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Quantum Cosmology*

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Abstract

In these lectures we would like to introduce some basic ideas on the quantization of general relativity with emphasis on the application of the resulting theory to cosmology, which originates an area called quantum cosmology.

Quantum cosmology has been developed over the years, by many physicists, and some interesting results have been derived. In order to show some of these results and also to solidify the basic ideas introduced, in the first part of these lectures, we shall examine a simple quantum cosmology model.

Key-words: Quantum general relativity; No-boundary wave function; Pseudo-spherical universes

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Chapter 1

Introduction to quantum general relativity.

1.1 Introduction.

Many physicists have contributed since P. A. M. Dirac [1], towards a consistent quantized version of general relativity. In the hands of J. A. Wheeler [2], and B. S. DeWitt [3], all this material led to the first attempt on the quantization of a cosmological model.

The application of the quantum theory of general relativity to the study of models of the universe is given the generic name of quantum cosmology. Although this is a very important application of the theory, recently the study of the quantum theory of other spacetime configurations has been developed. We may mention the study of the interior of Schwarzschild black holes [4], and the Euclidean section of these black holes [5], [6] and [7]. Here we shall restrict our attention to quantum cosmology.

In this Chapter 1, we shall study few basic points of the formalism of quantum general relativity and the 'no-boundary' boundary conditions [8]. In Sec. 1.2, the variables of the gravitational field that one has to quantize, are identified. The path integral approach of general relativity is discussed in Sec. 1.3. Finally, in Sec. 1.4 we introduce the 'no-boundary' proposal.

1.2 Variables to be quantized.

The most important concept introduced by Wheeler and DeWitt is the superspace S. For Wheeler, superspace is the arena for geometrodynamics. In order to understand its meaning one has to identify the variables describing the gravitational field. Wheeler [2], emphasized that a 'dynamical' description of general relativity can be given if one splits the full four-dimensional metric tensor $\mathbf{g}$, in three parts [9], [10]. This split is known as the ADM formalism of general relativity.

The first part, a three-dimensional tensor $\mathbf{h}$ which includes the physical degrees of freedom of the gravitational field, is the metric of space-like hypersurfaces. The 'evolution' of $\mathbf{h}$, in order to recover $\mathbf{g}$, is determined by the two remaining parts of $\mathbf{g}$: a three-dimensional vector $\mathbf{N}$, the shift vector, and a scalar N, the lapse function. $\mathbf{N}$ and N are not physical degrees of freedom of the gravitational field and their presence in $\mathbf{g}$ is due to the invariance of the theory under coordinate transformations [9], [10].

It is natural, then, that for a 'geometrodynamical' description of the theory, very important for the canonical quantization approach, one identifies the superspace elements with the metrics $\mathbf{h}$. In a more mathematical definition, each element of S is an equivalent class of metrics $\mathbf{h}$ that are transformable, one into another, by diffeomorphisms [2].

The superspace, as defined above, has an infinite number of components which in a loose estimate by Wheeler, add up to $(\infty^3)^{\infty^3}$ [2]. The metrics composing S are not restricted to be regular. Indeed, for D. Brill, singular metrics are essential for the description of topology changes [11].

As pointed out by DeWitt [3], as one integrates in the spatial coordinates second derivative terms appearing in the gravitational action, in order to obtain a first order theory, one get surface terms which are expressions of the total gravitational energy. By simplicity, although some work has been done in this area [4], [12], we shall restrict our attention to compact space-like hypersurfaces which eliminates all surface terms in the spatial directions. In particular, it means that the total gravitational energy of these spacetimes are nill.

Therefore, based upon the above discussion and also in a deeper study [13] one may identify the variables to be quantized as the metrics $\mathbf{h}$. More precisely, the equivalence classes of metrics $\mathbf{h}$, mentioned above.

1.3 Path integral quantization.

1.3.1 The formalism

In Hawking's point of view, the two important motivations for the investigation of the general relativity path integral approach are:
(i) the implementation of a quantum description of general relativity displaying a greater equivalence between space and time, than the canonical quantization; and
(ii) the possibility of having more complex four-dimensional topologies, than the ones furnished by the canonical quantization, contributing to transition amplitudes.

We start by writing the amplitude Ψ, to go from a three-dimensional hypersurface M_1 with a metric h^1_{ij} and matter fields ϕ_1 to another three-dimensional hypersurface M_2 with a metric h^2_{ij} and matter fields ϕ_2, as a sum of all field configurations g and ϕ which take the given values on M_1 and M_2,

$$\Psi \equiv < h^2_{ij}, \phi_2 | h^1_{ij}, \phi_1 > = \int D[g_{\alpha\beta}, \phi] \exp(iS[g_{\alpha\beta}, \phi]), \tag{1.1}$$

where $D[g_{\alpha\beta}, \phi]$ is a measure on the space of all field configurations $g_{\alpha\beta}$ and ϕ; $S[g_{\alpha\beta}, \phi]$ is the action for the fields; and the integral is taken over all fields which have the given values on M_1 and M_2.

