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An exact soluble path integral model for stochastic Beltrami fluxes and its string properties

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ABSTRACT

We propose an exactly soluble path integral model for stochastic Beltrami fluxes in three-dimensional space-time with a fixed eddie scale. We show further the appearance of a three-dimensional self avoiding random surface structure for the *spatial* vortex loop in our exactly soluble turbulence reduced model.

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One of the most interesting problems in the path integral formalism for turbulence is related to the evaluation of the associated turbulence path integral in a perturbative or non-perturbative framework ([1]). Our aim in this letter is to present an exactly soluble path integral model for stochastic hydrodynamic motions defined here to be random regime of the *physical Navier-Stokes* equation in the incompressible case dominated by generalized Beltrami fluxes defined by the condition $rot\ \vec{v} = \lambda\vec{v}$ with λ a positive parameter ([2]) and according to Leonardo da Vinci who described turbulence of fluid fluxes (low viscosity $\nu \to 0$) as an amalgamation of "rough" roll up (fixed scale) fluid motions.

Let us, thus, start with the usual Navier-Stokes equation, i.e.,

$$\frac{\partial\vec{v}}{\partial t} + \left(\frac{1}{2}grad(\vec{v}^2) - (\vec{v}\times rot\vec{v})\right) = -\frac{gradP}{\rho} + \nu\Delta\vec{v} + \vec{F}^{ext} \tag{1}$$

where, the random stirring force is such that its satisfies the following spatially non-local Gaussian statistics in our reduced model for turbulence, i.e.,

$$\langle (F^{ext})_i(\vec{r},t)(F^{ext})_j(\vec{r}\,',t')\rangle = \lambda^2\delta_{ij}((\Delta^{-1})\delta(r-r'))\delta(t-t') \tag{2}$$

where Δ_r^{-1} denotes the Laplacean Green function.

At this point we take the curl of Eq. (1) and consider the already mentioned Beltrami flux condition and its direct consequence, namelly:

$$\lambda^2\vec{v} = rot(rot\ \vec{v}) = grad\ (div\ \vec{v}) - \Delta\vec{v} = -\Delta\vec{v} \tag{3.a}$$

$$\vec{v}\times rot\ \vec{v} = \vec{v}\times(\lambda\vec{v}) = 0 \tag{3.b}$$

in order to replace the Navier-Stokes equation, Eq. (1), by the exactly soluble Langevin equation for the fluid flux stirred by the external force $\vec{\Omega}^{ext} = rot(\vec{F}^{ext})$ in our proposed model of Navier-Stokes turbulence dominated by generalized Beltrami fluxes.

$$\frac{\partial\vec{v}(\vec{r},t)}{\partial t} = -\nu\lambda^2\vec{v}(\vec{r},t) + \frac{1}{\lambda}\vec{\Omega}^{ext}(\vec{r},t). \tag{4}$$

The new external stirring $\vec{\Omega}^{ext} = rot(\vec{F}^{ext})(\vec{r},t)$ satisfies a Gaussian process with the following two-point correlation function

$$\langle \Omega_\ell^{ext}(\vec{r},t)\Omega_{\ell'}(\vec{r}',t')\rangle =$$
$$= (\varepsilon^{\ell jk}\partial_j^{(r)})(\varepsilon^{\ell' j' k'}\partial_{j'}^{(r')})\langle F_k^{ext}(\vec{r},t)F_{k'}^{ext}(\vec{r}',t')\rangle$$
$$= \lambda^2(\delta^{\ell\ell'}\delta^{jj'} - \delta^{\ell j'}\delta^{\ell' j})\partial_j^{(r)}\partial_{j'}(\Delta_r^{-1}\delta^{(3)}(\vec{r}-\vec{r}')) \times \delta(t-t')$$
$$= \lambda^2\delta^{\ell\ell'}\delta^{(3)}(r-r')\delta(t-t') - \lambda^2\partial_\ell^{(r)}\partial_{\ell'}^{(r')}(\Delta_r^{-1}\delta(\vec{r}-\vec{r}'))\delta(t-t') \quad (5)$$

It is obvious that eq. (5) satisfies the incompressibility condition necessary for the incompressibility consistency of our Brownian-Langevin fluid equation (4) and its stochastic version below.

