, from the fact that the terms $\int dS\, L_{1K} W(A)$ are linearly independent and that $W(A)$ is the only genuinely chiral piece of the classical action.

The supersymmetric version of the nonrenormalization theorem for the axial anomalies also holds for the coefficients r_{0a} in (8.11) as it did for the coefficient r in (8.1): the r_{0a} are exactly given by one-loop graphs. They are indeed the coefficients of the axial anomalies which break the conservation of the currents associated to the chiral invariances[31] (8.9). From now on we shall restrict ourselves to the theories for which all axial anomalies vanish:

$$r = 0 \ , \quad r_{0a} = 0 \ . \tag{8.15}$$

This can be achieved by a suitable choice of the representation in which the matter fields live – this choice must of course also assure the vanishing of the gauge anomaly, which is also a 1-loop problem (see the first remark at the end of Subsection 6.3).

It can moreover be shown [30] that the second of Eqs. (8.15) implies

$$r_k = 0 \ . \tag{8.16}$$

Due to (8.12), (8.15) and (8.16), the equation (8.14) becomes homogeneous in the β-functions:

$$\beta_g \left(\frac{1}{128g^2} + r_g \right) + \beta_{ijk} r^{ijk} = 0 \ . \tag{8.17}$$

One sees that the first of Eqs. (8.15) corresponds to the vanishing of β_g at the one-loop order. Demanding the vanishing of the β_{ijk} at this order implies that the coupling constants λ_{ijk} have to be functions of the gauge coupling g (see [63]). We are therefore motivated to demand such a dependence to all orders. In order to be consistent this dependence must be given by functions $\lambda_{ijk}(g)$ which are solutions of the "reduction equations" [64]

$$\beta_{ijk} = \beta_g \frac{d}{dg} \lambda_{ijk} \ . \tag{8.18}$$

The perturbative existence of a solution to (8.18) is assured if the solution at lowest order is "isolated", i.e. does not belong to a continuous family of solutions [65]. At this stage the theory depends of the single coupling constants g.

The substitution of (8.18) into (8.17) yields

$$\beta_g \left(\frac{1}{128g^3} + r_g + r_{ijk} \frac{d}{dg} \lambda^{ijk}(g) \right) = \beta_g \left(\frac{1}{128g^3} + O(\hbar) \right) = 0 \ , \tag{8.19}$$

which implies the vanishing of the β-function:

$$\beta_g = 0 \ . \tag{8.20}$$

The resulting theory is thus "finite", the only "infinite" renormalizations being those of the field amplitudes, characterized by the anomalous dimensions, which may or may not vanish, but do not correspond to observables anyhow.

[31] Eq. (8.11) for $A = 0a$ represents the anomalous conservation laws for the axial Noether currents J^μ_a associated to the chiral invariances (8.9). More precisely, J^μ_a is the $\theta\sigma_\mu\bar\theta$ - component of the superfield $J^{\rm inv}_{0a}$, the anomalous conservation law is the θ^2 - component of (8.11) for $A = 0a$. The anomaly is contained in the term $\bar D^2(r_{0a}K^0)$.

APPENDICES

A Notations and Conventions

The notations and conventions are those of [7].

A.1 Weyl Spinors and Pauli Matrices

Units: $\hbar = c = 1$

Space-time metric: $(g_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$, $(\mu, \nu, \dots = 0,1,2,3)$

Fourier transform:

$$f(x) = \frac{1}{2\pi}\int dk\ e^{ikx}\tilde{f}(k) \ , \quad \tilde{f}(k) = \int dx\ e^{-ikx} f(k) \ , \qquad (\ \partial_\mu \leftrightarrow ik_\mu\ ) \ .$$

Weyl spinor: $(\psi_\alpha\ ,\ \alpha = 1,2)\ \in$ repr. $(\frac{1}{2}, 0)$ du groupe de Lorentz.
The spinor components are Grassmann variables: $\psi_\alpha \psi'_\beta = -\psi'_\beta \psi_\alpha$

Complex conjugate spinor: $(\bar{\psi}_{\dot\alpha} = (\psi_\alpha)^*,\ \dot\alpha = 1,2)\ \in$ repr. $(0, \frac{1}{2})$

raising and lowering of spinor indices:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \ , \quad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \ ,$$

$$\text{with } \varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha} \ , \quad \varepsilon^{12} = 1 \ , \quad \varepsilon_{\alpha\beta} = -\varepsilon^{a\beta} \ , \quad \varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma \ ,$$

(the same for dotted indices).

Derivative with respect to a spinor component:

$$\frac{\partial}{\partial\psi^\alpha}\psi^\beta = \delta^\beta_\alpha \ , \quad \frac{\partial}{\partial\psi_\alpha} = \varepsilon^{\alpha\beta}\frac{\partial}{\partial\psi^\beta} \ ,$$

(the same for dotted indices)

Pauli matrices:

$$\left(\sigma^\mu_{\alpha\dot\beta}\right) = \left(\sigma^0_{\alpha\dot\beta},\ \sigma^1_{\alpha\dot\beta},\ \sigma^2_{\alpha\dot\beta},\ \sigma^3_{\alpha\dot\beta}\right)$$

$$\bar\sigma^{\dot\alpha\beta}_\mu = \sigma^{\beta\dot\alpha}_\mu = \varepsilon^{\beta\alpha}\varepsilon^{\dot\alpha\dot\beta}\sigma_{\mu\,\alpha\dot\beta} \ ,$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{2}[\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu]_\alpha{}^\beta \ , \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \frac{i}{2}[\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu]^{\dot\alpha}{}_{\dot\beta} \ ,$$

with

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

$$\bar\sigma^0 = \sigma^0 \ , \quad \bar\sigma^i = -\sigma^i = \sigma_i \ , \sigma^{0i} = -\bar\sigma^{0i} = -i\sigma^i \ , \quad \sigma^{ij} = \bar\sigma^{ij} = \varepsilon^{ijk}\sigma^k \ ,$$

$$i, j, k = 1, 2, 3 \ .$$

Summation conventions and complex conjugation: Let ψ and χ be two Weyl spinors.

$$\psi\chi = \psi^\alpha\chi_\alpha = -\chi_\alpha\psi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \ ,$$
$$\bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\chi^{\dot\alpha}\bar\psi_{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi \ ,$$
$$\psi\sigma^\mu\bar\chi = \psi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} \ , \quad \bar\psi\bar\sigma_\mu\chi = \bar\psi_{\dot\alpha}\bar\sigma_\mu^{\dot\alpha\alpha}\chi_\alpha \ ,$$
$$(\psi\chi)^* = \bar\chi\bar\psi = \bar\psi\bar\chi \ ,$$
$$(\psi\sigma^\mu\bar\chi)^* = \chi\sigma^\mu\bar\psi = -\bar\psi\bar\sigma^\mu\chi \ ,$$
$$(\psi\sigma^{\mu\nu}\chi)^* = \bar\chi\bar\sigma^{\mu\nu}\bar\psi \ .$$