The action $S[g_{\alpha\beta}, \phi]$, is given by the sum of two factors: one for the gravitational field (S_G), and other for matter and/or gauge fields present in the model (S_M).

The action for the gravitational field, including boundaries, has the expression [14]:

$$S_G = \frac{1}{16\pi}\int_M (R - 2\Lambda)(-g)^{1/2}\, d^4x + \frac{1}{8\pi}\int_{\partial M} K h^{1/2}\, d^3x, \tag{1.2}$$

where R is the curvature scalar, Λ is the cosmological constant, g is the determinant of the four-dimensional metric, K is the trace of the extrinsic curvature or second fundamental form of the boundaries, and h is the determinant of the three-dimensional metrics induced on the boundaries.

The specific expression for the action of the matter and/or gauge fields S_M, will depend on the particular fields one chooses to couple to gravity. Its generic form in terms of the density of Lagrangian L_m, of those fields is:

$$S_M = \int L_m\, (-g)^{1/2}\, d^4x. \tag{1.3}$$

1.3.2 Euclideanization.

Hawking and Gibbons proposed in [15], for the first time, the application of the Wick rotation (in other words the transformation $t \to -i\tau$) in the study of quantum gravitational systems. It means that they were interested only in space-times with the Euclidean signature $(+ + ++)$, rather than the Lorentzian one $(- + ++)$. They had two basic motivations:
(i) The hope of transforming the oscillatory terms in the path integral in convergent ones; and
(ii) The formal analogy between the expressions for the Euclidean path integral and for the statistical mechanical canonical ensemble partition function [16].

Even though the analytic continuation of the Lorentzian time proved to be a success in quantum field theory [17], some problems appeared in its application for the gravitational case. The most important of them, as pointed out by Hawking [15], is that the gravitational field differs from other fields by the fact that its action is not positive definite, not even for real positive-definite metrics. Indeed, the gravitational action can be made arbitrarily negative by the use of conformal transformations.

One solution to this problem was given by Gibbons, Hawking and M. Perry [18], who proposed the analytic continuation of the conformal factor and the positive action conjecture. This conjecture asserts that: any asymptotically Euclidean, positive-definite metric with $R = 0$, has positive or zero action. This result has two important limitations: it constraints the spacetimes to have zero curvature scalar, and it does not apply to space-times with compact spatial hypersurfaces [19].

In order to deal with the case of compact hypersurfaces, J. J. Halliwell and J. B. Hartle have proposed a more pragmatic procedure to identify physically and mathematically consistent contours of integration for (1.1) [19]. Among the criteria proposed by them one can mention the following ones: the integral defining transition amplitudes (1.1), should converge; and the wave-function should imply classical space-time on familiar scales, when the universe is large. Another important requirement of the Halliwell and Hartle proposal is the use of complex metrics [19]. This proposal has been successfully used by some authors in the quantization of some models of the universe, see [20] (and references therein).

Another important motivation for the Wick rotation is the possibility of topology changes of compact three-dimensional hypersurfaces [21]. Unlike the Lorentzian case, where topology changes imply the development of closed time-like curves [22], and the impossibility of an $SL(2, C)$ spinor structure [23], one has not found yet comparable constraints for topology changes in Euclidean spacetimes [21]. This property of Euclidean space-times gives a more definite meaning to Wheeler's idea of taking in account different topologies when computing the wave-function of the Universe at the Planck scale (space-time foam) [2].

We may re-write the transition amplitude Ψ (1.1), after applying the analytic continuation to Euclidean metrics,

$$\Psi(h_{ij}^2, \phi_2; h_{ij}^1, \phi_1) = \sum_M \int D[g_{\alpha\beta}^E, \phi] \exp(-I[g_{\alpha\beta}^E, \phi]), \tag{1.4}$$

where,

$$\begin{aligned} I[g_{\alpha\beta}^E, \phi] &= -\frac{1}{16\pi} \int_M (R^E - 2\Lambda)(g^E)^{1/2}\, d^4x - \int_M L_m^E (g^E)^{1/2}\, d^4x \\ &- \frac{1}{8\pi} \int_{\partial M} K h^{1/2}\, d^3x. \end{aligned} \tag{1.5}$$

The superscript E in the last two equations, means that they are evaluated using Euclidean metrics, and the sum is over manifolds of different topologies.

1.4 The 'no-boundary' boundary conditions.

In the case of quantum general relativity of closed spatial hypersurfaces, we have seen how to compute quantum states of given systems using the path integrals (1.1) and (1.4). Due to the mathematical nature of path integrals, the precise determination of those quantum states requires the specification of boundary conditions. Those boundary conditions must be given for two different spatial hypersurfaces and they must define, there, the gravitational and other fields.

