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Algebraic Bethe Ansatz for a Spin-1/2 Quantum Linear Chain with Competing Interactions

by

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Abstract

We propose a generalization of the formula that gives an integrable quantum one-dimensional spin Hamiltonian with nearest-neighbor interactions as a logarithmic derivative of a vertex model transfer matrix in order to include in this scheme more realistic integrable models. We compute exactly this generalized formula using the R-matrix of the XXX-model obtaining the Majumdar-Ghosh Hamiltonian plus a charge-like interaction term. We diagonalize this Hamiltonian using the Quantum Inverse Scattering Method and present the Bethe ansatz equations of the model.

Key-words: integrable spin models; Bethe ansatz equations; next-nearest-neighbor interactions; competing interactions; Majumdar-Ghosh model.

Since the pioneering work of Bethe in 1931 [1], low-dimensional integrable spin models have been subject of increasing interest. Nowadays, there is a long list of one-dimensional ($1D$) integrable spin models solved with Bethe ansatz or with other methods [2]. Nevertheless, few solutions have been found in higher dimensions and in low-dimensional models involving more complicated interactions such as nearest- and next-nearest-neighbor interactions.

In this letter we propose a generalization of the well-known connection [3] between the transfer matrix of vertex lattice models and quantum $1D$ spin Hamiltonians with nearest-neighbor interactions without spoiling integrability. The purpose of this extension is to accommodate, within this picture, more realistic integrable quantum $1D$ spin Hamiltonians presenting nearest-neighbor as well as next-nearest-neighbor interactions.

In our generalization we work with two transfer matrices; one of them is constructed taking the trace of products of L-operators with alternating values of the spectral parameter and the other is given by a shift of the previous one. Apart a trivial constant the spin Hamiltonian is given by the difference of the logarithmic derivative of both transfer matrices.

We compute exactly this generalized formula using the R-matrix of the XXX-model obtaining a spin-1/2 quantum $1D$ Hamiltonian with isotropic nearest- and next-nearest-neighbor interactions (usually called Majumdar-Ghosh Hamiltonian [4]) plus a charge-like interaction term. Thus, for the R-matrix under consideration our approach provides a model of great interest: a quantum linear chain with competing interactions [5].

Due to the special way this Hamiltonian is given through a transfer matrix we show that we can diagonalize it using the Quantum Inverse Scattering Model (QISM) [6] obtaining the Bethe vectors, energy eigenvalues and the algebraic Bethe ansatz equations (BAE) for this model, generalizing in a non-trivial way the respective well-known expressions of the XXX Heisenberg model.

The results of this paper indicate that associated to each model in the large class of integrable spin models obtained by means of transfer matrix of vertex models there is a descendant integrable model with nearest- and next-nearest-neighbor interactions obtained following the approach we are going to describe.

We start presenting the main points of a generalization of the QISM that it is suitable for the discussion of the Hamiltonian with competing interactions we are going to study. The main formula of the QISM is a variation of the Yang-Baxter relation [7] usually called RLL-relation

$$R_{a_1a_2}(\lambda-\mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1a_2}(\lambda-\mu), \tag{1}$$

where λ is a complex parameter called spectral parameter, $R_{a_1a_2}(\lambda)$ is the well-known R-matrix of the XXX-model [6] that acts in the tensor product of two auxiliary spaces given by $C^2 \otimes C^2$ and satisfies the quantum Yang-Baxter relation [7]. $L_{n_1,a_1}(\lambda)$ is an operator acting in the tensor product of a local space C^2 and an auxiliary C^2,

$$L_{n,a}(\lambda) = \lambda I_n \otimes I_a + \frac{i}{2}\sum_{\alpha} \sigma_n^{\alpha} \otimes \sigma^{\alpha} \ , \tag{2}$$

with "I" denoting the unit matrix in the respective space and $\vec{\sigma}$ the Pauli matrices.