Infinitesimal Lorentz transformations of the Weyl spinors:

$$\text{if} \quad \delta^{\rm L} x^\mu = \omega^\mu{}_\nu x^\nu \ , \quad \text{with } \omega^{\mu\nu} = -\omega^{\nu\mu} \qquad (\omega^{\mu\nu} = g^{\nu\rho}\omega^\mu{}_\rho) \ , \quad \text{then:}$$

$$\delta^{\rm L}\psi_\alpha(x) = \frac{1}{2}\omega^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\psi_\alpha(x) - \frac{i}{2}(\sigma_{\mu\nu})_\alpha{}^\beta\psi_\beta \right) \ ,$$
$$\delta^{\rm L}\bar\psi^{\dot\alpha}(x) = \frac{1}{2}\omega^{\mu\nu}\left((x_\mu\partial_\nu - x_\nu\partial_\mu)\bar\psi^{\dot\alpha}(x) + \frac{i}{2}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar\psi^{\dot\beta} \right) \ .$$

A.2 Superfields

Superspace: may be defined as a "space" whose "points" are characterized by even and odd coordinates

$$\left\{ (x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha}), \ \mu = 0, \cdots 3, \ \alpha = 1, 2, \ \dot\alpha = \dot 1, \dot 2 \right\}$$

where the odd coordinates θ are anticommuting constant Weyl spinors.

Superfields: A (classical) superfield is a function in superspace $\phi(x, \theta, \bar\theta)$, transforming under infinitesimal translations P_μ and supersymmetry transformations Q_a, $\bar Q_{\dot\alpha}$ with the differential operators defined by:

$$\begin{aligned} \delta^P_\mu\phi &= \partial_\mu\phi \ , \\ \delta^Q_\alpha\phi &= \left(\frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \right)\phi, \\ \delta^{\bar Q}_{\dot\alpha}\phi &= \left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \right)\phi \ , \end{aligned} \tag{A.1}$$

obeying the algebra

$$\left\{ \delta^Q_\alpha, \delta^{\bar Q}_{\dot\alpha} \right\} = -2i\sigma^\mu_{\alpha\dot\alpha}\delta^P_\mu \ , \qquad \text{(the other (anti)commutators vanishing)} \ . \tag{A.2}$$

Due to the anticommutivity of the θ_α and $\bar\theta_{\dot\alpha}$, one can write a finite expansion for the superfield ϕ (we take it real, $\bar\phi = \phi$):

$$\begin{aligned}\phi(x,\theta,\bar\theta) = C(x) + \theta\chi(x) + \bar\theta\bar\chi(x) + \frac{1}{2}\theta^2 M(x) + \frac{1}{2}\bar\theta^2\bar M(x)\\ + \theta\sigma^\mu\bar\theta v_\mu(x) + \frac{1}{2}\bar\theta^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar\theta\bar\lambda + \tfrac{1}{4}\theta^2\bar\theta^2 D(x) \ ,\end{aligned} \tag{A.3}$$

where the components are ordinary space-time fields.

Covariant derivatives They are defined such as to anticommute with the supersymmetry transformation rules (A.1):

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \ , \quad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \ . \tag{A.4}$$

They obey the algebra

$$\begin{aligned}&\left\{D_\alpha, \bar D_{\dot\alpha}\right\} = 2i\sigma^\mu_{\alpha\dot\alpha}\delta^P_\mu \ ,\\ &\text{(the other anticommutators vanishing)} \ .\end{aligned} \tag{A.5}$$

Chiral superfields: A chiral superfield A is defined through the constraint

$$\bar D_{\dot\alpha} A(x,\theta,\bar\theta) = 0 \ ,$$

The complex conjugate constraint defines an antichiral superfield $\bar A$:

$$D_\alpha \bar A(x,\theta,\bar\theta) = 0 \ .$$

These constraints can be solved *algebraically*, with the help of the commutation rules (A.27). The result is the following, expanded in component fields:

$$\begin{aligned}A(x,\theta,\bar\theta) &= e^{-i\theta\sigma^\mu\bar\theta\partial_\mu} A_{\rm chiral}(x,\theta) = e^{-i\theta\sigma^\mu\bar\theta\partial_\mu}\left(A(x) + \theta\psi(x) + \theta^2 F(x)\right) \ ,\\ \bar A(x,\theta,\bar\theta) &= e^{i\theta\sigma^\mu\bar\theta\partial_\mu} \bar A_{\rm antichiral}(x,\bar\theta) = e^{i\theta\sigma^\mu\bar\theta\partial_\mu}\left(\bar A(x) + \bar\theta\bar\psi(x) + \bar\theta^2 \bar F(x)\right) \ .\end{aligned} \tag{A.6}$$

Note that the same symbol A is used for both the chiral superfield $A(x,\theta,\bar\theta)$ and its $\theta = 0$ component $A(x)$.

Chiral and antichiral representations: It is possible to perform changes of superspace coordinates in such a way that the covariant derivatives, either $\bar D$ or D, take a simple form. This leads to the two following representations for the superfields.

1. The chiral representation, defined by

$$\phi_{(\text{chir rep})}(x,\theta,\bar\theta) = \phi(x + i\theta\sigma\bar\theta, \theta, \bar\theta) \ . \tag{A.7}$$

In this representation, the transformation laws (A.1) and the covariant derivatives (A.4) take the form

$$\begin{aligned}\delta^Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} \ , & \delta^{\bar Q}_{\dot\alpha}\phi &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \ ,\\ D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \ , & \bar D_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \ .\end{aligned} \tag{A.8}$$

In the chiral representation, a chiral superfield is independent of $\bar\theta$: its form is given by the first of Eqs. (A.6) without the exponential factor.

2. The antichiral representation, defined by

$$\phi_{(\text{antichir rep})}(x,\theta,\bar\theta) = \phi(x - i\theta\sigma\bar\theta, \theta, \bar\theta) \ . \tag{A.9}$$

In this representation, the transformation laws (A.1) and the covariant derivatives (A.4) take the form

$$\begin{aligned} \delta^Q_\alpha \phi &= \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \ , \qquad & \delta^{\bar Q}_{\dot\alpha}\phi &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \ , \ , \\ D_\alpha &= \frac{\partial}{\partial\theta^\alpha} \ , & \bar D_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \ . \end{aligned} \tag{A.10}$$

In the antichiral representation, an antichiral superfield is independent of θ: its form is given by the second of Eqs. (A.6) without the exponential factor.

"Tensor calculus": Products of superfields are superfields.
Products of chiral superfields of the same chirality are chiral.
The double derivative $\bar D^2\phi$ of a superfield is a chiral superfield.

Superspace integration: The integral with respect of a Grassmann variable θ being defined [66] by the derivative $\partial/\partial\theta$, one defines the integral of a superfield ϕ, or of a (anti)chiral superfield A ($\bar A$) by

$$\int dV\ \phi = \int d^4x\, D^2\bar D^2\phi \ , \quad \int dS\ A = \int d^4x\, D^2 A \ , \quad \int d\bar S\ \bar A = \int d^4x\, \bar D^2 \bar A \ . \tag{A.11}$$

The usual formula for the integration by part holds – but only for the full superspace measure dV – since

$$\int dV\ D_\alpha\phi = 0 \ , \tag{A.12}$$

provided $\phi(x,\theta,\bar\theta)$ decreases sufficiently rapidly at infinity in x-space. It also follows from the latter equation that the integrals (A.11) are *invariant* under the supersymmetry transformations.

