ISSN 0029-3865

CBPF-NF-072/88

INHOMOGENEOUS TWO-FLUID COSMOLOGIES WITH ELECTROMAGNETIC FIELD

by

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ABSTRACT

A new class of exact expanding inhomogeneous solutions of the Einstein-Maxwell equations is derived. These solutions generalize the dust filled models found by Ruban and the Doroshkevich "magnetic" universes. In the most general case the cosmological constant is non-zero and the matter content is a mixture of two interacting perfect fluids plus a sourceless electromagnetic field. The influence of the field near the singularity and at the latter stages of the expansion is examined. A subclass of the models approaches homogeneity and isotropy for large cosmological times.

Key words: Cosmology; Magnetic Fields; Inhomogeneous Models; General Relativity

1. INTRODUCTION

The assumption of a primeval magnetic field has some interesting consequences in astrophysical and cosmological problems. In principle, such a field could play an important role on the structure of formation process, in the origin of the galactic and intergalactic magnetic field, as well as to alter significantly the underlying geometric structure of the universe, at least in the early stages of the cosmic evolution[1-6].

Dynamical effects produced by magnetic fields were investigated by several authors, firstly in the framework of homogeneous axially-symmetric models[2-10]. Doroshkevich[2] derived a class of exact solutions and concluded that the magnetic field can exert a strong influence along the expansion for any matter equation of state. Generalizations of the Doroshkevich solutions are available in the literature[7-10]. An useful compendium of homogeneous axially--symmetric solutions with a simple fluid and magnetic field is given in the paper by Vajk and Eltgroth[7].

In the last decade, after the class of dust filled universes found by Szekeres[11], an increasing attention has been paid to inhomogeneous cosmological models[12-18]. Recently, the Szekeres-spacetime has been extended by introducing a new radiative component plus an electromagnetic field[19]. In this paper, Tomimura and Waga (hereafter referred to as TW) have shown that if an electromagnetic field is included as source term for the Szekeres metric of class II (for this notation see ref. (13)),self-consistent solutions are possible only if the spacetime gains symmetry. In this case, the Szekeres metric reduces to the inhomogeneous simple form first considered by Ruban[20]. In the present article we take the next step in the direction of determining exact solutions in the framework of Ruban's line element. We assume that the source of the gravitational field, in the most general case, is formed by a mixture of two interacting simple fluids plus an electromagnetic field. The work of TW has been extended to include several kinds of two fluids and a cosmological constant. This paper is organized as follows: in the next section, the basic equations are deduced and an algorithm, enclosing both the one- and two-fluid description, is given for obtaining new solutions and recovering the known ones as particular cases. In section 3, the method is applied for deriving a new class of solutions in closed form. Finally, in section 4, the evolution of the models and the influence of the electromagnetic field is discussed.

2. THE MODELS

Let us consider Ruban's line element[20]

$$ds^2 = dt^2 - Q^2(x,t)\ dx^2 - R^2(t)(dy^2+h^2dz^2) \quad , \quad (2.1)$$

where

$$h(y) = \frac{\sin\sqrt{k}\ y}{\sqrt{k}} = \begin{cases} \sin y \ \text{ if } \ k = 1 & , \quad (2.2a) \\ y \ \text{ if } \ k = 0 & , \quad (2.2b) \\ \sinh y \ \text{ if } \ k = -1 & , \quad (2.2c) \end{cases}$$

and k is the curvature parameter of the homogeneous 2-spaces t and x constants. The functions Q and R are free and will be determined by the Einstein field equations (EFE) with cosmological constant (in our units $8\pi G = c = 1$)

$$R^{\mu\nu} - \frac{1}{2}\ g^{\mu\nu}R + \Lambda g^{\mu\nu} = T^{\mu\nu}_m + T^{\mu\nu}_f \quad , \quad (2.3)$$

where $T^{\mu\nu}_m$ is the energy-momentum tensor (EMT) of the material medium and

$$T^{\mu\nu}_f = -\ \frac{1}{4\pi}\ (F^{\mu}{}_{\alpha}F^{\nu\alpha} - \frac{1}{4}\ g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \quad (2.4)$$

is the EMT of the electromagnetic field.