One set of boundary conditions appropriate for the Euclidean path integral approach is the 'no-boundary' proposal, due to Hartle and Hawking [8]. It is a set of instructions on how to write down the ground-state wave-function of the gravitational system under investigation. As explained by Hartle and Hawking [8], in the case of quantum general relativity of closed spatial hypersurfaces the relationship between the ground-state and a state of lowest energy does not exist. Indeed, for these hypersurfaces the total energy is zero [3].

Then, another option is trying to use a different property of gravitational systems in order to define the ground-state wave-function. Hartle and Hawking selected the geometry to be such property. The ground-state wave-function must be, then, defined as a path integral over geometries of high symmetry because intuitively these geometries should give rise to a state of minimum excitation [8].

Another feature of classical gravitational systems one generally has to take in account is the presence of singularities of the gravitational field. These singularities may be of cosmological or astrophysical nature. One of the main motivations of the quantum version of general relativity is to try removing these singularities, in analogy with the hydrogen atom case where the quantum theory removed the singularity present in the classical theory.

With these two conditions in mind, Hartle and Hawking proposed that the Euclidean path integral should be performed over four-dimensional, regular, geometries which have only one boundary, being compact. Any other field present in the model should also be everywhere regular, assuming specified values at the boundary M_b:

$$\Psi_{NB}[h^b_{ij}, \phi_b] \; = \; \sum_M \int_C D[g^E_{\alpha\beta}, \phi] \; \exp(-I_{NB}[g^E_{\alpha\beta}, \phi]), \tag{1.6}$$

where I_{NB} is derived from (1.5), by using the set of geometries (C) allowed by the 'no-boundary' proposal.

Thinking in terms of the three-dimensional spatial hypersurfaces one may interpret this wave-function as being the probability amplitude for a given hypersurface to evolve from nothing. As pointed out by Hartle and Hawking, in the case of models where the cosmological singularity ('big bang') is present, the above proposal would remove altogether the singularity.

The 'no-boundary' proposal has been shown to furnish enough conditions to fix the general relativity wave-function, at least in some minisuperspace models. For a few examples and a literature of the applications of this proposal for quantum cosmological minisuperspace models see [20]. The 'no-boundary' boundary conditions have also been used in the study of the Schwarzschild black hole Euclidean sector [5], [6] and [7].

The application of the 'no-boundary' boundary conditions to some models of the universe has not been restricted to the issue of mathematical consistency. Some predictions have also been made:

(i) It was shown that some minisuperspace wave-functions correspond, in the classical limit, to a family of classical solutions which have a long period of exponential or 'inflationary' expansion [24], [25];

(ii) As another testable prediction of this proposal, Hawking and Page showed that in minisuperspace models the probability distribution for the density parameter Ω, at a given density, is entirely concentrated at $\Omega = 1$. Thus, the Universe should have exactly the critical density [26];

(iii) Using a model which takes in account the full infinite-dimensional superspace, Halliwell and Hawking have demonstrated how a 'no-boundary' wave-function may develop a scale-free spectrum of density perturbations, which could account for the origin of the galaxies and other structures in the Universe [27];

(iv) The behavior of the density perturbations, mentioned above, can explain the existence and direction of the thermodynamic arrow of time [28];

(v) Gibbons and Hartle devised some general statements about the predictions of the 'no-boundary' wave-function for the present large scale geometry and topology of the Universe [29]. They argued that among the complex solutions of the Euclidean Einstein's equations, important in the semi-classical limit, the real tunneling ones play a special role. The simplest example of these solutions occurs in a deSitter model, with a positive cosmological constant and no matter or gauge fields. In this model, the real tunneling

solution is given by the joining of an Euclidean metric with a Lorentzian metric along a surface characterized by the vanishing of its second fundamental form. One important point is that the Euclidean region is related to the early Universe, in the sense that the scale factor assumes its smallest value there. After that, it grows up until one well defined value which depends on the cosmological constant. On the other hand, the Lorentzian region has the property of starting off with the final scale factor from the foregoing region and increases up to the value assigned by the final boundary condition;
(vi) Finally, S. Coleman proposed a mechanism which by using the 'no-boundary' wave-function of an universe composed of many wormholes at high energies, one may modify the physical constants of this universe at much lower energy scales [30]. In particular, he demonstrates that this formalism yields a probability distribution which is infinitely peaked in a region of superspace where the cosmological constant is zero.

The semi-classical approximation to the 'no-boundary' wave-function (1.6) may be written in the following way,

$$\Psi_{NB}[h^b_{ij}, \phi^b] \;=\; N_0 \sum_i A_i \exp(-I_i), \tag{1.7}$$

where N_0 is a normalization constant, and I_i are the actions of the Euclidean-Einstein's field equations. These solutions are compact and have the given three-metric h^b_{ij} and matter and/or gauge fields configuration ϕ_b on the boundary. The prefactors A_i are given by determinants of small fluctuations about the classical solutions.