Superspace Dirac distributions: We use the notation:

$$F(1,2,\cdots) = F(x_1,\theta_1,\bar\theta_1,x_2,\theta_2,\bar\theta_2,\cdots) \ , \quad \theta_{12} = \theta_1 - \theta_2 \ .$$

The delta functions are defined by

$$\begin{aligned} &\int dV(2)\delta_V(1,2)\phi(2) = \phi(1) \\ &\int dS(2)\delta_S(1,2)A(2) = A(1) \\ &\int d\bar S(2)\delta_{\bar S}(1,2)\bar A(2) = \bar A(1) \ , \end{aligned} \tag{A.13}$$

and are expressed by

$$\begin{aligned} \delta_V(1,2) &= \frac{1}{16}\theta_{12}^2\bar\theta_{12}^2\,\delta^4(x_1 - x_2) \\ \delta_S(1,2) &= -e^{i(\theta_1\sigma\bar\theta_2 - \theta_2\sigma\bar\theta_1)\partial}\,\frac{1}{4}\theta_{12}^2\delta^4(x_1 - x_2) \\ \delta_{\bar S}(1,2) &= -e^{i(\theta_1\sigma\bar\theta_2 - \theta_2\sigma\bar\theta_1)\partial}\,\frac{1}{4}\bar\theta_{12}^2\delta^4(x_1 - x_2) \ . \end{aligned} \tag{A.14}$$

One has:

$$\begin{aligned}
&\delta_S(1,2) = \bar{D}^2\delta_V(1,2)\ , \quad \delta_{\bar{S}}(1,2) = D^2\delta_V(1,2)\ , \\
&\bar{D}^2\delta_{\bar{S}}(1,2) = e^{i(\theta_1\sigma\bar{\theta}_2 - \theta_2\sigma\bar{\theta}_1 - \theta_{12}\sigma\bar{\theta}_{12})\partial}\delta^4(x_1 - x_2)\ , \\
&D^2\delta_S(1,2) = e^{i(\theta_1\sigma\bar{\theta}_2 - \theta_2\sigma\bar{\theta}_1 + \theta_{12}\sigma\bar{\theta}_{12})\partial}\delta^4(x_1 - x_2)\ .
\end{aligned} \tag{A.15}$$

Functional differentiation:

$$\frac{\delta\phi(1)}{\delta\phi(2)} = \delta_V(1,2)\ , \quad \frac{\delta A(1)}{\delta A(2)} = \delta_S(1,2)\ , \quad \frac{\delta\bar{A}(1)}{\delta\bar{A}(2)} = \delta_{\bar{S}}(1,2)\ . \tag{A.16}$$

Supersymmetry transformations of the components: The components of a superfield, defined by the expansion (A.3), transform under supersymmetry as

$$\begin{array}{ll}
\delta_\alpha C = \chi & \bar{\delta}_{\dot\alpha} C = \bar{\chi} \\
\delta_\alpha \chi^\beta = \delta_\alpha^\beta M & \bar{\delta}_{\dot\alpha}\bar{\chi}^{\dot\beta} = -\delta_{\dot\alpha}^{\dot\beta}\bar{M} \\
\delta_\alpha \bar{\chi}_{\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}(v_\mu + i\partial_\mu C) & \bar{\delta}_{\dot\alpha}\chi_\alpha = -\sigma^\mu_{\alpha\dot\alpha}(v_\mu - i\partial_\mu C) \\
\delta_\alpha M = 0 & \bar{\delta}_{\dot\alpha}\bar{M} = 0 \\
\delta_\alpha \bar{M} = \lambda_\alpha - i(\sigma^\mu\partial_\mu\bar{\chi})_\alpha & \bar{\delta}_{\dot\alpha} M = \bar{\lambda}_{\dot\alpha} + i(\partial_\mu\chi\sigma^\mu)_{\dot\alpha} \\
\delta_\alpha v_\mu = \dfrac{1}{2}(\sigma_\mu\bar{\lambda})_\alpha - \frac{i}{2}(\sigma^\nu\bar{\sigma}_\mu\partial_\nu\chi)_\alpha & \bar{\delta}_{\dot\alpha} v_\mu = \dfrac{1}{2}(\lambda\sigma_\mu)_{\dot\alpha} + \frac{i}{2}(\partial_\nu\bar{\chi}\bar{\sigma}_\mu\sigma^\nu)_{\dot\alpha} \\
\delta_\alpha\lambda^\beta = \delta_\alpha^\beta D + i(\sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta\partial_\nu v_\mu & \bar{\delta}_{\dot\alpha}\bar{\lambda}^{\dot\beta} = -\delta_{\dot\alpha}^{\dot\beta} D + i(\bar{\sigma}^\mu\sigma^\nu)^{\dot\beta}{}_{\dot\alpha}\partial_\nu v_\mu \\
\delta_\alpha\bar{\lambda}_{\dot\alpha} = i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu M & \bar{\delta}_{\dot\alpha}\lambda_\alpha = i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar{M} \\
\delta_\alpha D = -i(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha & \bar{\delta}_{\dot\alpha} D = i(\partial_\mu\lambda\sigma^\mu)_{\dot\alpha}
\end{array} \tag{A.17}$$

For the components of the chiral and antichiral superfields (A.6), one has

$$\begin{array}{ll}
\delta_\alpha A = \psi_\alpha & \bar{\delta}_{\dot\alpha}\bar{A} = \bar{\psi}_{\dot\alpha} \\
\delta_\alpha\psi^\beta = 2\delta_\alpha^\beta F & \bar{\delta}_{\dot\alpha}\bar{\psi}^{\dot\beta} = -2\delta_{\dot\alpha}^{\dot\beta}\bar{F} \\
\delta_\alpha F = 0 & \bar{\delta}_{\dot\alpha}\bar{F} = 0 \\
\delta_\alpha\bar{A} = 0 & \bar{\delta}_{\dot\alpha} A = 0 \\
\delta_\alpha\bar{\psi}_{\dot\alpha} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar{A} & \bar{\delta}_{\dot\alpha}\psi_\alpha = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu A \\
\delta_\alpha\bar{F} = -i(\sigma^\mu\partial_\mu\bar{\psi})_\alpha & \bar{\delta}_{\dot\alpha} F = i(\partial_\mu\psi\sigma^\mu)_{\dot\alpha}
\end{array} \tag{A.18}$$

