CBPF-NF-068/98

A Simple Renormalization Scheme in Random Surface Theory

Luiz L.C. Botelho

Departamento de Física
Universidade Federal Rural do Rio de Janeiro
23851–180 – Itaguaí, RJ, Brasil

Abstract

We propose a simple Analytical Renormalization Scheme for the self-avoiding interaction of a Random Surface with the origin.

In last years, it was suggested in ref. [1] a new renormalization precedure for the general theory of Gaussian Manifolds with self-avoiding interaction with a fixed hyper-plane. Unfortunately, the above mentioned renormalization prescription is somewhat intrincate in its application to the important case of Random Surfaces ([4]).

In this Brief Report we propose the use of the Riesz-Analytical Regularization scheme ([2]) to address the problem of renormalization of the Self-Avoiding Random Surfaces with a fixed hyper-plane in a simple way.

Let us start our analysis by considering the Path Integral Expression for the Partition Functional of the Theory of Self-Avoiding Random Surfaces in an extrinsic Euclidean Space R^D interacting with the origin ([1],[4])

$$Z[\lambda_b] = 1/Z[0] \int D^F[X^a(\xi)]\ exp\left\{-\frac{1}{2}\int_{R^2} d^2\xi X^a(-\Delta)^{-\alpha}X_a(\xi)\right\}$$
$$exp\left\{-\lambda_b \int_{R^2} d^2\xi\ \delta^{(D)}(X^a(\xi))\right\} \tag{1}$$

where $\{X^a(\xi), a = 1, \cdots, D\}$ denotes the Random Surface vector position with $\xi \in R^2$ and λ_b the (positive) bare self-avoiding coupling constant (the "exclude volume" case).

It is instructive point out that the formal perturbation expansion around the massless 2D fluctuation field $\{X^a(\xi)\}$ is ill defined due to the severe infrared divergence of the associated Laplacean Green function in the surface parameter space (R^2) ([3]).

$$< X^a(\xi_1)X^b(\xi_2) >= \delta^{ab}\left\{\frac{1}{2\pi}\ell g(|\xi_1-\xi_2|)+\varepsilon\right\}+C \tag{2}$$

where C is an arbitrary constant related to the zero modes of the Laplacean $-\Delta$ in R^2.

At this point we propose our solution for all the above cited problems. We consider the Path integral Eq. (1) to be well defined by means of a distributional limit associated to the Riesz analytical regularized Path integral below

$$Z^{(\alpha)}[\lambda_b] = \int D^F[X^a(\xi)]\ exp\left\{-\frac{1}{2}\int_{R^2} d^2\xi(X^a(-\Delta)^{\alpha}X_a)(\xi)\right\}$$
$$exp\left\{-\lambda_b \int_{R^2} d^2\xi\ \delta^{(D)}(X^a(\xi))\right\} \tag{3}$$

Here $(-\Delta)^\alpha$ is the analytical Seeley-Riesz-Hadamard Power for $\alpha > 1$ of the two dimensional Laplacean with Green function given by ([2])

$$G_\alpha(\xi_1,\xi_2) \equiv (-\Delta)^{-\alpha}(\xi_1,\xi_2) = \frac{e^{-i\pi\alpha}\Gamma(1-\alpha)}{4^\alpha(\pi)^{1/2}\Gamma(\alpha)}|\xi_1-\xi_2|^{2(\alpha-1)}$$
$$= \int d^2k\ e^{ik(\xi_1-\xi_2)}\frac{1}{k^{2\alpha}} \tag{4}$$

and completelly differing from the dimensional (analitical) regularization of ref. [6].

We, thus, define Eq. (1) from Eq. (3) by means of the renormalization prescription

$$\lambda_b = \lambda_r/(1-\alpha)^{D/2} \tag{5}$$

with λ_R denoting the renormalized self-avoiding coupling constant and the distributional (finite-part) limit of the theory partition functional

$$Z_R[\lambda_R] \equiv \lim_{\substack{\alpha\to 1\\ \alpha>1}} Z^\alpha[\lambda_b] \tag{6}$$

Let us show that Eq. (6) is well defined in a formal power expansion in the renormalized coupling constant λ_R eq. (5). In order to show this result, we make the power expansion of the regularized have Partitional Functional

$$Z^\alpha[\lambda_b] = \sum_{N=0}^{\infty} \frac{(-\lambda_b)^N}{N!} Z_N^{(\alpha)} \tag{7}$$

where

$$Z_N^{(\alpha)} = \prod_{j=1}^{N} \int d^2\xi_j \int d^DP_j\ exp\left(-\frac{1}{2}\sum_{(i,j)=1}^{N}(P_i^a P_{j,a})[G_\alpha(\xi_i,\xi_j)]\right) \tag{8}$$

Here $[G_\alpha(\xi_i,\xi_j)]_{\substack{1\le i\le N\\ 1\le j\le N}}$ denotes the $N\times N$ symmetric matrix with the (i,j) element given by Eq. (4) ([1]).