In the next chapter we shall compute the 'no-boundary' wave-function of a simple model of the universe. There we shall see some of the above mentioned predictions.

Chapter 2

The 'no-boundary' wave-function for pseudo-spherical universes.

2.1 Introduction.

In this chapter we are interested in compute the 'no-boundary' wave-function for universes of negative constant curvature in a consistent way, such that we could, in the future, use them to describe a more complete picture, which would take in account contributions coming from general topologies. Examples of 'no-boundary' wave-functions, for spacetimes with constant positive and zero curvatures, have already been examined [8], [31].

We shall restrict our attention to the semi-classical approximation and we shall follow the logic of the ADM formalism. It means that we shall split the four-dimensional manifold in compact, three-dimensional, hypersurfaces with metric components q^{α} and a lapse function N [20].

In order to compute the 'no-boundary' wave-function for pseudospherical universes, first, In Sec. 2.2, we introduce a specific metric ansatz to foliate the spacetime of constant negative curvature. Then, we derive the solutions of the classical field equations for the chosen metric. The compactification of the spatial sections of the solutions is done in Sec. 2.3. Sec. 2.4 is devoted to the regularity analysis of those solutions. Finally, in Sec. 2.5, the 'no-boundary' wave-function for pseudo-spherical universes is shown.

2.2 Metric ansatz and solutions to the Euclidean - Einstein's equations.

The general expression for the 'no-boundary' wave-function for the universe [8] is simplified by the assumptions that the universe is homogeneous, isotropic and its unique source of mass-energy is a negative cosmological constant. For simplicity we shall consider the (2+1)-dimensional case. This leads to a metric ansatz, with the lapse function N and the scale factor a depending only on t and the spatial sections being two-dimensional surfaces with constant negative curvature:

$$ds^2 \; = \; + N^2(t)dt^2 \; + \; a^2(t)\,[\,d\chi^2 \; + \; \sinh^2\chi d\theta^2\,] \tag{2.1}$$

In order to obtain the semi-classical Ψ_{NB} we must solve the Euclidean-Einstein's equations for (2.1),

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0 \qquad (\Lambda < 0), \tag{2.2}$$

subjected to the 'no-boundary' conditions.

For our case they are given by [32]:
(A)The manifold must be regular at every point;
(B) There should be a point where the scale factor vanishes;
(C) We should supply the real valued scale factor of the final spatial slice.
The first condition we shall tackle after the other two are implemented. The second one is a particularity of the minisuperspace treatment, and we shall choose the point to be the zero value of the time scale. With this choice we shall be able to picture our universe starting at t=0, from this surface of zero volume and evolving until $t = t_1$, where we shall furnish the other required value of the scale factor, say, $a(t_1) = a_1$. Introducing those quantities in the above equation (2.2), and rescaling the time in order that $t_1 = 1$, we get the two solutions given below.

The Lorentzian Solution:

$$N = iN_I, \quad \text{where} \quad N_I = \alpha\frac{\pi}{2} \pm \arccos(a_1/\alpha); \tag{2.3}$$

and

$$a(t) = \pm\alpha \sin(N_I t/\alpha); \quad \alpha^2 = \frac{1}{|\Lambda|}; \tag{2.4}$$

valid for $a_1 < \alpha$. It is important to notice that among the various solutions for N, labeled by an integer m, we have chosen as a matter of simplicity, the case $m = 0$ (the same remark holds for the complex solution given below). This spacetime (2.3), (2.4) is know in the literature as anti-deSitter spacetime, but we must add that our solution does not cover the whole of anti-deSitter and there are other coordinate charts which perform this task in a more complete way [33].

The complex solution:

$$N = N_R + iN_I, \tag{2.5}$$

where

$$N_I = \alpha\frac{\pi}{2} \quad ; \quad N_R = \alpha \arcsin\beta \quad ; \quad \beta = \mp\sqrt{(\frac{a_1}{\alpha})^2 - 1} \tag{2.6}$$

and

$$a(t) = a_R(t) + ia_I(t) \tag{2.7}$$

where

$$\begin{aligned} a_R(t) &= \mp\alpha \sin(N_I t/\alpha) \cosh(N_R t/\alpha) \\ a_I(t) &= \pm\alpha \cos(N_I t/\alpha) \sinh(N_R t/\alpha) \end{aligned} \tag{2.8}$$

and this solution holds for $a_1 > \alpha$.

One may show that the complex solution (2.5)-(2.8) is a signature change space-time of the type $(- + ++) \rightarrow (+ + ++)$ [34]. Next to $t = 0$, this solution is an anti-deSitter space-time.