As noted in [8] the RLL-relation can be generalized as

$$R_{a_1 a_2}(\lambda-\mu) L_{n,a_1}\left(\lambda^{(n)}\right) L_{n_1,a_2}\left(\mu^{(n)}\right) = L_{n_1,a_2}\left(\mu^{(n)}\right) L_{n_1,a_1}\left(\lambda^{(n)}\right) R_{a_1,a_2}(\lambda-\mu), \tag{3}$$

where $\lambda^{(n)} = \lambda + v_n$ and $\mu^{(n)} = \mu + v_n$, with v_n a set of complex numbers. This allow us to define

$$T_a(\{\lambda\}) = L_{1,a}\left(\lambda^{(1)}\right) \cdot\cdot L_{n,a}\left(\lambda^{(n)}\right) \cdot\cdot L_{N,a}\left(\lambda^{(N)}\right) \ , \tag{4}$$

that can be written in matricial form as

$$T_a(\{\lambda\}) = \left(\begin{array}{cc} A(\{\lambda\}) & B(\{\lambda\}) \\ C(\{\lambda\}) & D(\{\lambda\}) \end{array} \right) \ . \tag{5}$$

It is easy to see that [8]

$$R_{a_1 a_2}(\lambda-\mu) T_{a_1}(\{\lambda\}) T_{a_2}(\{\mu\}) = T_{a_2}(\{\mu\}) T_{a_1}(\{\lambda\}) R_{a_1 a_2}(\lambda-\mu) \ , \tag{6}$$

that are the generalization of the usual RTT-equations.

¿From the above RTT-equations we get

$$\begin{aligned} [B(\{\lambda\}), B(\{\mu\})] &= 0 \ , \\ A(\{\lambda\})B(\{\mu\}) &= f(\lambda-\mu)B(\{\mu\})A(\{\lambda\}) + g(\lambda-\mu)B(\{\lambda\})A(\{\mu\}) \ , \\ D(\{\lambda\})B(\{\mu\}) &= h(\lambda-\mu)B(\{\mu\})D(\{\lambda\}) + k(\lambda-\mu)B(\{\lambda\})D(\{\mu\}) \ , \end{aligned} \tag{7}$$

where

$$\begin{array}{ll} f(\lambda) = \frac{\lambda-i}{\lambda} \ , & g(\lambda) = \frac{i}{\lambda} \ , \\ h(\lambda) = \frac{\lambda+i}{\lambda} \ , & k(\lambda) = \frac{i}{\lambda} \ . \end{array} \tag{8}$$

Consider the reference state $\Omega = \prod_{n=N}^{1} \otimes w_n$ with $w_n \equiv |\uparrow\rangle$. On this state we have

$$L_n(\lambda^{(n)}) w_n = \left(\begin{array}{cc} \lambda^{(n)} + i/2 & * \\ 0 & \lambda^{(n)} - i/2 \end{array} \right) w_n \tag{9}$$

with '$*$' denoting operator expressions which are not relevant for us. Thus we have

$$C(\{\lambda\})\Omega = 0 \quad , \quad A(\{\lambda\})\Omega = \alpha^N(\{\lambda\})\Omega \quad , \quad D(\{\lambda\})\Omega = \delta^N(\{\lambda\})\Omega \ , \tag{10}$$

with $\alpha^N(\{\lambda\}) \equiv \prod_{i=1}^{N} \alpha(\lambda^{(i)})$, $\delta^N(\{\lambda\}) \equiv \prod_{i=1}^{N} \delta(\lambda^{(i)})$, $\alpha(\lambda) = \lambda + i/2$ and $\delta(\lambda) = \lambda - i/2$. ¿From the above relations we easily see that Ω is an eigenstate of $F(\{\lambda\}) = A(\{\lambda\}) + D(\{\lambda\})$.

Now, we define vectors of the form

$$\phi(\{\lambda\}) = B(\{\lambda\}_1) \cdots B(\{\lambda\}_l)\Omega \ . \tag{11}$$

Using eqs. (7) we can show that

$$\left[A(\{\lambda\}) + D(\{\lambda\}) - A(\{\tilde\lambda\}) - D(\{\tilde\lambda\})\right] \phi(\{\lambda\}) = \left[\Lambda(\{\lambda\}) - \Lambda(\{\tilde\lambda\})\right] \phi(\{\lambda\}), \tag{12}$$

where $\tilde{\lambda}^{(n)} = \lambda^{(n)} + \psi$ with ψ complex and

$$\Lambda(\{\lambda\}) = \alpha^N(\{\lambda\}) \prod_{k=1}^{\ell} f(\lambda - \lambda_k) + \delta^N(\{\lambda\}) \prod_{k=1}^{\ell} h(\lambda - \lambda_k), \tag{13}$$

if each λ_k satisfies

$$\alpha^N(\{\lambda\}_k) \prod_{m \neq k}^{\ell} f(\lambda_k - \lambda_m) = \delta^N(\{\lambda\}_k) \prod_{m \neq k}^{\ell} h(\lambda_k - \lambda_m) \ . \tag{14}$$

