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Metastability of the $(\bar{\psi}_i\psi_i)^2_3$ model at finite temperature and density.

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Abstract

Using concurrently the dimensional and analytic regularization methods we re-examinate the Gross-Neveu model at finite temperature and density (chemical potential) in a D-dimensional spacetime. The renormalized effective potential is presented at the one-loop approximation. In the case of non-zero chemical potential we show that the effective potential acquires an imaginary part, which means that the system becomes metastable, indicating the possibility of a first order phase transition.

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1 Introduction

Besides computer simulations and the lattice approach there have been many attempts over the years to circunvect the limitations of perturbative renormalizability, in quantum field theory in the framework of its formulation on a continuous spacetime. For instance, many efforts have been done in this sense in the context of constructive field theory [1]. Also if a N-component ("colors") field with a quartic coupling of intensity λ is considered, the introduction of an ultralocal intermediate field leads to the possibility of the summation of an infinite number of Feynman diagrams with an effective three body coupling constant $\frac{\lambda}{\sqrt{N}}$, in the large N limit. In this case some of the more divergent diagrams are suppresed and an effective complete propagator is obtained. This is the basic idea underlying the so-called $\frac{1}{N}$ expansion. A well known example where this idea is implemented is the four fermion self-interaction model $(\bar{\psi}_i\psi_i)^2$, the Gross-Neveu model [2]. Although this model is perturbatively non-renormalizable in $D = 3$, at leading order in $\frac{1}{N}$, the resulting effective theory shares some of the basic requirements for a constructively renormalizable theory. This is due to the remarkable property of asymptotic safeness [3]. From the point of view of the renormalization group, the model has been studied by many authors. Among them some are listed in ref.[4].

In $D = 2$ the massless model is asymptotically free and in any spacetime dimension $D < 4$ a chiral symmetry may be spontaneously broken with dynamical generation of mass. In the absence of chemical potential and at finite temperature the spontaneous broken symmetry may be restored via a second order phase transition. For $D = 2$ the condensation of kinks means that the phase transition may occur at any non-zero temperature. The question that is raised is to known whether the presence of the chemical potential changes the order of the phase transition [5]. In the literature there is no agreement on the subject since some results from lattice calculation indicate that for non-zero chemical potential the transition becomes of first order while it is of second order for vanishing chemical potential [6]. Wolf obtained a different result in $D = 2$, with a tricritical point separating second order phase transition from first order one. Recently the four fermion model has also been investigated by Barducci et al. [7] where an explicitly broken chiral symmetry introduced by a fermionic mass term to the Lagrangian was analized. These authors suggest that depending on the values of the fermion mass the transition could be of first or of second order. Also indications for the existence of a tricritical phenomenon were found.

In this note our aim is twofold. First, to investigate the finite temperature behavior of the effective potential of the Gross-Neveu model in arbitrary spacetime dimension. Second is to analyse the order of the phase transition for non-zero chemical potential. For more generality, we treat the model in arbitrary spacetime dimension D, and to exhibit the analytic structure of the effective potential we use a mix between dimensional and zeta function analytic regularization methods. It results that the so regularized effective potential is free of any ultraviolet singularity in odd spacetime dimension. This somehow surprising result is to be compared to those obtained in the case of the calculation of the Casimir energy using similar regularization methods. In the case of non-zero chemical potential, we show that the effective potential acquires an imaginary part, which means that the system becomes metastable, indicating the possibility of a first order phase transition.

The outline of the paper is the following: in section 2 we briefly review the formalism of the effective potential. In section 3 the effective potential is obtained at zero and non zero chemical potential. Conclusions are given in section 4. In this paper we use $\hbar = k_B = c = 1$.

2 The one-loop effective potential of the Gross-Neveu model at zero and finite temperature.

Let us consider a system consisting fermion fields in thermal equilibrium with a reservoir at temperature β^{-1}. They are defined on a $D = d + 1$ dimensional flat spacetime with trivial topology of the spacelike sections. The Gross-Neveu model is described in terms of a $U(N)$ symmetric action for a set of N massless fermions

$$S(\bar{\psi}, \psi) = \int d^D x \left(\bar{\psi}(i \not{\partial})\psi + \frac{\lambda}{2N}(\bar{\psi}\psi)^2 \right). \tag{1}$$

The symmetry which prevents the fermions from acquiring a mass in perturbation theory is

$$\psi \to \gamma^5 \psi. \tag{2}$$

In a even dimension spacetime the above transformation defines a discrete chiral symmetry. In a odd dimensional spacetime a fermionic mass breaks space parity.