2.3 Compactification of the spatial sections.

¿From the line element (2.1) we see that the spatial sections are pseudo-spheres, or H^2's, which are open.

One way to compactify the H^2 is projecting it onto a plane, which results in a disc called the Poincaré Disc (P.D.) [35]. The conformal transformations which takes one from the coordinates (χ, θ) to the polar coordinates (r, θ) on the P.D., are,

$$r = \sqrt{\alpha^2 - v_0^2}\tanh(\chi/2); \qquad \theta \quad \text{is unchanged}, \tag{2.9}$$

where

$$v_0 = \sqrt{\alpha^2 - a^2(t)}. \tag{2.10}$$

This mapping from the H^2 to the P.D. induces a non-trivial line element on the disc, given by,

$$ds^2_{PD} = 4(\alpha^2 - v_0^2)^2 \frac{(dr^2 + r^2 d\theta^2)}{(\alpha^2 - v_0^2 - r^2)^2}. \tag{2.11}$$

The next step in the compactification process, is the selection of a fundamental region (in general a polygon) which tessellates [36] the P.D., and may be transformed into a double torus. Then, we shall take this fundamental region and construct the double torus, or in other words, the quotient surface H^2/Γ, where Γ is a certain symmetry transformation of the H^2.

In two dimensions, the unique symmetry transformations of the H^2 capable of generating compact surfaces are the hyperbolic ones [37], in other words, Lorentzian boosts. Our compact surface will be fabricates by the identification of the sides, through the relevant boost transformations, of the above selected polygon.

The simplest fundamental region which gives rise to a double torus is the regular octagon [35]. If we now identify the opposite sides of this polygon, with the aid of four different boost transformations (one for each relevant direction), we obtain a representation of the desired compact surface.

Therefore, the spatial sections of our space-time will be given by double torus evolving in time, and each having the line element (2.11).

2.4 Regularity Analysis.

In this section, we shall study the regularity of the Lorentzian and the complex solutions introduced in Sec. 2.2.

We start by writing down the line element of the space-time described by the evolution of the double tori, in Cartesian coordinates, with the aid of eqs. (2.1) and (2.11),

$$ds^2 = -N^2 dt^2 + \frac{4(\alpha^2 - v_0^2)^2 (dx^2 + dy^2)}{(\alpha^2 - v_0^2 - x^2 - y^2)^2}. \tag{2.12}$$

Where the ranges of x and y will depend on the octagon dimensions; the opposite sides of the octagon are identified; $t \in [0, 1]$; and N was left generic in order to account for both Lorentzian and complex solutions.

Observing eqs. (2.1), (2.4), (2.7) and (2.8), we notice that both the Lorentzian and complex solutions have singularities at $t = 0$. This singularity can, at most, be conical because this space-time has an everywhere constant curvature scalar (R),

$$R \;=\; -\,6\,|\Lambda|\,. \tag{2.13}$$

In fact, it is known that this singularity is a coordinate singularity [33].

The same conclusion does not directly apply to the space-time with line element (2.12) because at the event $t = 0$ ($v_0^2 = \alpha^2$), due to the identifications mentioned after eq. (2.12), a conical singularity may have formed.

Let us, then, apply the holonomy method [38] in order to investigate the presence, or not, of a conical singularity in the space-time with line element (2.12), at the event $t = 0$.

2.4.1 The holonomy method.

The holonomy method aims at identifying if a given singularity is conical or not.

In order to apply this method to our case one has to follow the below steps:
(i) Construct closed loops, taking in account the identifications, for different values of t;
(ii) Parallel transport a test vector, v, around the closed loops introduced in (i);
(iii) Write down the holonomy matrix (HM). This matrix relates the values of v at a given point p, before and after the parallel transport of v around closed loops which contain p;
(iv) Finally, take the limit as $t \to 0$ (singular event) of the closed loops. This limit implies a certain limit value for the HM. If, in this limit, HM goes to the identity we say that $t = 0$ is a regular event, otherwise we say that it is a conical singularity.

Following the instructions of the holonomy method, we start by writing down the parallel transport equations for a test vector v, along a closed loop (to be specified), in the space-time represented by eq. (2.12).

The easiest way to derive them, from eq. (2.12), is by working on the orthonormal basis, defined by the transformations,

$$w^t \;=\; dt\,; \qquad w^i \;=\; \frac{2\,(\alpha^2 - v_0^2)}{\alpha^2 - v_0^2 - x^2 - y^2}\,dx_i\,, \tag{2.14}$$

where $i = x, y$ and $x_i = x, y$.