Let us now consider the special T-matrix given by

$$T(\{\lambda\}) = L_{1,a}(\lambda^{(1)}) \cdot \cdot L_{n,a}(\lambda^{(n)}) \cdot \cdot L_{N,a}(\lambda^{(N)}) \tag{15}$$

where

$$\lambda^{(odd)} = \lambda \quad , \quad \lambda^{(even)} = \lambda + i\alpha \tag{16}$$

and

$$T(\{\tilde{\lambda}\}) = L_{1,a}(\tilde{\lambda}^{(1)}) \cdots L_{N,a}(\tilde{\lambda}^{(N)}) \ , \tag{17}$$

for $\tilde{\lambda}^{(n)} = \lambda^{(n)} + \psi$ with $\psi = -i - i\alpha$. Defining the transfer matrix F

$$F(\{\lambda\}) = tr_a \ T(\{\lambda\}) \ , \tag{18}$$

we have computed exactly

$$H = c \, \frac{d}{d\lambda} \left\{ \ln F(\{\lambda\}) - \ln F(\{\tilde{\lambda}\}) \right\} \Big|_{\lambda = i/2} + N(\alpha - 1)(\alpha + 2) \tag{19}$$

for $c = -i(\alpha - 1)(\alpha + 1)$, using Maple and Mathematica softwares (it can be used any algebraic computation software), for 4 and 6 sites and we have obtained

$$\begin{aligned} H &= \sum_{n=1}^{2M} \vec{\sigma}_n . \vec{\sigma}_{n+1} - \frac{\alpha^2}{2} \sum_{n=1}^{2M} \vec{\sigma}_n . \vec{\sigma}_{n+2} + \frac{i\alpha}{2} \sum_{n=1}^{2M} (-1)^n \varepsilon^{ijk} \sigma_n^i \sigma_{n+1}^j \sigma_{n+2}^k + \\ &+ \ N \, \frac{(7\alpha^2 + 4\alpha - 8)}{8} \ , \end{aligned} \tag{20}$$

with $\sigma_{n+N}^i = \sigma_n^i$, ε^{ijk} the totally antisymmetric Levi-Civita tensor, $M = N/2$.

In two cases the Hamiltonian in (20) can be computed exactly for arbitrary even $N \geq 4$: for $\alpha \gg 1$ and α infinitesimal. In that cases we obtain exactly the asymptotic values of the Hamiltonian in (20) (see appendix for the computation). As a consequence of the asymptotic analysis if there is any other term in (20) for $0 < \alpha < \infty$ and arbitrary even $N \geq 4$, this extra term would contribute to the matrix element of H for $\alpha \to 0$ (or $\alpha \to \infty$) as $h_{(i,j)}^{extra} \to \alpha^{r_{(i,j)}}$ for $1 < r_{(i,j)} < 2$. But, since by construction the matrix elements of T are polynomials with integer powers in α, the matrix elements of H as computed from eqns. (15-19) are in general fractions of polynomials with integer powers in α. Thus, as α goes to zero or infinity these matrix elements of H behave as $\alpha^{n_{(i,j)}}$ where $n_{(i,j)}$ are integers, implying that the extra terms are zero. Then, the Hamiltonian obtained in eq. (20) is the general result for arbitrary even $N \geq 4$.

The above Hamiltonian is the periodic Majumdar-Ghosh Hamiltonian plus a $SU(2)$-invariant charge-like interaction term. The Majumdar-Ghosh Hamiltonian is conjectured not to be integrable [9] and what comes out from our calculation is that, the additional charge-like interaction term we find in eq. (20) is essential to render it integrable.