A.3 Some Useful Formula

Algebra of Pauli matrices:

$$\begin{aligned}
&\varepsilon_{\alpha\beta}\sigma^\mu_{\gamma\dot\gamma} + \varepsilon_{\beta\gamma}\sigma^\mu_{\alpha\dot\gamma} + \varepsilon_{\gamma\alpha}\sigma^\mu_{\beta\dot\gamma} = 0\ , \\
&\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + \varepsilon_{\beta\gamma}\varepsilon_{\alpha\delta} + \varepsilon_{\gamma\alpha}\varepsilon_{\beta\delta} = 0\ .
\end{aligned} \tag{A.19}$$

$$
\begin{aligned}
&\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\beta\dot{\beta}}_{\mu} = 2\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}} \ , && \sigma^{\mu}_{\alpha\dot{\alpha}}\sigma_{\mu\beta\dot{\beta}} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} \ , \\
&\sigma_{\mu\alpha\dot{\alpha}}\sigma^{\mu\nu}_{\beta\gamma} = i(\varepsilon_{\alpha\beta}\sigma^{\nu}_{\gamma\dot{\alpha}} + \varepsilon_{\alpha\gamma}\sigma^{\nu}_{\beta\dot{\alpha}}) \ , && \sigma_{\mu\alpha\dot{\alpha}}\bar{\sigma}^{\mu\nu}_{\dot{\beta}\dot{\gamma}} = -i(\varepsilon_{\dot{\alpha}\dot{\beta}}\sigma^{\nu}_{\alpha\dot{\gamma}} + \varepsilon_{\dot{\alpha}\dot{\gamma}}\sigma^{\nu}{}_{\alpha\dot{\beta}}) \ , \\
&\sigma^{\alpha\beta}_{\mu\nu}\sigma^{\mu\nu}_{\gamma\delta} = -4(\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}) \ , && \bar{\sigma}^{\dot{\alpha}\dot{\beta}}_{\mu\nu}\bar{\sigma}^{\mu\nu}_{\dot{\gamma}\dot{\delta}} = -4(\delta^{\dot{\alpha}}_{\dot{\gamma}}\delta^{\dot{\beta}}_{\dot{\delta}} + \delta^{\dot{\alpha}}_{\dot{\delta}}\delta^{\dot{\beta}}_{\dot{\gamma}}) \ , \\
&\bar{\sigma}^{\dot{\alpha}\dot{\beta}}_{\mu\nu}\sigma^{\mu\nu}_{\gamma\delta} = 0 \ ,
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
&\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = -i\sigma^{\mu\nu} \ , \quad \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} = i\bar{\sigma}^{\mu\nu} \ , \\
&\varepsilon_{\mu\nu\rho}{}^{\tau}\sigma_{\tau\lambda} = i\sigma_{\mu\nu}g_{\rho l} - i\sigma_{\mu\rho}g_{\nu l} + i\sigma_{\nu\rho}g_{\mu l} \ , \\
&\varepsilon_{\mu\nu\rho}{}^{\tau}\bar{\sigma}_{\tau\lambda} = -i\bar{\sigma}_{\mu\nu}g_{\rho l} + i\bar{\sigma}_{\mu\rho}g_{\nu l} - i\bar{\sigma}_{\nu\rho}g_{\mu l} \ , \\
&\quad \text{with:} \quad \varepsilon_{0123} = 1 = -\varepsilon^{0123} \ , \\
&\qquad \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\rho\sigma\lambda\tau} = -(\delta^{\mu}_{\lambda}\delta^{\nu}_{\tau} - \delta^{\nu}_{\lambda}\delta^{\mu}_{\tau}) \ .
\end{aligned}
\tag{A.21}
$$

$$
\sigma_{\mu}\bar{\sigma}_{\nu} = g_{\mu\nu} - i\sigma_{\mu\nu} \ , \quad \bar{\sigma}_{\mu}\sigma_{\nu} = g_{\mu\nu} - i\bar{\sigma}_{\mu\nu} \ . \tag{A.22}
$$

$$
\begin{aligned}
\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho} &= g^{\mu\nu}\sigma^{\rho} + g^{\nu\rho}\sigma^{\mu} - g^{\mu\rho}\sigma^{\nu} - i\varepsilon^{\mu\nu\rho\lambda}\sigma_{\lambda} \ , \\
\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho} &= g^{\mu\nu}\bar{\sigma}^{\rho} + g^{\nu\rho}\bar{\sigma}^{\mu} - g^{\mu\rho}\bar{\sigma}^{\nu} + i\varepsilon^{\mu\nu\rho\lambda}\bar{\sigma}_{\lambda} \ .
\end{aligned}
\tag{A.23}
$$

$$
\begin{aligned}
\sigma^{\mu\nu}\sigma^{\rho} &= i\sigma^{\mu}g^{\nu\rho} - i\sigma^{\nu}g^{\mu\rho} + \varepsilon^{\mu\nu\rho\lambda}\sigma_{\lambda} \ , \\
\sigma^{\rho}\bar{\sigma}^{\mu\nu} &= i\sigma^{\nu}g^{\rho\mu} - i\sigma^{\mu}g^{\rho\nu} + \varepsilon^{\mu\nu\rho\lambda}\sigma_{\lambda} \ , \\
\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho} &= i\bar{\sigma}^{\mu}g^{\nu\rho} - i\bar{\sigma}^{\nu}g^{\mu\rho} - \varepsilon^{\mu\nu\rho\lambda}\bar{\sigma}_{\lambda} \ , \\
\bar{\sigma}^{\rho}\sigma^{\mu\nu} &= i\bar{\sigma}^{\nu}g^{\rho\mu} - i\sigma^{\mu}g^{\rho\nu} - \varepsilon^{\mu\nu\rho\lambda}\bar{\sigma}_{\lambda} \ .
\end{aligned}
\tag{A.24}
$$

$$
\begin{aligned}
&\sigma_{\mu}\bar{\sigma}_{\nu}\sigma^{\mu} = -2\sigma_{\nu} \ , && \bar{\sigma}_{\mu}\sigma_{\nu}\bar{\sigma}^{\mu} = -2\bar{\sigma}_{\nu} \ , \\
&\sigma_{\mu}\bar{\sigma}_{\rho\lambda}\bar{\sigma}^{\mu} = \bar{\sigma}_{\mu}\sigma_{\rho\lambda}\sigma^{\mu} = 0 \ , && \sigma^{\mu\nu}\sigma_{\rho\lambda}\sigma_{\mu\nu} = -4\sigma_{\rho\lambda} \ , \\
&\sigma_{\mu}\bar{\sigma}^{\mu\nu} = 3i\sigma^{\nu} \ , && \bar{\sigma}_{\mu}\sigma^{\mu\nu} = 3i\bar{\sigma}^{\nu} \ , \\
&\bar{\sigma}^{\mu\nu}\bar{\sigma}_{\nu} = 3i\bar{\sigma}^{\mu} \ , && \sigma^{\mu\nu}\sigma_{\nu} = 3i\sigma^{\mu} \ , \\
&\sigma_{\mu\nu}\sigma^{\mu\nu} = 12 \ .
\end{aligned}
\tag{A.25}
$$