The non-vanishing connection coefficient components in this basis are,

$$\Gamma_{txx} \;=\; \Gamma_{tyy} \;=\; \frac{-2\dot{v}_0 v_0 (x^2 + y^2)}{(\alpha^2 - v_0^2)(\alpha^2 - v_0^2 - x^2 - y^2)} \;=\; -\Gamma_{xtx} \;=\; -\Gamma_{yty}\,, \tag{2.15}$$

$$\Gamma_{ijj} \;=\; \frac{-\,x_i}{\alpha^2 - v_0^2} \;=\; -\,\Gamma_{iji}, \tag{2.16}$$

where $j = x, y$; $i \neq j$ and i and x_i vary as in eq. (2.14).

The parallel transport equation for the test vector v, is given by [38],

$$\frac{dv^\alpha}{d\lambda} \;+\; v^\beta\,\Gamma^\alpha_{\beta\gamma}\,\Omega^\gamma_\delta\,\frac{dx^\delta}{d\lambda} \;=\; 0, \tag{2.17}$$

where λ is an affine parameter describing the curve; the Greek indices vary over all the coordinates; and the matrix Ω relates the non-coordinate basis with the coordinate one.

In order to obtain the explicit expression of eq. (2.17), we have to introduce the closed loop around which we shall parallel transport v.

We start by noting that for a given instant t there are four independent, closed directions, one for each pair of opposite, identified, sides of the octagon. Each one of the four independent, close, direction gives rise to a set of closed loops, which we shall call S_i $(i = 1, ..., 4)$. All closed loops which may be constructed, taking in account the identifications, are made up of at least one element of the four S_i. Therefore, since the space-time event under investigation is in the t direction, the simplest closed loops we can choose are the elements of the four S_i at constant and different values of t. These loops will collapse to $t = 0$ when we take the limit $t \to 0$.

Then, the results derived by the use of the elements of the four S_i will be enough to draw conclusions about the regularity of our space-time.

In order to apply the holonomy method to one of the four S_i, we must choose an element of this set. Let us say the closed loop C formed by joining the middle points of each opposite sides. Now, we orient our axes so that the x axis coincide with C. In terms of our coordinates, the parametric equation of C, for a certain instant of t, is

$$t \;=\; \text{constant}\,; \qquad y \;=\; 0\,; \qquad x \;=\; g(\lambda), \tag{2.18}$$

where λ is a periodic parameter varying in the range $[\lambda_0, \lambda_f], \lambda_f \equiv \lambda_0 + p_\lambda$, and p_λ is the period of λ; $g(\lambda)$ is a function of λ, to be specified, which describes x in terms of λ and varies between the extreme values of x, $[-x_0(t), x_0(t)]$. The extreme values of x are the following time dependent functions from eq. (2.9),

$$x_0(t) \;=\; \sqrt{\alpha^2 - v_0^2}\,\tanh(\chi_0/2)\,; \qquad \theta \;=\; 0\,,\,\pi\,, \tag{2.19}$$

where χ_0 is one fixed value of χ.

So, with the aid of the relevant Ω components, we may write the parallel transport equations (2.17), for C eq. (2.18),

$$\frac{dv^y}{d\lambda} \;=\; 0, \tag{2.20}$$

$$\frac{dv^t}{d\lambda} + \frac{4\gamma f^2 (df/d\lambda)}{(1-f^2)^2}\, v^x \;=\; 0, \tag{2.21}$$

$$\frac{dv^x}{d\lambda} + \frac{4\gamma f^2 (df/d\lambda)}{(1-f^2)^2}\, v^t \;=\; 0, \tag{2.22}$$

where,

$$f(\lambda) \;\equiv\; \frac{g(\lambda)}{\sqrt{\alpha^2 - v_0^2}} \qquad \text{and} \qquad \gamma \;\equiv\; \frac{\dot{v}_0 v_0}{\sqrt{\alpha^2 - v_0^2}}\,. \tag{2.23}$$

Observing eq. (2.19), we note that the main motivation for the introduction of the new parametrization function $f(\lambda)$, eq. (2.23), is to restrict all the time dependence of our subsequent results to be concentrated in γ, eq. (2.23).

The solution of eq. (2.20) is given by,

$$v^y \;=\; v_0^y\,, \tag{2.24}$$

where v_0^y is the value of $v^y(\lambda)$ for $\lambda = \lambda_0$. The two remaining equations (2.21) and (2.22), form a coupled system of first order differential equations. In order to solve it we shall have to introduce the explicit value of $f(\lambda)$.

One of the simplest choices for $f(\lambda)$, which gives periodic solutions for the system (2.21)-(2.22), is obtained by demanding that,

$$\frac{4\, f^2\,(df/d\lambda)}{(1\ -\ f^2)^2}\ =\ \cot(\gamma\lambda). \tag{2.25}$$

Which gives, after an integration, the following implicit equation for f as a function of λ,

$$\left(\frac{1-f}{1+f}\right)^{\gamma}\ \exp\left(\frac{2f\gamma}{1-f^2}\right)\ =\ A\,\sin(\gamma\lambda)\,, \tag{2.26}$$

where A is an integration constant.