The Green's functions of the model are generated by the functional derivatives of the

generating functional,

$$Z[\eta,\bar{\eta}] = c\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\{i\int d^Dx(\mathcal{L}+\bar{\psi}\eta+\bar{\eta}\psi)\} \tag{3}$$

where $\bar{\psi(x)}$, $\psi(x)$, $\bar{\eta(x)}$ and $\eta(x)$ are elements of a Grassmann algebra. The analysis of the symmetry behavior of the model is simplified using the following device. Let us introduce the constraint field $\varphi(x)$ in the Lagrangian density,

$$\mathcal{L}'(\varphi,\psi,\bar{\psi}) = \mathcal{L}(\psi,\bar{\psi}) - \frac{1}{2}\left(\varphi + (\frac{\lambda}{N})^{\frac{1}{2}}\bar{\psi}\psi\right)^2 = \bar{\psi}(i\ \not{\partial})\psi - (\frac{\lambda}{N})^{\frac{1}{2}}\varphi\bar{\psi}\psi - \frac{1}{2}\varphi^2. \tag{4}$$

By the introduction of this constraint field $\varphi(x)$, we may write the generating functional given by eq.(3) equivalently as

$$Z(\eta,\bar{\eta}) = c'\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\varphi\exp i\int d^Dx(\mathcal{L}'+\bar{\psi}\eta+\bar{\eta}\psi) \tag{5}$$

where $\mathcal{L}'(\varphi,\psi,\bar{\psi})$ is given by eq.(4). To determine the equilibrium value of $\varphi(x)$ we define the generating functional of the connected Green functions, $W(J)$ by

$$e^{iW(J)} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\varphi\exp\{i\int d^Dx(\mathcal{L}'+J\varphi\}, \tag{6}$$

from which we get by a Legendre transform the effective action $\Gamma(\varphi_0)$

$$\Gamma(\varphi_0) = \int d^Dx\,(\varphi_0(x)J(x) - W(J)) \tag{7}$$

where

$$\varphi_0(x) = \frac{\delta W(J)}{\delta J(x)}. \tag{8}$$

If we assume translational invariance the effective potential is given by

$$V(\varphi_0) = -\sum_n^\infty \frac{1}{n!}\tilde{\Gamma}^{(n)}(0,0,..0)(\varphi_0-\varphi_{min})^n \tag{9}$$

or

$$V(\varphi_0) = \frac{1}{2}\varphi_0^2 - iN\sum_{s=1}^\infty \int \frac{d^Dk}{2\pi^D}\frac{1}{s}(\frac{\lambda\varphi_0^2}{Nk^2})^s. \tag{10}$$

As is well known at zero temperature the minimum of the effective potential is not at the origin. There is a spontaneous symmetry breaking with a dinamical generation of mass associated to a bound state. In order to compare with the results we will get in the case of non-zero chemical potential, let us briefly review some basic features of the model at zero chemical potential.

If we assume that the system is in thermal equilibrium with a reservoir at temperature β^{-1} we may use the Matsubara formalism. In this case we have to perform the replacements $\omega \to \omega_n = \frac{2\pi}{\beta}(n + \frac{1}{2})$ and $\frac{1}{2\pi}\int dq_E^0 = \frac{1}{\beta}\sum_n$. Defining $(\frac{\lambda}{N})^{\frac{1}{2}} = g$ the the effective potential is given by

$$V(\varphi_0, \beta) = \frac{1}{2}\varphi_0^2 + 2N\sum_{s=1}^{\infty} p(D,s)(g\varphi_0)^{2s}\beta^{2s-D}\zeta(2s-d, \frac{1}{2}), \tag{11}$$

where $p(D,s)$ is given by:

$$p(D,s) = \frac{\pi^{\frac{d}{2}}}{\Gamma(s+1)}\Gamma(s - \frac{d}{2})\frac{(-1)^s}{(2\pi)^{2s}}. \tag{12}$$

The Hurwitz zeta function being done by

$$\zeta(z,q) = \sum_{n=0}^{\infty}\frac{1}{(n+q)^z}, \tag{13}$$

which is analytic for $Re(z) > 1$.

Note that the effective potential is not yet well defined since the Hurwitz zeta function is defined only in a open connected set of points in the s complex plane. By a standard procedure it is possible to find the analytic extension of the Hurwitz zeta function to the whole s plane. This analytic extension is a meromorphic function with simple poles. To analytically extend the Hurwitz zeta function we go along the following steps. First we use the Euler representation for the Gamma function to express it as

$$\zeta(z,q) = \frac{1}{\Gamma(z)}\int_0^{\infty} dt\ t^{z-1}\frac{e^{-tq}}{1-e^{-t}}. \tag{14}$$

Next we split the integral from zero to infinity in two integrals from zero to one and from one to infinity. The second one is analytic function, the divergences being associated to the zero limit of the first integration. Then using a Bernoulli representation for the integrand it is possible to get the following expression to the analytic extension of $\zeta(z, \frac{1}{2})$,

$$\zeta(z, \frac{1}{2}) = \frac{1}{\Gamma(z)}g_1(z) + \frac{1}{\Gamma(z)}\sum_{n=0}^{\infty}\frac{B_n(\frac{1}{2})}{n!}\frac{1}{(z+n-1)} \tag{15}$$

where $g_1(z)$ is given by

$$g_1(z) = \int_1^{\infty} dt\, t^{z-1}\frac{e^{\frac{t}{2}}}{e^t - 1}, \tag{16}$$