It is assumed that the material medium is a mixture of two perfect fluids, whose EMT is

$$T^{\mu\nu}_m = (\rho_m + p_m)u^{\mu}u^{\nu} - p_m g^{\mu\nu} \quad , \quad (2.5)$$

where $\rho_m = \rho_1+\rho_2$ and $p_m = p_1+p_2$ are respectively the net energy density and pressure of the mixture.

Now, by considering the magnetic and electric fields along the x-axis, it follows that the nonvanishing components of the Maxwell tensor $F_{\mu\nu}$ are $F_{01} = -F_{10} = E$ and $F_{32} = -F_{23} = H$. Moreover, since the metric functions depend on the coordinates t, x and y alone, it is reasonable to suppose ab initio that $E = E(t,x,y)$ and $H = H(t,x,y)$. In this case, it is easily shown by using the line element (2.1), that the Maxwell equations (square bracket means antisymmetrization)

$$\frac{1}{\sqrt{-g}} \left(\sqrt{-g}\; F^{\mu\nu}\right)_{;\nu} = 0 \tag{2.6}$$

and

$$F_{[\mu\nu;\lambda]} = 0 \tag{2.7}$$

give the following functional forms to the fields:

$$E = \frac{\sqrt{4\pi}\; E_0\; Q}{R^2} \tag{2.8}$$

and

$$H = \sqrt{4\pi}\; H_0\; \frac{\sin\sqrt{k}\; y}{\sqrt{k}} \quad , \tag{2.9}$$

where the factor $\sqrt{4\pi}$ was introduced to simplify the expression of $T_f^{\mu\nu}$. E_0 and H_0 are constants related with the intensity of the electric and magnetic fields respectively. Replacing the above equations into eq. (2.4) one obtains

$$T^f_{00} = -Q^{-2} T^f_{11} = R^{-2} T^f_{22} = R^{-2} h^{-2} T^f_{33} = \frac{E_0^2 + H_0^2}{2R^4} \; . \tag{2.10}$$

In the comoving frame $(u^\mu = \delta^\mu{}_0)$, using (2.5) and (2.10) the EFE given in Appendix A can be rewritten as

$$\rho_1 + \rho_2 + \frac{m_0^2}{R^4} = \frac{2\dot{R}\dot{Q}}{RQ} + \frac{\dot{R}^2+k}{R^2} - \Lambda \quad , \tag{2.11}$$

$$p_1 + p_2 - \frac{m_0^2}{R^4} = -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} + \Lambda \quad , \tag{2.12}$$

$$R\ddot{Q} + \dot{R}\dot{Q} - \left[\ddot{R} + \frac{\dot{R}^2+k}{R} - \frac{2m_0^2}{R^3}\right]Q = 0 \quad , \tag{2.13}$$

where $m_0^2 = \frac{E_0^2+H_0^2}{2}$ and an overdot means differentiation with respect to time. Note that the EFE are not modified if only one field is present. Since there is no observational basis for a primordial electric field, in what follows just the magnetic one will be considered. Incidentally, eqs. (2.9) and (2.10) show that such a field is "frozen-in" for any solution of (2.11) – (2.13). The lines of force are entrapped into the fluid and are carried along with it regardless the value of the electrical conductivity.

The system (2.11)–(2.13) is indeterminate since there are three differential equations and four unknown quantities, namely ρ, p, R and Q in the one-fluid description and six unknowns, namely ρ_1, ρ_2, p_1, p_2, R and Q for the case of two fluids. Moreover, the net pressure is a function of t alone whereas the net energy density is a function of t and x. Therefore, in the one fluid description the usual equation of state cannot be imposed without loss of generality. Essentially, this is the same problem appearing for the first time in Szekeres' type models. There, it was circumvented by Szafron[13] who suggested an algorithm for obtaining exact solutions. For a simple fluid it can be adapted as follows:

(i) specify the net pressure $p = p(R,\dot{R},\ddot{R},\Lambda)$ in eq. (2.12) and solve it for $R = R(t)$;

(ii) obtain $Q = Q(R,x)$ from (2.13);

(iii) from eq. (2.11), compute the net energy density of the fluid ρ_m by substituting the expressions of $Q(R,x)$ and $R(t)$.