Let us call the third term of eqn. (20) as

$$Q = \alpha \sum_{n=1}^{2M} (-1)^n \varepsilon^{\mu\nu\rho} \sigma_n^\mu \sigma_{n+1}^\nu \sigma_{n+2}^\rho \ .$$

Q is hermitian for $\alpha =$ pure imaginary. For complex α (not pure imaginary) we must make projections on the space of states of the system in order to obtain a model with a hermitian Hamiltonian. This projection will select among the eigenstates of the Hamiltonian given by the first two terms in eqn. (20) for complex α (not pure imaginary) those states belonging to the sector of zero eigenvalue of Q.

The case where $\alpha = i$ is particularly interesting since the ground state of the first two terms in eqn. (20) is known exactly and it has a dimerized form [4], [11]. If we introduce the notation for the singlet pair as

$$[l, m] \equiv \frac{1}{\sqrt{2}} (|\uparrow\rangle_l \otimes |\downarrow\rangle_m - |\downarrow\rangle_l \otimes |\uparrow\rangle_m) \tag{21}$$

and define $V_1(N)$ and $V_2(N)$ as

$$\begin{aligned} V_1(N) &\equiv [1,2][3,4][5,6]\cdots[N-1,N] \ , \\ V_2(N) &\equiv [2,3][4,5][6,7]\cdots[N,1] \ , \end{aligned} \tag{22}$$

we know that $V_{1,2}(N)$ are ground states of the first two terms of eqn. (20) for $\alpha = i$. It is possible to verify that there is no $c_{1,2}$ given in

$$V(N) = c_1 V_1(N) + c_2 V_2(N) \tag{23}$$

such that $V(N)$ is an eigenstate of Q. Then, the ground state of the Majumdar-Ghosh model at the Majumdar-Ghosh point (eqn. (20) with the first two terms for $\alpha = i$) does not remain the ground state of the Hamiltonian given in eqn. (20) for $\alpha = i$.

As this Hamiltonian is derived from the transfer matrix $F(\{\lambda\})$ defined by eqs. (18-19) it can be diagonalized as discussed previously with eigenvectors

$$\phi(\{\lambda\}) = B(\{\lambda\}_1) \cdot\cdot B(\{\lambda\}_l)\Omega \tag{24}$$

where $\lambda_k, \ k = 1, \cdot\cdot, \ell$, satisfy the algebraic Bethe ansatz equations

$$\frac{(\lambda_k + i/2)^M}{(\lambda_k - i/2)^M} \frac{(\lambda_k + i/2 + i\alpha)^M}{(\lambda_k - i/2 + i\alpha)^M} = \prod_{m\neq k}^{\ell} \frac{\lambda_k - \lambda_m + i}{\lambda_k - \lambda_m - i} \ , \tag{25}$$

with energy eigenvalue

$$E = (\alpha - 1)(\alpha + 1) \sum_{\beta=1}^{l} \left\{ \frac{1}{\lambda_\beta^2 + 1/4} + \frac{1}{\lambda_\beta^2 + 2i\alpha + (1/2 + \alpha)\ (1/2 - \alpha)} \right\}. \tag{26}$$

Of course if we perform the limit $\alpha \to 0$ eqns. (20, 25 and 26) become the Hamiltonian, BAE and energy eigenvalue of the Heisenberg XXX-quantum-chain respectively.

It is possible to prove that,

$$\begin{aligned} \left[S^3, B(\{\lambda\})\right] &= -B(\{\lambda\}) \;, \\ \left[S^+, B(\{\lambda\})\right] &= A(\{\lambda\}) - D(\{\lambda\}) \;. \end{aligned} \tag{27}$$

Since for the reference state Ω we have

$$S^+\Omega = 0 \quad , \quad S^3\Omega = \frac{N}{2}\,\Omega \;, \tag{28}$$

using eqns. (27) and repeating the procedure that was used to derive the BAE [12] we can show that if the BAE, eq. (25), are satisfied we have

$$S^+\phi(\{\lambda\}) = 0 \;, \tag{29}$$

which means that $\phi(\{\lambda\})$ are all highest weight states.