$$
\begin{aligned}
&\sigma^{\mu\nu}\sigma^{\rho\lambda} = g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho} + i\varepsilon^{\mu\nu\rho\lambda} + i\sigma^{\mu\lambda}g^{\nu\rho} - i\sigma^{\nu\lambda}g^{\mu\rho} + i\sigma^{\nu\rho}g^{\mu\lambda} - i\sigma^{\mu\rho}g^{\nu\lambda} \ , \\
&\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\lambda} = g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho} - i\varepsilon^{\mu\nu\rho\lambda} + i\bar{\sigma}^{\mu\lambda}g^{\nu\rho} - i\bar{\sigma}^{\nu\lambda}g^{\mu\rho} + i\bar{\sigma}^{\nu\rho}g^{\mu\lambda} - i\bar{\sigma}^{\mu\rho}g^{\nu\lambda} \ , \\
&\{\sigma^{\mu\nu}, \sigma^{\rho\lambda}\} = \mathrm{Tr}\,\sigma^{\mu\nu}\sigma^{\rho\lambda} = 2(g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho} + i\varepsilon^{\mu\nu\rho\lambda}) \ , \\
&\{\bar{\sigma}^{\mu\nu}, \bar{\sigma}^{\rho\lambda}\} = \mathrm{Tr}\,\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\lambda} = 2(g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho} - i\varepsilon^{\mu\nu\rho\lambda}) \ , \\
&[\sigma^{\mu\nu}, \sigma^{\rho\lambda}] = 2i(\sigma^{\mu\lambda}g^{\nu\rho} - \sigma^{\nu\lambda}g^{\mu\rho} + \sigma^{\nu\rho}g^{\mu\lambda} - \sigma^{\mu\rho}g^{\nu\lambda}) \ , \\
&[\bar{\sigma}^{\mu\nu}, \bar{\sigma}^{\rho\lambda}] = 2i(\bar{\sigma}^{\mu\lambda}g^{\nu\rho} - \bar{\sigma}^{\nu\lambda}g^{\mu\rho} + \bar{\sigma}^{\nu\rho}g^{\mu\lambda} - \bar{\sigma}^{\mu\rho}g^{\nu\lambda})
\end{aligned}
\tag{A.26}
$$

Algebra of covariant derivatives:

$$
\begin{aligned}
&D_\alpha \left(e^{i\theta\sigma^\mu\bar\theta\partial_\mu}\phi\right) = e^{i\theta\sigma^\mu\bar\theta\partial_\mu}\frac{\partial}{\partial\theta^\alpha}\phi \ , \\
&\bar D_{\dot\alpha} \left(e^{-i\theta\sigma^\mu\bar\theta\partial_\mu}\phi\right) = e^{-i\theta\sigma^\mu\bar\theta\partial_\mu}\left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\right)\phi \ .
\end{aligned}
\tag{A.27}
$$

$$
\begin{aligned}
&[D_\alpha, \bar D^2] = 4i(\sigma^\mu\bar D)_\alpha\partial_\mu \ , \quad [\bar D_{\dot\alpha}, D^2] = -4i(D\sigma^\mu)_{\dot\alpha}\partial_\mu \\
&[D^2, \bar D^2] = 8iD\sigma^\mu\bar D\partial_\mu + 16\partial^2 = -8i\bar D\bar\sigma^\mu D\partial_\mu - 16\partial^2 \\
&D\bar D^2 D = \bar D D^2 \bar D \\
&D\bar D_{\dot\alpha} D = -\frac{1}{2}\bar D_{\dot\alpha} D^2 - \frac{1}{2}D^2\bar D_{\dot\alpha} \ , \quad \bar D D_\alpha \bar D = -\frac{1}{2}D_\alpha\bar D^2 - \frac{1}{2}\bar D^2 D_\alpha
\end{aligned}
\tag{A.28}
$$

The following operators are projectors:

$$
\begin{aligned}
&P^{\mathrm{T}} = \frac{D\bar D^2 D}{8\partial^2} \ , \quad P^{\mathrm{L}} = -\frac{D^2\bar D^2 + \bar D^2 D^2}{16\partial^2} \\
&\left(P^{\mathrm{T}}\right)^2 = P^{\mathrm{T}} \ , \quad \left(P^{\mathrm{L}}\right)^2 = P^{\mathrm{L}} \ , \quad P^{\mathrm{T}}P^{\mathrm{L}} = 0 \ , \quad P^{\mathrm{T}} + P^{\mathrm{L}} = 1 \ .
\end{aligned}
\tag{A.29}
$$

Applied to the superspace Dirac distribution δ_V (A.14) they give

$$
\begin{aligned}
&P^{\mathrm{T}}\delta_V(1,2) = \frac{1}{8\partial^2}\left(1 + \frac{1}{4}\theta_{12}^2\bar\theta_{12}^2\partial^2\right) e^{i(\theta_1\sigma\bar\theta_2 - \theta_2\sigma\bar\theta_1)\partial}\delta^4(x_1 - x_2) \ , \\
&P^{\mathrm{L}}\delta_V(1,2) = \frac{1}{8\partial^2}\left(-1 + \frac{1}{4}\theta_{12}^2\bar\theta_{12}^2\partial^2\right) e^{i(\theta_1\sigma\bar\theta_2 - \theta_2\sigma\bar\theta_1)\partial}\delta^4(x_1 - x_2) \ .
\end{aligned}
\tag{A.30}
$$

B Generating Functionals

The content of this appendix is taken from ref. [7]. Let us consider a theory involving a set of fields $\phi_i(x)$ in D–dimensional space–time[32], with the index i denoting the species as well as the spin and internal degrees of freedom. The (classical) dynamics is defined by the action

$$S(\phi) = \int dx \mathcal{L}(x) = S_0(\phi) + S_{\text{int}}(\phi) \ . \tag{B.1}$$

The Lagrangian has the general form

$$\mathcal{L}(x) = \frac{1}{2}\phi_i(x) K^{ij}(\partial)\phi_j(x) + \mathcal{L}_{\text{int}} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \ . \tag{B.2}$$

$K^{ij}(\partial)$ is some invertible differential operator, usually a polynomial of second order in ∂ for the bosonic fields and of first order for the fermionic ones: the quadratic piece $\mathcal{L}_0$ of the Lagrangian corresponds to the free theory whereas $\mathcal{L}_{\text{int}}$ describes the interactions.

B.1 The Green Functional

The objects of the corresponding quantum theory one wants to compute are the Green functions, i.e., the vacuum expectation values of the time–ordered products of field operators:

$$G_{i_1\cdots i_N}(x_1, \cdots, x_N) = \langle T\phi_{i_1}(x_1)\cdots\phi_{i_N}(x_N)\rangle \ . \tag{B.3}$$

These Green functions may be collected together in the generating functional $Z(J)$, a formal power series in the "classical sources" $J^i(x)$[33]:

$$Z(J) = \sum_{N=0}^{\infty} \frac{(-1/\hbar)^N}{N!} \int dx_1 \cdots dx_N \, J^{i_1}(x_1)\cdots J^{i_N}(x_N) \, G_{i_1\cdots i_N}(x_1, \cdots, x_N) \ . \tag{B.4}$$

The Green functions are tempered distributions. The sources $J^i(x)$ thus belong to the set of Schwartz fast decreasing C^∞ functions ("test functions").