The Gaussian $\{P_k\}$ – integrals in R^D are easily evaluated with the result

$$Z_N = \prod_{j=1}^{N}\int d^2\xi_j\ det^{-D/2}[G_\alpha(\xi_i,\xi_j)] \tag{9}$$

The finitess of Eq. (9) for each N is a straightforward consequence of the following properties (see Appendix A)

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}} det^{-D/2}[G_\alpha(\xi_i,\xi_j)] = (1-\alpha)^{+\frac{ND}{2}}C_N(1) \tag{10}$$

$$\lim_{\alpha\to 1}\ G_\alpha(\xi_i,\xi_i) \equiv 0 \tag{11}$$

(here $C_N(1)$ is a ξ-independent constant) and our minimal "finite-part" renormalization prescription Eq. (5).

As a consequence of the analysis above exposed we obtain our finite result for the renormalized Partition Functional Eq. (6)

$$Z_R[\lambda_R] = \sum_{N=0}^{\infty} \frac{(-\lambda_R)^N}{N!} C_N(1).A^N = \lim_{\alpha\to 1} Z^{(\alpha)}[\lambda_r/(1-\alpha)^{\frac{D}{2}}] < \infty \tag{12}$$

with $A = \int d^2\xi$ denoting the internal random surface area and $C_N(1) = \frac{e^{-\pi_i N\alpha}}{4^{N\alpha}.\pi^{N/2}} \times (-1)^N \times (1-N)$ (see the Appendix).

In the general case of self-avoiding interaction with the tangent plane at the surface point $X^a(\xi)$, namelly: $T_a(\xi) = t_a^0 \cdot \xi_0 + t_a^{(1)}\xi_1 + X^a(\bar{\xi})$, where $\{t_a^{(0)}, t_a^{(1)}\}$ are the surface tangent vectors at $X^a(\bar{\xi})$ ($t_a^{(0)} = \partial_{\xi_0} X_a(\bar{\xi}) \cdot \partial_{\xi_0} X_a(\bar{\xi})$; $t_a^{(1)} = \partial_{\xi_1} X_a(\bar{\xi}) \cdot \partial_{\xi_1} X_a(\bar{\xi})$; ([4]) the associated Partition Functional path integral is now written in the following form

$$Z[\lambda_b] = \frac{1}{Z(0)} \int D^F[X^a(\xi)] exp\left\{-\frac{1}{2}\int_{R^2} d^2\xi X^a(X^a(-\Delta)X_a)(\xi)\right\}$$
$$exp\left\{-\lambda_b \int_{R^2} d^2\xi \; \delta^{(D)}(X^a(\xi) - T^a(\xi))\right\} \tag{13}$$

Let us point out that it takes the same form of eq. (1), after the variable change $X^a(\xi) \to X^a(\xi) - T^a(\xi)$, since

$$D^F[X^a(\xi)] = D^F[X^a(\xi) - T^a(\xi)] \tag{14}$$

and

$$exp\left\{-\frac{1}{2}\int_{R^2} d^2\xi (X^a - T^a)(\xi)\Delta(X_a - T_a)(\xi)\right\} = exp\left\{-\frac{1}{2}\int_{R^2} d^2\xi X^a(\xi)\Delta\; X_a(\xi)\right\} \tag{15}$$

Finally, we point out that in the general self-avoiding case $\delta^D(X^a(\xi) - X^a(\xi'))$, one should first proceed as in ref. [5] to renormalize by pure geometrical procedure, the case $X^a(\xi) = X^a(\xi')$ with $\xi = \xi'$ and consider, thus, the above cited tangent plane interaction eq. (13) for the remaining non-trivial self-avoiding interaction supported now on surface self-interaction lines.

Acknowledgements

This research was supported by CNPq - Brasil. The author is grateful to the referee to point out the general result of eq. (A04) of the Appendix A.