An explicit example will be given in the next section. We remark that if $p = \Lambda = 0$ the solutions obtained are simple inhomogeneous generalizations of the Doroshkevich universes. In addition, if $m_0 = 0$ Ruban's models are recovered, and if $p_m(R) = \frac{C}{R^2}$ or $p_m(R) = 3(\gamma-1)(\frac{\dot{R}^2+k}{R^2})$ where C and γ are constants then the models stand for two subclasses of Szekeres' type solutions(21-22). Of course, in the two-fluid description several choices are possible since one needs to fix three unknown quantities, for instance p_1, p_2 and ρ_2. If $p_1 = \Lambda = k = 0$ and $p_2 = \frac{1}{3}\rho_2 = \frac{\tilde{C}}{R^4}$ the solutions of TW with dust, isotropic radiation and electromagnetic field are recovered and if $m_0 = 0$ these solutions reduce to a subclass of Pollock and Cadderni(14) work.

3. A NEW CLASS OF SOLUTIONS

Here, by using the conditions (i)–(iii) defined in the last section we will exhibit, in closed form, a set of exact solutions to the system (2.11)–(2.13). Firstly, note that if $m_0 = 0$, eq. (2.12) reduces to that of the FRW models. It thus suggests to the pressure the following expression[21]

$$p = 3(\gamma-1)(\dot{R}^2+k)/R^2 \qquad , \tag{3.1}$$

where the constant parameter γ will be identified with the adiabatic index of the asymptotic (in time) equation of state $p = (\gamma-1)\rho$. So, inserting (3.1) into (2.12) and considering from now on that Λ vanishes one finds

$$R\ddot{R} + \frac{(3\gamma-2)}{2}\dot{R}^2 + \frac{(3\gamma-2)}{2}k - \frac{1}{2}\frac{m_0^2}{R^2} = 0 \quad , \tag{3.2}$$

the first integral of which is given by

$$\dot{R}^2 = \left(\frac{R_0}{R}\right)^{3\gamma-2} - k - \frac{1}{(4-3\gamma)}\left(\frac{m_0}{R}\right)^2 \quad , \text{ if } \gamma \neq \frac{4}{3} \ , \tag{3.3}$$

and

$$\dot{R}^2 = \left(\frac{R_0}{R}\right)^2 - k - \left(\frac{m_0}{R_0}\right)^2\left(\frac{R_0}{R}\right)^2 \ln\left(\frac{R_0}{R}\right) \ , \text{ if } \gamma = \frac{4}{3} \ , \tag{3.4}$$

where R_0 is a constant γ-independent.

If $m_0 = 0$ eq. (3.3) is the first integral for any γ, whereas (3.2) reduces to the FRW differential equation. In this case, the solution of (3.2) valid for any values of γ and k was given by Assad and Lima[23] in terms of hypergeometric functions. However, if m_0 and k are both different from zero the method developed there cannot be applied. Henceforth, for the sake of simplicity we consider just the quasi-Euclidean models ($k = 0$) with $\gamma \neq \frac{4}{3}$. In this case, following the ansatz of the ref. (23), it is easy to show that the solution of (3.2) or equivalently (3.3) is given by[24]

$$t-t_0 = \frac{2R_0}{4-3\gamma} (R/R_0)^{3\gamma/2} \left[1 - \frac{(m_0/R_0)^2}{4-3\gamma} (R/R_0)^{3\gamma-4}\right]^{1/2} F_1 - \frac{2R_0}{4-3\gamma} \left[1 - \frac{(m_0/R_0)^2}{4-3\gamma}\right]^{1/2} F_2 \quad , \tag{3.5}$$

where $t_0 = t(R_0)$, $F_2 = F_1(R_0)$ and $F_1(R)$ is the hypergeometric function

$$F_1 = F\left[\frac{3\gamma-2}{3\gamma-4} , 1; \frac{3}{2} ; 1 - \frac{(m_0/R_0)^2}{4-3\gamma}(R/R_0)^{3\gamma-4}\right]. \tag{3.6}$$