The Green functional (B.4) is *formally* given by the Feynman path integral

$$Z(J) = \mathcal{N} \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{1}{\hbar}\left(S(\phi) + \int dx J^i(x)\phi_i(x)\right)} \ , \tag{B.5}$$

where $\mathcal{N}$ is some (generally ill–defined) numerical factor. The solution for the free theory ($\mathcal{L}_{\text{int}} = 0$ in (B.2)) is given by

$$Z_{\text{free}}(J) = \mathrm{e}^{\frac{1}{2\hbar^2}\int dx_1 dx_2 J^{i_1}(x_1) J^{i_2}(x_2) \Delta_{i_1 i_2}(x_1, x_2)} \ , \tag{B.6}$$

where $\Delta_{i_1 i_2}(x_1, x_2)$ is the Stueckelberg–Feynman free causal propagator, obtained by inverting[34] the wave operator $K^{ij}(\partial)$ of (B.2):

$$K^{ij}\Delta_{jk}(x, y) = \hbar\delta^i_k \delta^D(x - y) \ . \tag{B.7}$$

In the case of the full interacting theory a formal solution is given by [67]

$$Z(J) = \mathcal{N} \mathrm{e}^{-\frac{1}{\hbar}S_{\text{int}}\left(-\hbar\frac{\delta}{\delta J}\right)} Z_{\text{free}}(J) \ . \tag{B.8}$$

This expression leads to the well–known perturbative expansion of the Green functions in terms of Feynman graphs.

[32] Space–time point coordinates are denoted by (x^μ, $\mu = 0, \cdots, D-1$).
[33] The Planck constant $\hbar$ is set equal to 1 in the main text.
[34] Signs in the following equations correspond to the case where the fields ϕ_i are all bosonic. The reader may generalize to the case where fermionic fields are also present.

B.2 The Connected and the Vertex Functionals

Let us introduce two more functionals. The total contribution of the *connected* graphs to a Green function is called a truncated or a connected Green function. The generating functional of the connected Green functions

$$\begin{aligned} &Z^{\mathrm{c}}(J) = \\ &\sum_{N=1}^{\infty} \frac{(-1/\hbar)^{N-1}}{N!} \int dx_1 \cdots dx_N \, J^{i_1}(x_1) \cdots J^{i_N}(x_N) \, \langle T\phi_{i_1}(x_1) \cdots \phi_{i_N}(x_N) \rangle_{\mathrm{conn}} \end{aligned} \tag{B.9}$$

is related to the Green functional $Z(J)$ by

$$Z(J) = \mathrm{e}^{-\frac{1}{\hbar} Z^{\mathrm{c}}(J)} \; . \tag{B.10}$$

A *one–particle irreducible* (1PI) graph is a connected graph, amputated from its external legs, which remains connected after cutting any internal line. The total contribution of these graphs to an (amputated) connected Green function is called a 1PI or vertex function. The generating functional of the vertex functions reads

$$\begin{aligned} &\Gamma(\phi^{\mathrm{class}}) = \sum_{N=2}^{\infty} \frac{1}{N!} \int dx_1 \cdots dx_N \, \phi^{\mathrm{class}}_{i_1}(x_1) \cdots \phi^{\mathrm{class}}_{i_N}(x_N) \, \Gamma^{i_1 \cdots i_N}(x_1, \cdots, x_N) \\ &\Gamma^{i_1 \cdots i_N}(x_1, \cdots x_N) = \langle T\phi_{i_1}(x_1) \cdots \phi_{i_N}(x_N) \rangle_{\mathrm{1PI}} \; , \end{aligned} \tag{B.11}$$

where the arguments ϕ^{class}, the "classical fields", are Schwartz fast decreasing test functions. Later on we shall suppress the superscript "class", no confusion between the classical field and the corresponding quantum field being expected. The vertex functional is related to the connected functional by a Legendre transformation:

$$\Gamma(\phi) = Z^{\mathrm{c}}(J) - \int dx J^i(x)\phi_i(x) \Big|_{\phi_i(x) = \frac{\delta Z^{\mathrm{c}}}{\delta J^i(x)}} \; . \tag{B.12}$$

In the right–hand side, $J(x)$ is replaced by the solution $J(\phi)(x)$ of the equation

$$\phi_i(x) = \delta Z^{\mathrm{c}} / \delta J^i(x) \; .$$

The inverse Legendre transformation is given by

$$Z^{\mathrm{c}}(J) = \Gamma(\phi) + \int dx J^i(x)\phi_i(x) \Big|_{J^i(x) = -\frac{\delta \Gamma}{\delta \phi_i(x)}} \; . \tag{B.13}$$

We have assumed that the vacuum expectation values of the field variables are zero. One has thus

$$\left. \frac{\delta Z^{\mathrm{c}}}{\delta J^i(x)} \right|_{J=0} = 0 \; , \qquad \left. \frac{\delta \Gamma}{\delta \phi_i(x)} \right|_{\phi=0} = 0 \; . \tag{B.14}$$

Remark. For the two–points functions the Legendre transformation yields:

$$\int dy \Gamma^{ij}(x, y) \, \langle T\phi_j(y)\phi_k(z) \rangle_{\mathrm{conn}} = -\delta^i_k \delta^D(x - y) \; , \tag{B.15}$$

which is the all order generalization of (B.7).

B.3 Expansion in $\hbar$

From its definition and the formula (B.8) for the perturbative expansion of the Green functional one can easily check that the vertex functional can be written as a formal power series in $\hbar$:

$$\Gamma(\phi) = \sum_{n=0}^{\infty} \hbar^n \Gamma^{(n)}(\phi) \ , \tag{B.16}$$

the order n corresponding to the contributions of the n–loop graphs. In order to prove this statement, let us consider the contribution of a 1PI diagram consisting of I internal lines, V vertices and L loops. Counting a factor $\hbar$ for each internal line, a factor $\hbar^{-1}$ for each vertex and an overall factor $\hbar$ due to the factor $\hbar^{-1}$ in (B.10), we find the value $I - V + 1$ for the total power in $\hbar$. The result then follows from the topological identity

$$L = I - V + 1 \ , \tag{B.17}$$

due to Euler. The zeroth order

$$\Gamma^{(0)}(\phi) = S(\phi) \tag{B.18}$$

is the classical action (B.1). This is obvious since the only 1PI zero-loop graphs – the 1PI tree graphs – are the trivial ones, i.e., those containing a single vertex, and this vertex corresponds to a term of the interaction Lagrangian. In this approximation the Legendre transform $Z^{c(0)}(J)$ of $\Gamma^{(0)}(\phi)$ generates the connected Green functions, given by the connected tree Feynman graphs. $\Gamma^{(n)}$ corresponds to the contributions of the n–loop graphs.

B.4 Composite Fields

We are also interested in Green functions involving composite field operators. Such operators appear in particular in theories invariant under field transformations which depend nonlinearly on the fields – e.g. the BRS transformations in (super) Yang-Mills theories. Let us thus consider field operators $Q^p(x)$, corresponding to local field polynomials $Q^p_{\mathrm{class}}(x)$ in the classical theory. If one performs again the construction above, but starting with the new classical interaction

$$S_{\mathrm{int}}(\phi, \rho) = S_{\mathrm{int}}(\phi) + \int dx \rho_p(x) Q^p_{\mathrm{class}}(x) \tag{B.19}$$

depending on the "external fields" $\rho_p(x)$, one obtains a new Green functional [44]

$$\begin{aligned} Z(J,\rho) = \sum_{N=0}^{\infty}\sum_{M=0}^{\infty} \frac{(-1/\hbar)^N}{N!} \quad & \frac{(-1/\hbar)^M}{M!} \int dx_1 \cdots dx_N \int dy_1 \cdots dy_M \\ & J^{i_1}(x_1) \cdots J^{i_N}(x_N)\, \rho_{p_1}(y_1) \cdots \rho_{p_M}(y_M) \\ & \langle T \phi_{i_1}(x_1) \cdots \phi_{i_N}(x_N)\, Q^{p_1}(y_1) \cdots Q^{p_M}(y_M) \rangle \ , \end{aligned} \tag{B.20}$$

which generates the Green functions with insertions of the local composite quantum fields $Q^p(x)$.