It is important to remark that the formal wave vectors of the Beltrami hydrodynamical motions have eddies of a fixed scale $|\vec{k}| = \gamma$ in our reduced model. As a consequence of this fact, we assume implictly the same wave vector constraint in our random stirrings eq. (2) and eq. (5).

A simple functional integral shift leads to the exactly generating functional path integral for our Brownian reduced model, where we have used the incompressibility constraint $\partial_i^{(r)}v^i(\vec{r},t) = 0$ to see that the spatially non-local piece of eq. (5) does not contributes to the final path integral weight eq. (6).

$$Z[\vec{j}(\vec{r},t)] = \int D[\vec{v}(\vec{r},t)]\ exp\ \left(i\int_{-\infty}^{+\infty} d^3\vec{r}\int_0^{\infty} dt(\vec{j}\cdot\vec{v})(\vec{r},t)\right) \times$$
$$det\left[\frac{\partial}{\partial t} - \nu\lambda^2\right]\delta^{(F)}(div\ \vec{v})exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} d^3\vec{r}\ d^3\vec{r}'\ \int_0^{+\infty} dt\ dt'\right.$$
$$\left(\frac{\partial v}{\partial t}i + \nu\lambda^2 v_i\right)(\vec{r},t)[\delta^{ii'}\delta^{(3)}(\vec{r}-\vec{r}')\delta(t-t') - \partial_i^{(r)}\partial_{i'}^{(r')}(\Delta_r^{-1}\delta(\vec{r}-\vec{r}'))\delta(t-t')]$$
$$\left.\left(\frac{\partial v_{i'}}{\partial t} + \nu\lambda^2 v_{i'}\right)(\vec{r}',t')\right\}$$
$$= \int D[\vec{v}(\vec{r},t)]exp\left(i\int_{-\infty}^{+\infty} d^3\vec{r}\int_0^{+\infty} dt(\vec{j}\cdot\vec{v})(\vec{r},t)\right)$$
$$exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} d^3\vec{r}\int_0^{+\infty} dt\left(\frac{\partial\vec{v}}{\partial t} + \nu\lambda^2\vec{v}\right)^2(\vec{r},t)\right\} \quad (6)$$

At this point it is worth to compare the exactly soluble path integral above written (note the fixed wave vector $|\vec{k}| = \gamma$ imposed implicitly on eq. (6)) with that one associated to the complete Navier-Stokes equation for ultra-local random external stirring with strengh D namelly: $\langle F_i(\vec{r},t)F_j(\vec{r}\,',t')\rangle = D\delta^{(3)}(\vec{r}-\vec{r}\,')\delta(t-t')\delta_{ij}$ and full range scale $0 \leq |\vec{k}| < \infty$ ([3]).

$$\begin{aligned} Z[\vec{j}(\vec{r},t)] &= \int D[\vec{v}(\vec{r},t)] det \left[\left(\frac{\partial}{\partial t} - \nu\Delta \right) \delta_{\ell k} + \sqrt{D}\ \frac{\delta}{\delta v_\ell}((\vec{v}\cdot\vec{\Delta})v)_k \right] \\ & exp\left\{ -\frac{1}{2}\int_{-\infty}^{+\infty} d^3r \int_0^{+\infty} dt \left(\frac{\partial}{\partial t} - \nu\Delta\vec{v} + \sqrt{D}(\vec{v}\cdot\vec{\nabla})v + \frac{gradP}{\rho} \right)^2 (\vec{r},t) \right\} \\ & exp\left\{ i\sqrt{D}\int_{-\infty}^{+\infty} d^3r \int_0^{+\infty} (\vec{j}\cdot\vec{v})(\vec{r},t) \right\} \end{aligned} \tag{7.a}$$

Let us remark that it is possible to eliminate the pressure term $-\frac{1}{\rho}\ gradP$ in this path-integral framework by using the incompressibility condition $div(\vec{v}) = 0$, which, by its turn leads one to consider only the transverse part of the external force and of the non-linear term in the effective action in eq. (7.a) ([3]).