The connection between the transfer matrix of vertex lattice models and quantum $1D$ spin Hamiltonians with nearest-neighbor interactions is well-known. Several integrable quantum $1D$ spin models with nearest-neighbor interactions are within this framework. We believe that the case analyzed in this letter: acquisition of the integrable model given in eqn. (20) having nearest-neighbor as well as next-nearest-neighbor interactions through a vertex transfer matrix and the first steps in the proof of complete integrability of the model, is not a singular case. We conjecture that, associated to each model in the large class of integrable spin models with nearest-neighbor interactions obtained by means of a transfer matrix of vertex models there is a descendant integrable spin Hamiltonian with nearest-neighbor as well as next-nearest-neighbor interactions obtained using the approach described in this letter.

Finally, it would be interesting to investigate if eqs. (18-19) could be further generalized in order to accommodate integrable descendants with interactions up to nth-neighbor interactions.

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Appendix

In this appendix we are going to prove that asymptotic limits of the Hamiltonian in (20) are obtained using equations (15-19) in two asymptotic cases. Details are given just for λ-part of the transfer matrix in eqn. (19) since the $\hat{\lambda}$-part is obtained in a similar way. Moreover, for simplicity we will focus our analysis on the terms proportional to α.

It is convenient to rewrite the L-matrices for odd and even sites as

$$L_{n,a}(\lambda) = (\lambda - i/2)I_{n,a} + iP_{n,a} \tag{30}$$

and

$$\hat{L}_{n,a}(\lambda) = (\lambda - i/2)I_{n,a} + iP_{n,a} + i\alpha I_{n,a}, \tag{31}$$

where we denote by $L_{n,a_1}(\lambda)$ the L-matrix for odd sites, $\hat{L}_{n,a_1}(\lambda)$ the L-matrix for even sites and $P_{n,a}$ is the twist matrix for quantum and auxiliary spaces indicated by the sub-indices n and a respectively. Moreover, in components these matrices are written as

$$L_{n,a}(\lambda) \longrightarrow L_{\alpha_n}^{\bar{\alpha}_n}(\gamma_n\gamma_{n+1})(\lambda), \tag{32}$$

where $(\alpha_n, \bar{\alpha}_n)$ are the quantum indices and $(\gamma_n\gamma_{n+1})$ the auxiliary indices. Using these notations we write the transfer matrix as

$$F_{\{\alpha\}}^{\{\bar{\alpha}\}}(\{\lambda\}) = \sum_{\{\gamma\}} L_{\alpha_1}^{\bar{\alpha}_1}(\gamma_1\gamma_2)(\lambda)\hat{L}_{\alpha_2}^{\bar{\alpha}_2}(\gamma_2\gamma_3)(\lambda)\cdots L_{\alpha_{N-1}}^{\bar{\alpha}_{N-1}}(\gamma_{N-1}\gamma_N)(\lambda)L_{\alpha_N}^{\bar{\alpha}_N}(\gamma_N\gamma_1)(\lambda) \tag{33}$$

In two cases the Hamiltonian in (20) can be computed exactly for arbitrary even $N \geq 4$. In the first case we take $\alpha \gg 1$ in eqns. (15-19) and since in eqn. (31) the dominant term is the unity in the auxiliary and quantum spaces we trivially obtain, apart a trivial constant that can be easily computed, the isotropic interaction spanning over odd sites. It can be easily verified that the $\tilde{\lambda}$-part of the transfer matrix in eqn. (19) gives the isotropic interaction spanning over even sites.

The second case, obtained by considering α infinitesimal in eqns. (15-19) is less trivial. Consider the transfer matrix when $\lambda = i/2$ for α infinitesimal

$$F(\{i/2\}) = A + \alpha \sum_{i=1}^{N/2-1} B(i), \tag{34}$$

where

$$A_{\{\alpha\}}^{\{\bar{\alpha}\}} = i^N \sum_{\{\gamma\}} P_{\alpha_1}^{\bar{\alpha}_1}(\gamma_1\gamma_2)\cdots P_{\alpha_N}^{\bar{\alpha}_N}(\gamma_N\gamma_1) \tag{35}$$

and

$$B_{\{\alpha\}}^{\{\bar{\alpha}\}}(i) = i^N \sum_{\{\gamma\}} \quad P_{\alpha_1}^{\bar{\alpha}_1}(\gamma_1\gamma_2)\cdots P_{\alpha_{2i-1}}^{\bar{\alpha}_{2i-1}}(\gamma_{2i-1}\gamma_{2i})I_{\alpha_{2i}}^{\bar{\alpha}_{2i}}(\gamma_{2i}\gamma_{2i+1})$$
$$P_{\alpha_{2i+1}}^{\bar{\alpha}_{2i+1}}(\gamma_{2i+1}\gamma_{2i+2})\cdots P_{\alpha_N}^{\bar{\alpha}_N}(\gamma_N\gamma_1). \tag{36}$$