The connected functional $Z^c(J,\rho)$ and the vertex functional $\Gamma(\phi,\rho)$ involving these composite fields are related to $Z(J,\rho)$ via the generalizations of (B.10), (B.12) and (B.13):

$$Z(J,\rho) = \mathrm{e}^{-\frac{1}{\hbar} Z^c(J,\rho)} \ , \tag{B.21}$$

$$\Gamma(\phi,\rho) = Z^c(J,\rho) - \int dx J^i(x)\phi_i(x) \Big|_{\phi_i(x) = \frac{\delta Z^c}{\delta J^i(x)}} \ , \tag{B.22}$$

and

$$Z^c(J,\rho) = \Gamma(\phi,\rho) + \int dx J^i(x)\phi_i(x)\bigg|_{J^i(x)=-\frac{\delta\Gamma}{\delta\phi_i(x)}} \; . \tag{B.23}$$

In particular

$$-\,\hbar\,\frac{\delta Z}{\delta\rho_p(y)}\bigg|_{\rho=0} := Q^p(y)\cdot Z(J)\;, \tag{B.24}$$

generates the Green functions

$$\langle T Q_p(y)\,\phi_{i_1}(x_1)\cdots\phi_{i_N}(x_N)\rangle\;\;, \tag{B.25}$$

whose Feynman graphs contain a new vertex corresponding to the insertion of the field polynomial $Q^p_{\rm class}(y)$ (with possible quantum corrections). In the same way

$$\frac{\delta Z^c}{\delta\rho_p(y)}\bigg|_{\rho=0} = Q^p(y)\cdot Z^c(J)\;, \tag{B.26}$$

and

$$\frac{\delta\Gamma}{\delta\rho_p(y)}\bigg|_{\rho=0} = Q^p(y)\cdot\Gamma(\phi)\;, \tag{B.27}$$

generate connected and 1PI Green functions, respectively, with the insertion of the operator $Q^p(y)$. The zeroth order term of the loop ($\hbar$) expansion of (B.27) coincides with the classical field polynomial which is the starting point of the perturbative construction of the quantum insertion Q^p:

$$Q^p(y)\cdot\Gamma(\phi) = Q^p_{\rm class}(y) + O(\hbar)\;. \tag{B.28}$$

REFERENCES

[1] O. Piguet and S.P. Sorella, "Algebraic Renormalization", *Lecture Notes in Physics, vol m28 (Springer Verlag, Berlin, Heidelberg, 1995)*;

[2] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, "Superspace: or One Thousand and One Lessons in Supersymmetry", *(Benjamin/Cummings, London, 1983)*;

[3] J. Wess and J. Bagger, "Supersymmetry and Supergravity", (Princeton University Press, 1983);

[4] P. West, "Introduction to Supersymmetry and Supergravity", *(World Scintific, Singapore, 1987)*;

[5] I.L. Buchbinder and S.M. Kuzenko, "Ideas and Methods of Supersymmetry and Supergravity", *(Institute of Physics Publishing, Bristol and Philadelphia, 1995)*;

[6] M. Jacob, "Supersymmetry and Supergravity", *(North Holland, Amsterdam, 1987*;

[7] O. Piguet and K. Sibold, "Renormalized Supersymmetry", series *"Progress in Physics", vol. 12 (Birkhäuser Boston Inc., 1986)*;

[8] R. Slansky, *Phys. Rep.* 79 (1981) 1;
H. Georgi, "Lie Algebras in Particle Physics: From Isospin to Unified Theories", *(Benjamin/Cummings, 1982)*;

[9] S. Coleman and J. Mandula, *Phys. Rev.* 159 (1967) 1251;

[10] R. Haag, "Local Quantum Physics", *Springer Verlag, Berlin, 1992*;

[11] R. Haag, J. Łopuszański and M. Sohnius, *Phys. Lett.* B88 (1975) 257;

[12] M.B. Green, J.H. Schwartz and E. Witten, "Superstring Theory" *(2 vol., Cambridge University Press, 1987)*;

[13] S. Raby, "SUSY and the Gauge Hierarchy Problem", in *"Supersymmetry and Supergravity", p. 304 ed. S. Ferrara, J.G. Taylor and P. van Nieuwenhuizen (World Scientific, Singapore, 1983)*;

[14] L. Girardello and M.T. Grisaru, *Nucl. Phys.* B194 (1982) 65;

[15] G.R. Ross, "Grand Unified Theories", *(Benjamin/Cummings, 1984)*;

[16] J.F. Donoghue, E. Golowich and B.R. Holstein, "Dynamics of the Standard Model", *(Cambridge Univ. Press, 1994)*;

[17] W.Lucha and H. Neufeld, *Phys. Rev.* D34 (1986) 1089;
W. Lucha, *Phys. Lett.* B191 (1987) 404;

[18] M.Böhm and A. Denner, *Nucl. Phys.* B282 (1987) 206;

[19] S. Mandelstam, *Nucl. Phys.* B213 (1983) 149;
L. Brink, O. Lindgren and B. Nilsson, *Nucl. Phys.* B212 (1983) 401;

[20] P.S. Howe, K.S. Stelle and P.C. West, *Phys. Lett.* B124 (1983) 55;

[21] P.S. Howe, K.S. Stelle and P.K. Townsend, *Nucl. Phys.* B236 (1984) 125;

[22] A.J. Parkes and P.C. West, *Phys. Lett.* B138 (1984) 99;
Nucl. Phys. B256 (1985) 340;
D.R.T. Jones and A.J. Parkes, *Phys. Lett.* B160 (1985) 267;
A.J. Parkes, *Phys. Lett.* B156 (1985) 73;

[23] S. Hamidi J. Patera and J.H. Schwartz, *Phys. Lett.* B141 (1984) 349;

[24] S. Rajpoot and J.G. Taylor, *Int. J. Theor. Phys.* 25 (1986) 117;

[25] A.V. Ermushev, D.I. Kazakov and O.V. Tarasov, *Nucl. Phys.* B281 (1987) 72;
D.I. Kazakov, *Mod. Phys. Lett.* A9 (1987) 663;
D.R.T. Jones, *Nucl. Phys.* B277 (1986) 153;
D.I. Kazakov, M.Yu. Kalmykov, I.N. Kondrashuk and A.V. Gladyshev, JINR E2-95-482, Dubna 1995, hep-ph/9511419;