$$\begin{aligned} Z[\vec{j}(\vec{r},t)] &= \int D^F[\vec{v}(\vec{r},t)] \\ & exp\left\{ -\frac{1}{2}\int_{-\infty}^{-\infty} d^3r \int_0^{+\infty} dt \left(\frac{\partial}{\partial t}\vec{v} - \nu\Delta\vec{v} + \sqrt{D}((\vec{v}.\vec{\nabla})\vec{v})^{Tr} \right)^2 \right\} \end{aligned} \tag{7.b}$$

Here the transverse part of a generic vector field is defined by the expression

$$(\vec{W}(\vec{r},t))^{Tr} = \vec{W}(\vec{r},t) - \frac{1}{4\pi}\ grad_r(\Delta^{-1}(div\ \vec{W})) \tag{8}$$

Note that now one should postulate the non-local two-point correlation function in order to get eq. (7.b) $\langle F_i^{Tr}(\vec{r},t)F_j^{Tr}(\vec{r}(\vec{r}\,',t')\rangle = \mathcal{D}\delta^{(3)}(\vec{r}-\vec{r}\,')\delta(t-t')\delta_{ij}$.

It is worth remark that eqs. (7.a)-(7.b), applied to the Burger equation leads to a different path integral than that proposed in ref.[1] since in the path-integral framework the viscosity is not a perturbative parameter which, in our case, is $\sqrt{D}$. Besides, the propagator in the free case for *the time parameter in the range* $0 \leq t \leq \infty$ is given by ([3])

$$\left(\left(\frac{\partial}{\partial t} - \nu\Delta \right)^{-1} * \left(-\frac{\partial}{\partial t} - \nu\Delta \right)^{-1} \right)(k,t,t') = -\frac{1}{\nu k^2}\left[e^{-\nu k^2|t-t'|} - e^{-\nu k^2|t+t'|} \right] \tag{9}$$

and differing significantly from the Dominicis-Martin propagator suitable for the range $-\infty \leq t \leq \infty$ ([1]):

$$\left(\left(\frac{\partial}{\partial t} - \nu\Delta \right)^{-1} * \left(-\frac{\partial}{\partial t} - \nu\Delta \right)^{-1} \right)(k,t,t') = \int_{-\infty}^{+\infty} dw (e^{iw(t-t')}) \frac{1}{w^2 + \nu^2|\vec{k}|^4} \tag{10}$$

Let us now evaluate the vortex phase factor defined by a fixed-time spatial loop $\ell = \{\vec{\ell}(\sigma), a \leq \sigma \leq b\}$ in our exactly soluble model eq. (6) in order to see the connection with strings (random surfaces) ([7])

$$\left\langle exp(i \oint \vec{v}(\vec{\ell}(\sigma), t) d\vec{\ell}(\sigma)) \right\rangle_{\vec{v}} \equiv \int D^F[\vec{v}(\vec{r}, t)] exp\left\{ -\frac{1}{2} \int_{-\infty}^{+\infty} d^3\vec{r} \int_0^{+\infty} dt \right.$$

$$\left. \left(\frac{\partial \vec{v}}{\partial t} + \nu\lambda^2 \vec{v} \right)^2 (\vec{r}, t) \right\} exp(i \oint \vec{v}(\vec{\ell}(\sigma), t) d\vec{\ell}(\sigma)) \tag{11}$$

Since the flux is of a Beltrami type in our soluble model eq. (6), we propose to re-write the circulation phase factor as a sum over all surfaces bounding the fixed loop ℓ by making use of Stokes theorem and by taking into account again the Beltrami condition, i.e.,

$$\left\langle e^{i \oint_c \vec{v} \cdot d\vec{\ell}} \right\rangle_v =$$

$$\int \mathcal{D}^F[\vec{v}(\vec{r}, t)] \, e^{-\frac{1}{2} \int_{-\infty}^{+\infty} d^3 r \int_0^{+\infty} dt \left[\vec{v} \left(-\frac{\partial}{\partial t^2} + \nu\lambda^2 \right) \vec{v} \right]}$$

$$\times \left(\sum_S exp(i\lambda \iint_S \vec{v}(x, t) \cdot d\vec{A}(x)) \right), \tag{12}$$