It should be noticed that taking the limit $m_0 \to 0$ and using the identity[25] $F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, eq. (3.5) gives the expected result

$$R = R_0 \left[1 + \frac{3\gamma}{2}(t-t_0)\right]^{2/3\gamma} \quad , \tag{3.7}$$

of the FRW flat models[23].

Consider now the equation of Q as given in (2.13) with $\Lambda = 0$. Replacing in it (3.2) one finds that

$$R\ddot{Q} + \dot{R}\dot{Q} + \left[\left(\frac{4-3\gamma}{3\gamma-2}\right)\ddot{R} + \left(\frac{6\gamma-5}{3\gamma-2}\right) \frac{m_0^2}{R^3}\right]Q = 0 \quad , \tag{3.8}$$

the solution of which, as shown in Appendix B, can be written as

$$Q = \beta R F_3 + \mu R_0 (R/R_0)^{(3\gamma-4)/2} F_4 \quad , \tag{3.9}$$

where β and μ are arbitrary functions of x, and F_3, F_4 are two new hypergeometric functions

$$F_3 = F\left[\frac{1-D}{2(3\gamma-4)} , \frac{1+D}{2(3\gamma-4)} ; \frac{3\gamma-2}{2(3\gamma-4)} ; \frac{(m_0/R_0)^2}{4-3\gamma} (R/R_0)^{3\gamma-4}\right] \quad , \tag{3.10}$$

$$F_4 = F\left[\frac{3\gamma-5-D}{2(3\gamma-4)} , \frac{3\gamma-5+D}{2(3\gamma-4)} ; \frac{9\gamma-14}{2(3\gamma-4)} ; \frac{(m_0/R_0)^2}{4-3\gamma} (R/R_0)^{3\gamma-4}\right] \quad , \tag{3.11}$$

where $D = \sqrt{33-24\gamma}$. Note that if $m_0 = 0$ the function Q reduces to

$$Q = \beta R + \mu R_0 (R/R_0)^{(3\gamma-4)/2} \quad . \tag{3.12}$$

The full solution for the metric (2.1) with k = 0 is given implicitly by (3.5)—(3.6) together with (3.9)—(3.11). To complete the solutions we obtain the net energy density

$$\rho = \rho_m + \rho_B \quad , \tag{3.13}$$

where $\rho_B = \frac{m_0^2}{R^4}$ is the energy density of the magnetic field. From eqs. (2.11) and (3.9) one finds that

$$\rho = \frac{\dot{R}^2}{R^2} \left(1 + \frac{2RQ'}{Q}\right) \quad , \tag{3.14}$$

where $\dot{R}^2$ is given by (3.3) and $Q' \equiv \partial Q/\partial R$. This completes the solution in the case k = 0. Note that since $\rho_B \geq 0$, the weak energy condition will be ensured by the positiveness of the matter energy density itself. By combining eqs. (3.12) and (3.13) it follows that $\rho_m > 0$ only if

$$\frac{Q'}{Q} > \frac{1}{2R} + \frac{2m_0^2}{R^2\dot{R}^2} \quad . \tag{3.15}$$

Thus, since the first member of this equation is a function of t and x whereas the second one depends only on t, as in the Szekeres type models, the positivity of ρ may be closely related with the choice of the arbitrary functions[12,15,19].

If $m_0 = 0$, these solutions stand for a class of models recently derived in the framework of a two fluid interpretation[22]. In addition, if $p = 0$ ($\gamma = 1$), the quasi-Euclidean Ruban's model is recovered. If $m_0 \neq 0$ and the arbitrary functions are made constants, the case $\gamma = 1$ is just the dust-filled electromagnetic model found by Doroshkevich (see Appendix C).