References

[1] B. Duplantier, Phys. Rev. Lett. 62, 2337 (1989).
F. David, B. Duplantier and E. Guilter, Phys. Rev. Lett. 70, 2205 (1993).
B. Duplantier, T. Hwa, M. Kardar, Phys. Rev. Lett. 64, 2022 (1990).

[2] I.M. Guelfan, G.E. Chilov, Les Distributions, vol. 1 (Dunod–Paris), 1962.

[3] Y. Kantor, M. Kardar and D.R. Nelson, Phys. Rev. A35, 3050 (1987).

[4] Luiz C.L. Botelho, J. Math. Phys. 30, 2160 (1989).
Phys. Rev. 41D, 3283 (1990).
Phys. Rev. 52D, 6941 (1995).
Phys. Rev. 56D, 1338 (1997).

[5] Luiz C.L. Botelho, Brazilian Journal of Physics, vol. 22, n. 4, 323 (1992).
Mod. Phys. Lett. 6B, n. 4, 203 (1992).

[6] F. David, B. Duplantier and J. Guitter, Nucl. Phys. B394 (1993), 555-664.

Appendix A

In this Appendix we present detailed calculations leading to eq. (10) in the text.

Firstly,

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}}[G_\alpha(\xi_1,\xi_1)] = \lim_{\substack{\alpha\to 1\\ \alpha>1}}\left\{\frac{e^{-i\pi\alpha}\Gamma(1-\alpha)}{4^\alpha\pi^{\frac{1}{2}}\Gamma(\alpha)}(0)^{2(\alpha-1)}\right\} = 0 = C_1(1) \tag{A.1}$$

Secondly,

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}} \det_{N=2}\begin{bmatrix} G_\alpha(\xi_1,\xi_1) & G_\alpha(\xi_1,\xi_2)\\ G_\alpha(\xi_2,\xi_1) & G_\alpha(\xi_2,\xi_2)\end{bmatrix} = \lim_{\substack{\alpha\to 1\\ \alpha>1}}\{(0\times 0) - G_\alpha(\xi_1,\xi_2)G_\alpha(\xi_2,\xi_1)\}$$

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}}\left[\frac{-e^{-2\pi i\alpha}}{4^{2\alpha}\cdot\pi}\left(\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)}\right)^2\right](|x_1-\xi_2||\xi_2-\xi_1|)^{2(\alpha-1)}$$

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}}\frac{(|\xi_1-\xi_2||\xi_2-\xi_1|)^0}{16\pi(1-\alpha)^2} = \frac{C_2(1)}{(1-\alpha)^2} \tag{A.2}$$

Thirdly,

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}} \det_{N=3}\begin{bmatrix} 0 & G_\alpha(\xi_1,\xi_2) & G_\alpha(\xi_1,\xi_2)\\ G_\alpha(\xi_2,\xi_1) & 0 & G_\alpha(\xi_2,\xi_3)\\ G_\alpha(\xi_3,\xi_1) & G_\alpha(\xi_3,\xi_2) & 0\end{bmatrix} =$$

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}}\{-G_\alpha(\xi_1,\xi_2)(-G_\alpha(\xi_2,\xi_3)G_\alpha(\xi_3,\xi_1))G_\alpha(\xi_1,\xi_3)(G_\alpha(\xi_2,\xi_1)G_\alpha(\xi_3,\xi_2))\} =$$

$$\frac{-e^{-3\pi i\alpha}}{4^{3\alpha}\cdot\pi^{\frac{3}{2}}}\left(\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)}\right)^3(1+1) = \frac{e^{-\pi i}}{32\pi^{\frac{3}{2}}}\cdot\frac{1}{(1-\alpha)^3} = \frac{C_3(1)}{(1-\alpha)^3} \tag{A.3}$$

Finally,

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}} \det_{N\times N}[G_\alpha(\xi_i,\xi_j)] = \frac{e^{-N\pi i\alpha}}{4^{N\alpha}\cdot\pi^{\frac{N}{2}}}\cdot\frac{1}{(1-\alpha)^N}\, det[A_{i,j}] \tag{A.4}$$

where $[A_{i,j}]$ is the matrix where entries are

$$[A_{i,j}] = \begin{cases} 0 & \text{if} \quad i\neq j\\ 1 & \text{if} \quad i\neq j\end{cases}$$

which has the general result

$$det[A_{i,j}] = -(N-1)(-1)^N \tag{A.5}$$

It is worth to remark the convergence of eq. (12) for all values of the internal random surface area A_∞.