Now, introducing our parametrization choice eq. (2.25), in the system (2.21)-(2.22) and solving it, we find the general solutions,

$$v^t(\lambda)\ =\ \frac{D}{2}\left[\frac{B^2+\sin^2(\gamma\lambda)}{B\sin(\gamma\lambda)}\right]\quad \text{and}\quad v^x(\lambda)\ =\ \frac{D}{2}\left[\frac{B^2-\sin^2(\gamma\lambda)}{B\sin(\gamma\lambda)}\right]\,, \tag{2.27}$$

where B and D are integration constants to be determined by the initial conditions.

Observing the solutions (2.27), we note that (for non-vanishing C and D) they are singular whenever $\lambda = 2n\pi/\gamma$, n being an integer. This property comes directly from our choice for the parametrizing function $f(\lambda)$, eq. (2.25). This singularity of the solutions do not prevent us from use them in the holonomy method, because the holonomy matrix is computed by comparing the test vector v at the same point [38]. Before and after the parallel transport of v around the closed loop. Therefore, we must only avoid choosing as the initial point of the loop, one of the singular points of the solutions (2.27).

For the initial conditions,

$$v^j(\lambda = \lambda_0)\ =\ v_0^j\,, \tag{2.28}$$

where $j\ =\ t, x$, and the solutions (2.24) and (2.27), we obtain the following holonomy matrix M,

$$\begin{pmatrix} v^t(\lambda_f) \\ v^x(\lambda_f) \\ v^y(\lambda_f) \end{pmatrix} = \begin{pmatrix} M_{tt} & M_{tx} & 0 \\ M_{xt} & M_{xx} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_0^t \\ v_0^x \\ v_0^y \end{pmatrix}\,, \tag{2.29}$$

where,

$$M_{tt}\ =\ M_{xx}\ =\ \frac{\sin^2(\gamma\lambda_0)\ +\ \sin^2(\gamma\lambda_f)}{2\sin(\gamma\lambda_0)\sin(\gamma\lambda_f)}\ \equiv\ M_+\,, \tag{2.30}$$

$$M_{tx}\ =\ M_{xt}\ =\ \frac{\sin^2(\gamma\lambda_0)\ -\ \sin^2(\gamma\lambda_f)}{2\sin(\gamma\lambda_0)\sin(\gamma\lambda_f)}\ \equiv\ M_-\,. \tag{2.31}$$

2.4.2 Regularity conditions.

The next step in the holonomy method is the determination of the limit of the holonomy matrix elements as the closed loops collapse to the event $t = 0$. It means that we must take the limit of the non-trivial elements of M, given in eqs. (2.30) and (2.31), as $t \to 0$.

The limits of M_+, eq. (2.30), and M_-, eq. (2.31), as $t \to 0$ are easy to determine because they depend on t only through γ. With the aid of eqs. (2.10) and (2.23) we compute the limit of γ as,

$$\lim_{t\to 0} \gamma \;=\; -\,N\,. \tag{2.32}$$

¿From eq. (2.32), the desired limits of $M_\pm$ are,

$$\lim_{t\to 0} M_\pm \;=\; \frac{\sin^2(N\lambda_0)\,\pm\,\sin^2(N\lambda_f)}{2\sin(N\lambda_0)\sin(N\lambda_f)}\,. \tag{2.33}$$

where the $+$ and $-$ signs in the right hand side of eq. (2.33) are associated, respectively, with the limits of M_+ and M_-.

Since we would like to obtain the conditions to have regular space-times, following the holonomy method, we have to impose that the limits of M_+ and M_- are,

$$\lim_{t\to 0} M_+ \;=\; 1 \qquad \text{and} \qquad \lim_{t\to 0} M_- \;=\; 0\,. \tag{2.34}$$

Before we impose that the limits of M_+ and M_- have the values given by eq. (2.34), we must rewrite them in terms of the other free parameter (besides N) of the model. It is the period p_λ of the parameter λ which describes the closed loop C, eq. (2.18).

Using the expression which relates λ_f with λ_0 and p_λ, shown just after eq. (2.18), we may introduce p_λ in the M_+ and M_- limits eq. (2.33). If, after doing that, we impose that the resulting expressions have the values given in eq. (2.34), we obtain the following independent equations for generic values of λ_0,

$$\sin^2(Np_\lambda) \;=\; 0\,, \tag{2.35}$$

$$\sin(Np_\lambda)\,\cos(Np_\lambda) \;=\; 0\,, \tag{2.36}$$

$$\sin(Np_\lambda)\,[\,\cos(Np_\lambda)\,-\,1\,] \;=\; 0\,, \tag{2.37}$$

$$\cos^2(Np_\lambda)\,-\,2\cos(Np_\lambda)\,+\,1 \;=\; 0\,. \tag{2.38}$$

The most general solution to this system of equations (2.35)-(2.38), in the variables N and p_λ, is

$$N\,p_\lambda \;=\; 2\pi\,n\,, \tag{2.39}$$

where n is a non-zero, positive, integer.