4. SINGULARITIES AND EVOLUTION

In the framework of the homogeneous Bianchi type models it is widely accepted that an electromagnetic field does not remove the initial singularity present in the evolutionary models obeying the usual energy conditions[9-10]. However, a comprehensive analysis of the role played by a "field-term" near the singularity as well as its influence in the isotropization process taking place in anisotropic and inhomogeneous cosmologies is far from being complete. These questions will be discussed next, by comparing the asymptotic behavior of the solutions presented in the last section for the models with and without electromagnetic field.

Consider first the kinematic quantities of the fluid. Since the pressure is a function of time alone and the coordinates system is synchronous and comoving, the 4-acceleration and vorticity are zero. The rate of shear scalar σ and the expansion parameter θ are

$$\sigma^2 = \frac{1}{2}\,\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{1}{3}\,\left(\frac{R\dot{Q}-Q\dot{R}}{RQ}\right)^2 \qquad , \tag{4.1}$$

and

$$\theta = \frac{3\dot{R}}{R} - \sqrt{3}\,\sigma \qquad . \tag{4.2}$$

It thus follows that if $Q \to \alpha R$ where α is an arbitrary function of x, the models approach the FRW ones. However, the asymptotic behavior is strongly dependent of the γ-parameter. Next, the limits have been computed retaining only the leading terms in all physical quantities.

4.1 - Behavior for Large Cosmological Times ($R \gg R_0$)

The conditions under which the present FRW phase is attained are easily determined from eq. (3.3). The first term of the right-hand side (r.h.s.) of eq. (3.3) will fastly be dominant if the parameter $b = m_0/R_0$ is not very large and $\gamma < 4/3$. In this case the function R given in eq. (3.5) approaches the FRW form $R \sim R_0 \ (t/t_0)^{2/3\gamma}$. Moreover, from eq. (3.10)–(3.11) one finds that in this limit

$$F_3 \sim 1 + c(\gamma)(R/R_0)^{3\gamma-4} \quad , \tag{4.3}$$

and

$$F_4 \sim 1 + d(\gamma)(R/R_0)^{3\gamma-4} \quad , \tag{4.4}$$

where c and d are constants γ-dependent. Substituting (4.3) and (4.4) into (3.9) one finds that $Q \sim \beta R$. Thus, as remarked before, this result suggests that the models approach the homogeneity and isotropy along each fluid line. The equation of state $p \sim (\gamma-1)\rho$ may, in fact, be established since the limit forms of the pressure and the energy density are

$$p \sim \frac{3}{R_0^2} \ (\gamma-1)(R_0/R)^{3\gamma} \quad , \tag{4.5}$$

and

$$\rho \sim \frac{3}{R_0^2} \ (R_0/R)^{3\gamma} \quad . \tag{4.6}$$

Hence, it follows that the γ-parameter plays the role of an adiabatic index in the latter stages. The line element itself, after a trivial variable change, can be rewritten as

$$ds^2 = dt^2 - R^2(dx'^2 + dy'^2 + dz'^2) \quad , \tag{4.7}$$

just as one should expect in a FRW phase.

Of course, if $\gamma > \frac{4}{3}$ the models approach anisotropy rather than isotropy, since the field term in (3.3) will be dominant. Note also that if $b = 0$ the models evolve to the FRW ones regardless the value of γ. These results make evident the influence of the field along the expansion. Formally its main effect, at the latter stages, is to restrict the admissible values of the adiabatic index over the interval $1 \leq \gamma < \frac{4}{3}$.

4.2 - Approach to the Initial Singularity

In general, the models derived in the preceding section present singularities when $Q = 0$ or $R = 0$. The "pancake" $Q = 0$ singularity is established for those positive values of the transverse scale-factor R which are solutions, for each value of γ, of the equation $Q(R,x) = 0$. Of course, due to the x dependence it does not occur simultaneously in the comoving frame. The character of the $R = 0$ singularity depends of the values assumed by the "adiabatic index" γ. If $\gamma > \frac{4}{3}$, for instance, it is analogous to the "point-like singularity" which arises in the FRW models. Here as there, it is simultaneous for the comoving observers and one may adjust the arbitrary time scale t_0 so that it occurs when $t = 0$. The role played by the electromagnetic field in such early times is also strongly dependent of the considered interval of the γ-parameter. For the sake of simplicity we discuss its influence only near the initial singularity $R = 0$.