By observing now that the two-point correlation of our Brownian-Beltrami turbulent flux is exactly given by

$$\langle v_i(\vec{r}, t) v_j(\vec{r}\,', t') \rangle_v = \int_{|\vec{k}|=\lambda} e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \frac{e^{-\nu\lambda^2|t-t'|}}{\nu\lambda^2} \theta(t-t')\delta_{ij}, \tag{13}$$

with $t, t'; \in [0, \infty]$ and $\theta(0) = 1/2$ in this initial value problem, we can easily evaluate the average Eq. (6) and producing a strongly coupled ($\nu \to 0$) area dependent functional for the spatial vortex phase factor ([5]), in our proposed turbulent flux regime

$$W[\vec{\ell}, \vec{v}] \equiv \left\langle e^{i \oint_\ell \vec{v}(\ell, t) \cdot d\vec{\ell}} \right\rangle = \sum_{\{S\}} exp\left\{ -\frac{\lambda}{\nu} \iint_S d\vec{A}(x) \frac{sen(\lambda|x-y|)}{|x-y|} d\vec{A}(y) \right\} \tag{14}$$

The above obtained results rise hopes that a strongly coupled string theory, as used to understand Q.C.D. and Quantum Gravity ([5], [6]) may be relevant to modelling structures with non trivial geometry in turbulence ([4], [7]) at least for the restrict family of stochastic Beltrami motions (with a eddie of a fixed scale $|\vec{k}| = \lambda$) (see Appendix A for a toy model for $|\vec{k}| \neq \lambda$).

Appendix A – The case of non fixed eddies scale

Just for completeness of our study and in order to generalize the Beltrami flux turbulence analysis represented in the main text, for the physical case of the complete wave vector range $0 \leq |\vec{k}| < \infty$ in our turbulent path integral soluble model studies, we propose to consider a kind of generalized Beltrami condition to overcome this possible drawback of our turbulence modeling, namelly:

$$rot\vec{v}(\vec{r},t) = \lambda(\vec{r})\vec{v}|\vec{r},t| \tag{A.15}$$

where $\lambda(\vec{r})$ is a positive function varying in the space and to be determined from a phenomenological point of the view ([4]). Note that the Fourier transformed (wave-vector) condition takes now the general form

$$|\vec{k}| \cdot |\tilde{\vec{v}}(\vec{k},t)| = \int_{R^3} d^3\vec{p} |\tilde{\lambda}(\vec{p}-\vec{q})| \cdot |\tilde{\vec{v}}(\vec{p},t)| \tag{A.16}$$

which, by its turn, leads to the full range scale $0 < |\vec{k}| < \infty$ for the eddies hydrodynamical motions under study. By supposing that the "vortical" stirring eq. (5) is a pure white noise process with strenght D,

$$\langle \Omega_\ell^{ext}(\vec{r},t)\Omega_{\ell'}^{ext}(\vec{r},t')\rangle = D \cdot \delta^{\ell\ell'}\delta^{(3)}(\vec{r}-\vec{r}\,')\delta(t-t') \tag{A.17}$$

It is a straightforward deduction by following our procedures as exposed in the text to arrive at an analogous Gaussian path integral for the Generalized Beltrami random hydrodynamical defined by equation (A.15). The generalized effective motion equation is given, in this new situation, by

$$\begin{aligned} &\left[\left(\frac{\partial}{\partial t} - \nu\left(\frac{\Delta\lambda}{\lambda}\right)(\vec{r}) - \frac{\nu}{\lambda(\vec{r})}\,\frac{\partial\lambda(\vec{r})}{\partial x^e}\,\frac{\partial}{\partial x_e} + \nu\lambda^2(\vec{r})\right)\delta^{ik} \right.\\ &\left. +\nu\left(\varepsilon^{ijk}\frac{\partial\lambda(\vec{r})}{\partial x_j}\right)\right] v_k(\vec{r},t) = \Omega_i^{ext}(\vec{r},t) \end{aligned} \tag{A.18}$$