and $B_n(x)$ are the Bernoulli polynomials. Taking $z = 2s-d$ and defining $c_n = \frac{B_n}{n!}(\frac{1}{2^{n-1}} - 1)$ we replace this analytic extension in the effective potential $V(\varphi_0, \beta)$. A straightforward calculation gives,

$$V(\varphi_0, \beta) = \frac{1}{2}\varphi_0^2 + 2N\sum_{s=1}^{\infty} q(D,s)[g_1(2s-d) + \sum_{n=0}^{\infty}\frac{c_n}{2s-D+n}](g\varphi_0)^{2s}\beta^{2s-D}, \tag{17}$$

where $q(D,s)$ is given by

$$q(D,s) = \frac{(-1)^s}{\Gamma(s+1)} \frac{1}{\Gamma(s-\frac{D}{2}+1)} \frac{1}{(4\pi)^{2s-\frac{D}{2}}} \tag{18}$$

A direct consequence of the regularization method employed is the following: we see from the above equation that $h(D,s)$ has no singularities for odd D and interger s. This means that for odd D the effective potential is finite. No renormalization procedure is needed. As we will see in the conclusions it is interesting to note that this fact has a mathematical connection with similar situations in the Casimir effect and for anomalies in unbounded odd dimensional spacetimes [8]. We will show, that with the introduction of a chemical potential the effective potential develop an imaginary part.

Let us introduce a chemical potential μ and define

$$\sigma = \frac{\mu\beta}{2\pi}. \tag{19}$$

Note that the only change is that instead $\zeta(z,\frac{1}{2})$, we have to analytically extend $\zeta(z,\frac{1}{2}+i\sigma)$. Let us first write a integral representation of the Hurwitz zeta function as follows

$$\zeta(z,\frac{1}{2}+i\sigma) = \frac{1}{\Gamma(z)} \int_0^\infty dt\ t^{z-1} \frac{e^{-t(\frac{1}{2}+i\sigma)}}{1-e^{-t}}. \tag{20}$$

Proceeding as we have done to get eq.(15) above, the integrand of eq.(20) may be expressed in terms of the Bernoulli polynomials. Using the following expansion

$$B_n(x+h) = \sum_{k=0}^{n} C_n^k B_k(x) h^{n-k} \tag{21}$$

we have

$$B_n(\frac{1}{2}-i\sigma) = \sum_{k=0}^{n} A_{nk} \sigma^{n-k} \tag{22}$$

where

$$A_{nk} = -(-i)^{n-k} C_n^k (1-2^{1-k}). \tag{23}$$

Substituting eqs.(21),(22) and (23) in eq.(20) we have for the analytic extension

$$\zeta(2s-d,\frac{1}{2}+i\sigma) = \frac{1}{\Gamma(2s-d)} g_2(z) + \frac{1}{\Gamma(2s-d)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A_{nk}}{(2s-D+n)} \sigma^{n-k} \tag{24}$$

where $g_2(z)$ is given by

$$g_2(z) = \int_1^\infty dt\, t^{2s-D} \frac{e^{t(\frac{1}{2}-i\sigma)}}{e^t-1}. \tag{25}$$

The effective potential is then expressed as

$$V(\varphi_0,\beta)=\frac{1}{2}\varphi_0^2+2N\sum_{s=1}^{\infty}q(D,s)\left(g_2(2s-d,\sigma)+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{A_{nk}}{2s-D+n}\sigma^{n-k}\right)(g\varphi_0)^{2s}\beta^{2s-D} \tag{26}$$

Since from eq.(23) for odd $(n-k)$, A_{nk} is imaginary, the effective potential develops an imaginary part. As it was shown by many authors [9] this imaginary part has a natural interpretation as the decay rate per unit volume of some unstable homogeneous quantum state. We conclude that the presence of the chemical potential introduce physical instability in the model. This is indicative of a first order phase transition.

3 Conclusion

We obtained two results in the paper. First if we consider that there is a non-zero fermion density we showed that the effective potential acquires an imaginary part. This may be interpreted as an indication of a first order phase transition. The second is expressed in eqs.(18) and (26). The effective potential is finite in any odd dimensional space time, in particular in $D=3$. There are no ultraviolet divergences. In the Yukawa model the same mathematical phenomenon has been showed [10] This situation is very similar to the one encountered in the calculation of the renormalized vacuum energy of scalar fields confined in boxes (Casimir energy) [11]. Dolan and Nash used the zeta function analytic regularization method to obtain the Casimir energy of conformally coupled scalar field confined in odd and even dimensional spheres [12]. They obtained that for odd dimensional spheres (even space-time dimension) there is a pole in the point of interested, being necessary the introduction of a counterterm, while for even dimensional spheres (odd dimensional space-time) the result obtained is naturally finite. No renormalization is needed. The question that we have to answer is the link between the one-loop effective energy and the Casimir energy. This problem has been studied by some authors where it was proved that the effective energy and the Casimir energy differ, but this is a inherent renormalization-scheme ambiguity [13].

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