First consider briefly the case without field ($m_0=0$).

From eq. (3.12) it follows that if $R \to 0$ then $Q \sim \mu R_0 (R/R_0)^{(3\gamma-4)/2}$, and using eqs. (2.1), (3.1), (3.3) and (3.14) we readily obtain for the pressure, energy density and metric the expressions

$$p \sim \frac{3}{R_0^2} (\gamma-1)(R_0/R)^{3\gamma} \quad , \tag{4.8}$$

$$\rho \sim \frac{3}{R_0^2} (\gamma-1)(R_0/R)^{3\gamma} \quad , \tag{4.9}$$

and

$$ds^2 \sim dt^2 - \mu(x) R_0^2 (R/R_0)^{3\gamma-4} dx^2 - R^2 (dy^2 + y^2 dz^2) \quad , \tag{4.10}$$

where one can make $\mu = 1$ by a transformation of x. Thus, the models starts homogeneous but anisotropic if $\gamma \neq 2$. Moreover, by comparing eqs. (4.8) and (4.9) one can see that in this limit $p \sim \rho$ regardless the value of γ. In fact, the case $\gamma = 2$ starts as the stiff-matter FRW universe itself. Note that if $\gamma > \frac{4}{3}$ the singularity is "point-like", but if $\gamma < \frac{4}{3}$ it is a "cigar type singularity". In the latter case, since $\frac{\dot{Q}}{Q} \sim (3\gamma-4) \frac{\dot{R}}{R}$, the Hubble parameter is negative and a blue shift of the distant objects must be observed. This generalizes the result implicitly given in Ruban's work for $\gamma = 1$.

Now consider $m_0 \neq 0$. For $\gamma > \frac{4}{3}$, it is easily seen that the hypergeometric functions defined by (3.10) and (3.11) behave, in the limit $R \to 0$, as $F \sim 1 + \alpha (m_0/R_0)^2 (R/R_0)^{3\gamma-4}$ where α is a constant γ-dependent. Inserting this result into (3.10), as in the preceding case, one finds $Q \sim \mu R_0 (R/R_0)^{(3\gamma-4)/2}$. Therefore, for $\gamma > \frac{4}{3}$ the asymptotic behavior (in time) is fully analogous to the case without field. Following Doroshkevich, we say that if $\gamma > \frac{4}{3}$ the field is "written into" the solution but it does not modify its characteristics near the initial singularity. However, the field can exert a notable influence if $\gamma < \frac{4}{3}$. In this case, it may be easily shown that the initial "cigar-like singularity" which occurs for $m_0 = 0$ can be avoided. In fact, since $\dot{R}^2 \geq 0$, the first integral (3.3) shows that there is, for each value of γ, a critical value of R given by $R_c = R_0 \left[\frac{(m_0/R_0)^2}{4-3\gamma}\right]^{1/(4-3\gamma)}$ in which $\dot{R} = 0$. In principle, R_c should be the minimal allowed value of R since eq. (3.2) yields $\ddot{R}_c = \frac{1}{2}\frac{m_0^2}{R_c^3}$. But, these models cannot start with such a minimal value of R because if $\dot{R} = 0$ the net energy vanishes (see eq. (3.19)). It thus follows that $R > R_c$ if one wishes to ensure the positiveness of the matter energy density. Of course, this result remains valid for the Doroshkevich solution ($\gamma = 1$).