Equation (2.39) introduces conditions upon p_λ and N. Since N depends on Λ, this equation introduces conditions on Λ.

Depending whether a_1 is smaller or greater than α, we have to solve eq. (2.39) for the appropriate Lorentzian or complex solution.

As an example, if we set $n = 1$ and $p_\lambda = 2\pi$ in eq. (2.39), we get the following equation for the Lorentzian solution (2.3),

$$a_1 \sqrt{|\Lambda|} \;=\; \sin \sqrt{|\Lambda|}. \qquad (2.40)$$

Equation (2.40) has solutions for $a_1 < 2/\pi$, and they form a discrete set. The number of solutions will depend on a_1 due to its presence in the superior limit of x.

We may write an approximate rule to the solutions of eq. (2.40), if we separate them in two sets. The first set has the even order solutions and the second the odd order solutions.

Calling m the order of the solutions, we have the following approximate rules for the two sets of solutions,

$$\text{Even order}: \quad |\Lambda|_k \;\approx\; (1-a_1)^2\,(2k-1)^2\,\pi^2\,, \quad k = 1,2,3,\ldots\,, \qquad (2.41)$$

$$\text{Odd order}: \quad |\Lambda|_l \;\approx\; (1+a_1)^2\,(2l\pi)^2\,, \quad l = 1,2,3,\ldots\,, \qquad (2.42)$$

where $k = m/2$; $l = (m-1)/2$.

Therefore, we can have regular space-times, for appropriate values of p_λ and $|\Lambda|$, from the solutions (2.3)-(2.4) and (2.5)-(2.8).

2.5 Wave-function.

Since we saw that the Lorentzian and the complex solutions satisfy all the 'no-boundary' conditions, we may write down the semi-classical 'no-boundary' wave function Ψ_{NB} (1.7). Due to equations (2.41) and (2.42), we understand that our wave-function must be a sum over all Complex and Lorentzian solutions with different and allowed values for the cosmological constant. As a matter of simplicity we shall consider here the wave-function for just one generic value of the cosmological constant, say, $|\Lambda|_k$.

For the present situation, the Euclidean action (1.5) is given by,

$$I_k[N, a(t)] \;=\; -\mathcal{A} \int [\,\dot{a}^2(t) \;-\; N^2 \;+\; a^2(t)N^2|\Lambda|_k\,]\,\frac{1}{N}dt, \qquad (2.43)$$

where: $\mathcal{A}$ is a finite defined number proportional to the volume of the compact, spacelike hypersurfaces.

The Ψ_{NB} (1.7), valid for a_1 smaller than α, which corresponds to the Lorentzian solution is, (up to normalization)

$$\Psi^L_{NBk} \;=\; \cos\left[\frac{2\mathcal{A}}{|\Lambda|_k}\,(1 \;-\; |\Lambda|_k a_1^2)^{3/2}\right]. \qquad (2.44)$$

The Ψ_{NB} (1.7), valid for a_1 greater than α, which corresponds to the complex solution is, (up to normalization)

$$\Psi^C_{NBk} \;=\; \exp\left[-\frac{2\mathcal{A}}{|\Lambda|_k}\,(a_1|\Lambda|_k \;-\; 1)^{3/2}\right]. \qquad (2.45)$$

Let us describe the behavior of the two wave-functions above in terms of the variable $x \equiv a_1|\Lambda|_k$. Note that for a fixed value of the cosmological constant this variable gives a direct measure of the scale factor size.

We notice that each of the wave-functions above correspond to a distinct region, whether x is smaller or greater than 1. The Ψ^L_{NBk}, corresponds to the region where $x < 1$. The Ψ^C_{NBk}, corresponds to the region where $x > 1$.

Since Ψ^L_{NBk} is a oscillatory function of x and Ψ^C_{NBk} is a decreasing exponential function of x, we may interpret the above result in the following way.

The universe has a probability proportional to $\Psi^{L^*}_{NBk}\Psi^L_{NBk}$ of be in the region where $x < 1$. This probability varies with $\cos^2$, from (2.44). On the other hand, the universe has an exponentially decreasing probability, proportional to $\Psi^{C^*}_{NBk}\Psi^C_{NBk}$ (2.45), to be found in the region where $x > 1$.

As an example, for fixed value of $|\Lambda|_k$, we would say that it is easier to find pseudo-spherical universes with $a_1 < \alpha$, than with $a_1 > \alpha$.

It is not difficult to see that Ψ_{NB}, derived here, correctly represents the anti-deSitter space-time [33]. This space-time is the classical vacuum solution of highest symmetry. For this reason Ψ_{NB} is sometimes called the ground-state wave-function.

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