It is easy to see that the inverse transfer matrix for $\lambda = i/2$ in the α infinitesimal case is

$$F^{-1}(\{i/2\}) = A - \alpha \sum_{i=1}^{N/2} B(i). \tag{37}$$

Moreover,

$$\frac{d}{d\lambda}F(\{\lambda\})\Big|_{\lambda=i/2} = \sum_{n=1}^{N} C(n) + \alpha \left(\sum_{i=0}^{N/2-1} D(i) + \sum_{i=1}^{N/2} E(i) \right), \tag{38}$$

where,

$$C_{\{\alpha\}}^{\{\bar{\alpha}\}}(n) = i^{N-1} \sum_{\{\gamma\}} \quad P_{\alpha_1}^{\bar{\alpha}_1}(\gamma_1\gamma_2)\cdots P_{\alpha_{n-1}}^{\bar{\alpha}_{n-1}}(\gamma_{n-1}\gamma_n)\dot{L}_{\alpha_n}^{\bar{\alpha}_n}(\gamma_n\gamma_{n+1})$$
$$P_{\alpha_{n+1}}^{\bar{\alpha}_{n+1}}(\gamma_{n+1}\gamma_{n+2})\cdots P_{\alpha_N}^{\bar{\alpha}_N}(\gamma_N\gamma_1), \tag{39}$$

$$
\begin{aligned}
D^{\{\bar\alpha\}}_{\{\alpha\}}(i) &= i^{N-1}\sum_{n=1}^{N/2}\sum_{\{\gamma\}} P^{\bar\alpha_1}_{\alpha_1}(\gamma_1\gamma_2)\cdots P^{\bar\alpha_{2i}}_{\alpha_{2i}}(\gamma_{2i}\gamma_{2i+1})\dot{L}^{\bar\alpha_{2i+1}}_{\alpha_{2i+1}}(\gamma_{2i+1}\gamma_{2i+2})P^{\bar\alpha_{2i+2}}_{\alpha_{2i+2}}(\gamma_{2i+2}\gamma_{2i+3}) \\
&\quad \cdots P^{\bar\alpha_{2n-1}}_{\alpha_{2n-1}}(\gamma_{2n-1}\gamma_{2n})I^{\bar\alpha_{2n}}_{\alpha_{2n}}(\gamma_{2n}\gamma_{2n+1})P^{\bar\alpha_{2n+1}}_{\alpha_{2n+1}}(\gamma_{2n+1}\gamma_{2n+2})\cdots P^{\bar\alpha_N}_{\alpha_N}(\gamma_N\gamma_1) \qquad (40)
\end{aligned}
$$

and

$$
\begin{aligned}
E^{\{\bar\alpha\}}_{\{\alpha\}}(i) &= i^{N-1}\sum_{n=1,n\neq i}^{N/2}\sum_{\{\gamma\}} P^{\bar\alpha_1}_{\alpha_1}(\gamma_1\gamma_2)\cdots P^{\bar\alpha_{2i-1}}_{\alpha_{2i-1}}(\gamma_{2i-1}\gamma_{2i})\dot{L}^{\bar\alpha_{2i}}_{\alpha_{2i}}(\gamma_{2i}\gamma_{2i+1})P^{\bar\alpha_{2i+1}}_{\alpha_{2i+1}}(\gamma_{2i+1}\gamma_{2i+2}) \\
&\quad \cdots P^{\bar\alpha_{2n-1}}_{\alpha_{2n-1}}(\gamma_{2n-1}\gamma_{2n})I^{\bar\alpha_{2n}}_{\alpha_{2n}}(\gamma_{2n}\gamma_{2n+1})P^{\bar\alpha_{2n+1}}_{\alpha_{2n+1}}(\gamma_{2n+1}\gamma_{2n+2})\cdots P^{\bar\alpha_N}_{\alpha_N}(\gamma_N\gamma_1). \qquad (41)
\end{aligned}
$$

with

$$
\dot{L}^{\bar\alpha_m}_{\alpha_m}(\gamma_m\gamma_{m+1}) = \frac{d}{d\lambda}L^{\bar\alpha_m}_{\alpha_m}(\gamma_m\gamma_{m+1})(\lambda)\Big|_{\lambda=i/2}. \qquad (42)
$$