5. FINAL COMMENTS

The existence of spatially inhomogeneous solutions of the Einstein-Maxwell equations in the framework of Ruban's metric has been examined. By adding another fluid component plus an electromagnetic field the algorithm suggested by Szafron has been extended for determining all exact solutions in the considered background. The previously known models are recovered by a straightforward application of it and a new class of solutions containing matter plus a "frozen-in" magnetic field have been derived. In general, the models are spatially inhomogeneous but there are three Killing vectors in their bidimensional sections t and x constants. The influence of the magnetic field near the singularity and at the latter stages of the expansion is strongly dependent on the "adiabatic index" γ of the asymptotic equation of state. If $\gamma > \frac{4}{3}$ the field is "written into" the solution but it does not change its characteristics in the neighborhood of the singular point. The models evolve to the FRW ones with an arbitrary γ-law only if $\gamma < \frac{4}{3}$. In any case, these solutions show that the "frozen-in" condition is, in fact, unrelated with the spatial homogeneity property.

Finally, we remark that if $m_0 = 0$ the one-fluid solutions have been interpreted as a mixture of two interacting simple fluids[22]. However, it is not clear for us, if such an interpretation would be applied in the presence of the electromagnetic field.

APPENDIX A

THE EINSTEIN FIELD EQUATIONS

For Ruban's line element (2.1), the Einstein field equations with cosmological constants $R_{\mu\nu} - (\frac{R}{2} - \Lambda)g_{\mu\nu} = T_{\mu\nu}$, reduces to

$$QR^2T_{00} = Q\dot{R}^2 + 2R\dot{Q}\dot{R} + kQ - \Lambda QR^2 \quad , \qquad (A.1)$$

$$Q^{-2}R^2T_{11} = -2\ddot{R}R - \dot{R}^2 - k + \Lambda R^2 \quad , \qquad (A.2)$$

$$QR^{-1}T_{22} = -Q\ddot{R} - \dot{Q}\dot{R} - \ddot{Q}R + \Lambda QR \quad , \qquad (A.3)$$

$$QR^{-1}h^{-2}T_{33} = -Q\ddot{R} - \dot{Q}\dot{R} - \ddot{Q}R + \Lambda QR \quad , \qquad (A.4)$$

where an overdot means time derivative.

APPENDIX B

THE SOLUTION OF THE Q FUNCTION

We now consider the differential equation (3.8) to the Q function

$$R\ddot{Q} + \dot{R}\dot{Q} + \left[\left(\frac{4-3\gamma}{3\gamma-2}\right)\ddot{R} + \left(\frac{6\gamma-5}{3\gamma-2}\right) \frac{m_0^2}{R^3} \right] Q = 0 \quad . \tag{B.1}$$

In order to obtain Q directly as a function of R, we define for $\gamma \neq \frac{4}{3}$ the following transformation

$$Q = R\psi(\xi,x) \quad , \quad \xi = \frac{1}{(4-3\gamma)} \left(\frac{m_0}{R_0}\right)^2 \left(\frac{R}{R_0}\right)^{3\gamma-4} \quad . \tag{B.2}$$

Substituting (B.2) into (B.1) and using (3.2) and (3.3) we find that ψ satisfies the Gaussian hypergeometric differential equation[26]

$$\xi(\xi-1) \frac{\partial^2\psi}{\partial\xi^2} + \left[\frac{(3\gamma-2)}{2(3\gamma-4)} - \frac{3(\gamma-1)}{3\gamma-4}\,\xi\right] \frac{\partial\psi}{\partial\xi} + \frac{2}{3\gamma-4}\,\psi = 0 \quad , \tag{B.3}$$

with parameters $a = \frac{1-D}{2(3\gamma-4)}$, $b = \frac{1+D}{2(3\gamma-4)}$ and $c = \frac{3\gamma-2}{2(3\gamma-4)}$, where $D = \sqrt{33-24\gamma}$. For $\gamma \neq \frac{2(4n-1)}{3(2n-1)}$, where n is an integer, the general solution of (B.3) is given by[27]

$$\psi(\xi,x) = \beta(x)\ F\left[\frac{1-D}{2(3\gamma-4)}\ ,\ \frac{1+D}{2(3\gamma-4)}\ ;\ \frac{3\gamma-2}{2(3\gamma-4)}\ ;\ \xi\right]$$

$$+\ \mu(x)\,\xi^{(3\gamma-6)/2}\ F\left[\frac{3\gamma-5-D}{2(3\gamma-4)}\ ,\ \frac{3\gamma-5+D}{2(3\gamma-4)}\ ;\ \frac{9\gamma-14}{2(3\gamma-4)}\ ;\ \xi\right]\ , \qquad \text{(B.4)}$$

where β and μ are arbitrary functions of x, and the F's are hypergeometric functions.