The Gaussian path-integral, thus, is exactly written below

$$\begin{aligned} &Z[j_i(\vec{r},t)] = \int \prod_{i=1}^{3} D^F[v_i(\vec{r},t)]\; exp\left(i\int_{\infty}^{+\infty} d^3\vec{r}\int_0^\infty dt\; (j^i v_i)(\vec{r},t)\right) \\ &exp\left[-\frac{1}{2D}\int_{-\infty}^{+\infty} d^3\vec{r}\int_0^\infty dt\; v_k(\vec{r},t)(M^{*ki}\cdot M^{is})v_s(\vec{r},t)\right] \end{aligned} \tag{A.19}$$

Here, the differential operators entering in the kinetic term of the our turbulent path integral are

$$M^{*ki} = \left(-\frac{\partial}{\partial t} + \frac{\nu}{\lambda(\vec{r})} \frac{\partial \lambda(\vec{r})}{\partial x_e} \frac{\partial}{\partial x^e} + \frac{\nu}{\lambda(\vec{r})} \cdot \Delta\lambda(\vec{r}) \right.$$
$$\left. -\nu \frac{\Delta\lambda(\vec{r})}{\lambda(\vec{r})} + \nu\lambda^2(\vec{r}) \right) \delta^{ki} + \nu\varepsilon^{kji} \frac{\partial \lambda(\vec{r})}{\partial x_j} \tag{A.20}$$

and

$$M^{is} = \left(+\frac{\partial}{\partial t} - \frac{\nu}{\lambda(\vec{r})} \frac{\partial \lambda(\vec{r})}{\partial x^e} \frac{\partial}{\partial x_e} + \nu\lambda^2(\vec{r}) - \nu \frac{\Delta\lambda(\vec{r})}{\lambda(\vec{r})} \right) \delta^{is} + \nu\varepsilon^{ijs} \frac{\partial \lambda(\vec{r})}{\partial x_j} \tag{A.21}$$

It is worth point out that the exact evaluation of the variance eq. (A-19) depends on the *exact* form of our rotation function $\lambda(\vec{r})$ defining the Beltrami condition (A.15).

The vortex phase factor eq. (11), takes now a form closelly related to the pure self-avoiding string theory of refs. [5]-[7] in the case of a slowly varying function $|\text{grad}\lambda(\vec{r})| << \lambda(\vec{r})$ and $\lambda(\vec{r}) \sim 1$ (a very slowly $\vec{r}$-varying function: for instance as $\lambda(\vec{r}) = \lambda_0\, exp(-10^{-5}|\vec{r}|^2)$)

$$\langle e^{i\oint \vec{v}(\ell,t)d\vec{\ell}} \rangle = \sum_{\{S\}} exp\left\{ -\frac{1}{\nu} \int_S \int_S d\vec{A}(x) \cdot \delta^{(3)}(x-y) \cdot d\vec{A}(y) \right\} \sim exp\left(-\frac{1}{\nu}\text{area}(S)\right) \tag{A.22}$$

Now, if we follows refs. [5] and [6], it is an easy task to deduce that the above written *time-fixed* vortex phase factor satisfies the famous loop wave equation for Abelian Q.C.D. at very low energy and a large number of colors. It may be written in the geometrical (infinitely differentiable loops $\vec{\ell}(\sigma)$ notation of ref. [7] as the following

$$\partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} \left(\langle e^{i\oint \vec{v}(\ell,t)d\vec{\ell}} \rangle \right)$$
$$\frac{1}{\nu} \oint d\vec{y}\,\delta^{(3)}(\vec{x}-\vec{y}) \langle e^{i\oint \vec{v}(\ell,t)d\vec{\ell}} \rangle \tag{A.23}$$

The above obtained results rise hopes again that a string theory may be relevant to understand turbulence modeled as an amalgamation of "though" roll up of random stirred fluid motions. Namelly: $rot\vec{v}(\vec{r},t) = \lambda(\vec{r})\vec{v}(\vec{r},t)$.

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