Now, we are going to compute the product $F^{-1}(\lambda)\frac{d}{d\lambda}F(\{\lambda\})\Big|_{\lambda=i/2}$ from eqns. (37-38). The product of the first terms of the right hand side of eqns. (37-38) is a well known calculation and gives the first term of the right hand side of eqn. (20). It can be verified that, the first term on the right hand side of eqn. (37) times the second term on the right hand side of eqn. (38) plus the second term on the right hand side of eqn. (37) times the terms with n odd in the first term on the right hand side of eqn. (38) gives

$$
i^{2N-1}\sum_{i=0}^{N/2-1}\delta^{\bar\alpha_1}_{\alpha_1}\cdots\delta^{\bar\alpha_{2i}}_{\alpha_{2i}}\left(\delta^{\bar\alpha_{2i+1}}_{\alpha_{2i+2}}\dot{L}^{\bar\alpha_{2i+2}}_{\alpha_{2i+3}}(\alpha_{2i+1}\bar\alpha_{2i+3}) - \delta^{\bar\alpha_{2i+2}}_{\alpha_{2i+1}}\dot{L}^{\bar\alpha_{2i+1}}_{\alpha_{2i+3}}(\alpha_{2i+2}\bar\alpha_{2i+3})\right)\delta^{\bar\alpha_{2i+4}}_{\alpha_{2i+4}}\cdots\delta^{\bar\alpha_N}_{\alpha_N}. \qquad (43)
$$

Moreover, the first term on the right hand side of eqn. (37) times the third term on the right hand side of eqn. (38) plus the second term on the right hand side of eqn. (37) times the n even terms in the first term on the right hand side of eqn. (38) gives

$$
-\,i^{2N-1}\sum_{i=1}^{N/2}\delta^{\bar\alpha_1}_{\alpha_1}\cdots\delta^{\bar\alpha_{2i-2}}_{\alpha_{2i-2}}\dot{L}^{\bar\alpha_{2i}}_{\alpha_{2i}}(\alpha_{2i-1}\bar\alpha_{2i-1})\delta^{\bar\alpha_{2i+1}}_{\alpha_{2i+1}}\cdots\delta^{\bar\alpha_N}_{\alpha_N}. \qquad (44)
$$

It is easy to see that

$$
\begin{aligned}
\delta^{\bar\alpha_{2i+1}}_{\alpha_{2i+2}}\dot{L}^{\bar\alpha_{2i+2}}_{\alpha_{2i+3}}(\alpha_{2i+1}\bar\alpha_{2i+3}) \;-\; &\delta^{\bar\alpha_{2i+2}}_{\alpha_{2i+1}}\dot{L}^{\bar\alpha_{2i+1}}_{\alpha_{2i+3}}(\alpha_{2i+2}\bar\alpha_{2i+3}) = \\
&-\frac{i}{2}\varepsilon^{lmn}\sigma^l_{\alpha_{2i+1}\bar\alpha_{2i+1}}\sigma^m_{\alpha_{2i+2}\bar\alpha_{2i+2}}\sigma^n_{\alpha_{2i+3}\bar\alpha_{2i+3}} \qquad (45)
\end{aligned}
$$

and

$$
\dot{L}^{\bar\alpha_{2i}}_{\alpha_{2i}}(\alpha_{2i-1}\bar\alpha_{2i-1}) = \delta^{\bar\alpha_{2i-1}}_{\alpha_{2i-1}}\delta^{\bar\alpha_{2i}}_{\alpha_{2i}}. \qquad (46)
$$

Using eqns. (39-42) in the λ-part of the transfer matrix in eqn. (19) we obtain the odd sites of the third term and half of the term proportional to α in the fourth term of eqn. (20). The even sites are obtained by a similar computation using the $\tilde\lambda$-part of the transfer matrix in eqn. (19).

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