Inserting (B.4) into (B.2) it is easily seen that the solution of Q is given by eqs. (3.9)–(3.11).

APPENDIX C

THE CASE $\gamma = 1$

Of particular interest within the class of solutions presented in the section 3 is the case $\gamma = 1$, which generalizes the homogeneous model found by Doroshkevich[(2)]. Setting $\gamma = 1$ in eqs. (3.9)-(3.11) we find

$$Q(R,x) = \beta(x) R F\left[1,-2;-\frac{1}{2}\ ;\ \left(\frac{m_0}{R_0}\right)^2 \left(\frac{R}{R_0}\right)^{-1}\right] +$$

$$+ \mu(x) R_0 \left(\frac{R}{R_0}\right)^{-1} F\left[\frac{5}{2}\ ;-\frac{1}{2};\ \frac{5}{2}\ ;\ \left(\frac{m_0}{R_0}\right)^2 \left(\frac{R}{R_0}\right)^{-1}\right] \quad ; \qquad \text{(C.1)}$$

this may be rewritten in terms of elementary functions as

$$Q(R,x) = \beta(x) R\left[1 + 4\left(\frac{m_0}{R_0}\right)^2 \left(\frac{R}{R_0}\right)^{-1} - 4\left(\frac{m_0}{R_0}\right)^2 \left(\frac{R}{R_0}\right)^{-2}\right] +$$

$$+ \mu(x) R_0 \left(\frac{R}{R_0}\right)^{-1/2} \left[1 - \left(\frac{m_0}{R_0}\right)^2 \left(\frac{R}{R_0}\right)^{-1}\right]^{1/2} \quad , \qquad \text{(C.2)}$$

where the scale factor R(t), implicitly given by eqs. (3.5)-(3.6) taking $\gamma = 1$, may be written in parametric form as[(7)]

$$R(\eta) = \frac{R_0}{4}\eta^2 + \frac{m_0^2}{R_0} \quad , \qquad \text{(C.3)}$$

$$t - t_0 = \frac{R_0}{12}\eta^3 + \frac{m_0^2}{R_0}\eta \quad , \qquad \text{(C.4)}$$

where η is the usual conformal time defined by $dt = Rd\eta$. Substituting (C.3) into (C.2) one finds

$$Q(\eta,x) = \frac{\beta(x)}{4R_0}\left[\frac{R_0^4\eta^4 + 24R_0^2m_0^2\eta^2 - 24m_0^4}{R_0^2\eta^2 + 4m_0^2}\right] + \frac{2\mu(x)R_0^3\eta}{R_0^2\eta^2+4m_0^2} \quad . \qquad \text{(C.5)}$$

As one should expect, identifying $m_0^2 = q^2$, $R_0 = C$ and taking the arbitrary functions β and μ as given by $\beta = \frac{2}{3}\mu_0 R_0$ and $\mu = \frac{E}{2R_0^3}$ where μ_0 and E are two dimensional constants, the solutions (C.3)–(C.5) reduce to the quasi-Euclidean Doroshkevich universe in the form presented by Vajk and Eltgroth(7).

ACKNOWLEDGMENTS: We would like to thank M.J.D. Assad, R.Portugal, J. Tiomno and I. Waga by helpful discussions and A.F. da F. Teixeira and N. Tomimura for a critical reading of the manuscript. Partial support from Coordenação de Aperfeiçoamento de Pessoal de Ensino Superior-CAPES (Brazilian Research Agency) is acknowledged.

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