[26] X.D. Jiang and X.J. Zhou, *Commun. Theor. Phys.* 5 (1986) 179;
F.X. Dong, X.D. Jiang and X.J. Zhou, *J. Phys.* A19 (1986) 3863;
X.D. Jiang and X.J. Zhou, *Nucl. Phys.* B281 (1987) 72;

[27] O. Piguet and K. Sibold, *Phys. Lett.* B177 (1986) 373;

[28] O. Piguet and K. Sibold, *Int. J. Mod. Phys.* A1 (1986) 913;

[29] C. Lucchesi, O. Piguet and K. Sibold, *Phys. Lett.* B201 (1988) 241;

[30] C. Lucchesi, O. Piguet and K. Sibold, *Helv. Phys. Acta* 61 (1988) 321;

[31] O. Piguet and K. Sibold, *Helv. Phys. Acta* 63 (1990) 71;

[32] O. Piguet, "Supersymmetry, Ultraviolet Finiteness and Grand Unification", Talk given at the International conference "Problems of Quantum Field Theory", Alushta (Crimea, Ukraine, May 1996), hep-th/9606045;

[33] D.J. Gross and F. Wilczek, *Phys. Rev.* D8 (1973) 3633;

[34] J. Kubo, M. Mondragon, M. Olechowski and G. Zoupanos, MPI-PH-95-133, Dec. 1995, hep-ph/9512435;

[35] C. Lucchesi and G. Zoupanos, "All-order Finiteness in N=1 SYM Theories: Criteria and Applications", *Proceedings of the Workshop on "Recent Developments in theoretical Physics"*, CERN, Geneva, Dec. 1995, hep-ph/9604216;

[36] J. Wess and B. Zumino, *Phys. Lett.* B49 (1974) 52;

[37] S. Ferrara and B. Zumino, *Nucl. Phys.* B79 (1979) 413;
A. Salam and J. Strathdee, *Phys. Lett.* B51 (1974) 353;

[38] J. Wess and B. Zumino, *Nucl. Phys.* B78 (1974) 1;
B. de Wit and D.Z. Freedman *Phys. Rev.* D12 (1975) 2286;

[39] P. Breitenlohner and D. Maison, "Renormalization of supersymmetric Yang–Mills theories", *in Cambridge 1985, Proceedings: "Supersymmetry and its applications"*, p.309;
and "N=2 Supersymmetric Yang–Mills theories in the Wess–Zumino gauge", *in "Renormalization of quantum field theories with nonlinear field transformations", ed.: P. Breitenlohner, D. Maison and K. Sibold, Lecture Notes in Physics, Vol.* 303, p.64 (*Springer Verlag, Berlin, Heidelberg*, 1988);

[40] P.L. White, *Class. Quantum Grav.* 9 (1992) 1663; *Class. Quantum Grav.* 9 (1992) 413;
N. Maggiore, *Int. J. Mod. Phys.* A10 (1995) 3937,3781; N. Maggiore, O. Piguet and S. Wolf, *Nucl. Phys.* B458 (1996) 403;

[41] O. Piguet and K. Sibold, *Nucl. Phys.* B253 (1985) 517;
O. Piguet and K. Sibold, *Nucl. Phys.* B248 (1984) 301;

[42] H.P. Nilles, *Phys. Rep.* 110 (1984) 1;

[43] P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, *Nucl. Phys.* 258 (1985) 46;
S. Ferrara, C. Kounnas, M. Porrati and F. Zwirner, *Nucl. Phys.* B318 (1989) 75;
S. Ferrara, C. Kounnas and M. Porrati, *Phys. Lett.* B206 (1988) 25;
J.P. Derendinger, in "Electroweak Interactions and Unified Theories", 401, proceedings of the conference at "Les Arcs", 1991;

[44] K. Symanzik, "Renormalization of theories with broken symmetry", in *"Cargèse Lectures in Physic 1970", p.179, ed. D. Bessis (Gordon & Breach, NY 1972)*;

[45] O. Piguet and K. Sibold, *Nucl. Phys.* B247 (1984) 484;

[46] O. Piguet and K. Sibold, *Nucl. Phys.* B248 (1984) 336;
Nucl. Phys. B249 (1984) 396;

[47] Y.M.P. Lam, *Phys. Rev.* D6 (1972) 2145; *Phys. Rev.* D7 (1973) 2943;
T.E. Clark and J.H. Lowenstein, *Nucl. Phys.* B113 (1976) 109;

[48] O. Piguet, M. Schweda and K. Sibold, *Nucl. Phys.* 174 (1980) 183;

[49] J. Dixon, *Commun. Math. Phys.* 140 (1991) 169;

[50] F. Brandt, *Nucl. Phys.* B392 (1993) 428;

[51] S. Ferrara, L. Girardello, O. Piguet and R. Stora, *Phys. Lett.* B157 (1985) 179;

[52] G. Girardi, R. Grimm and R. Stora, *Phys. Lett.* B156 (1985) 203;
L. Bonora, P. Pasti and M. Tonin, *Phys. Lett.* B156 (1985) 341;, *Nucl. Phys.* B261 (1985) 249;

[53] A. Blasi, O. Piguet and S.P. Sorella, *Nucl. Phys.* B356 (1991) 154;

[54] A. Brandhuber, M. Langer, O. Piguet, M. Schweda and S.P. Sorella,
Phys. Lett. B300 (1993) 92;
A. Brandhuber, S. Emery, M. Langer, O. Piguet, M. Schweda and S.P. Sorella,
Helv. Phys. Acta 66 (1993) 551;
O. Piguet and S. Emery, *Helv. Phys. Acta* 67 (1994) 22;

[55] O. Piguet, C. Lucchesi and S.P. Sorella, *Nucl. Phys.* B395 (1993) 325;

[56] O. Piguet and S.P. Sorella, *Nucl. Phys.* B395 (1993) 661;

[57] O. Piguet and S.P. Sorella, *Phys. Lett.* B371 (1996) 238;

[58] O. Piguet and K. Sibold, *Nucl. Phys.* B196 (1982) 428,447;

[59] P. Fayet and S. Ferrara, *Phys. Rep.* 32 (1977) 250;

[60] S. Ferrara and B. Zumino, *Nucl. Phys.* B87 (1975) 207;

[61] J. Wess and B. Zumino, *Nucl. Phys.* B70 (191974) 39;

[62] K.G. Wilson and W. Zimmermann, *Commun. Math. Phys.* 24 (1972) 87;

[63] A.J. Parkes and P.C. West, *Phys. Lett.* B138 (1984) 99;
D.R.T. Jones and L. Mezincescu, *Phys. Lett.* B138 (1984) 293;
D.R.T. Jones and A.J. Parkers, *Phys. Lett.* B160 (1985) 267;

[64] R. Oehme, K. Sibold and W. Zimmerman, *Phys. Lett.* B147 (1984) 115;

[65] R. Oehme, *Progr. Theor. Phys. Suppl.* 86 (1986) 215;

[66] F.A. Berezin, "The Method of Second Quantization", *Acad. Press, New York* (1966);

[67] P. Ramond, "Field Theory: a Modern Primer", *Frontiers in Physics, Benjamin/Cummings Publ. Comp.* (1981);