CBPF-NF-067/97

Conditions for regular space-times with compact spatial sections

G. Oliveira-Neto[*†]

Department of Physics, The University, Newcastle-Upon-Tyne, NE1 7RU, U. K.
and
Departamento de Campos e Partículas, Centro Brasileiro de Pesquisas Físicas,
R. Dr. Xavier Sigaud 150, Urca, CEP 22290-180, Rio de Janeiro, Brazil

Abstract

We apply the holonomy method to study the regularity of space-times which initially open spatial sections are made compact. We restrict our attention to a set of different expressions of the anti-deSitter space-time in (2+1)-dimensions. After we have computed the holonomy matrix to this set, we explicitly derive the regularity conditions for two elements of the set. The first expression, relevant to classical cosmology, will be regular if the periods of the compactified directions are 2π. The other expression, relevant to quantum cosmology, will be regular if the cosmological constant has a certain, well defined, discrete, spectrum.

Key-words: Compactification; Holonomy method; Conical singularities; Cosmology

PACS - 02.40, 04.20, 04.60

---

[*]Email: n04c7@symbcomp.uerj.br

[†]Present address: Departamento de Física Teórica, Instituto de Física, UERJ, Rua São Francisco Xavier 524, Maracanã, CEP 20550-013, Rio de Janeiro, Brazil

I. INTRODUCTION.

The need of different types of space-times with closed spatial sections, in classical [1] and quantum [2] models of the Universe, has produced over the last years an increasing number of works where one turns into compact, or compactify, open manifolds [3], [4].

An interesting motivation, at the classical level, for the use of space-times which initially open spatial sections were compactified, are the works by Fagundes [3]. There, he proposed an explanation to the so-called quasar redshift controversy (see Ref. [5] for a review), by means of multiply images of a single source produced in an universe with compact hyperbolic spatial sections.

Another motivation to consider space-times with compact spatial sections comes from quantum cosmology [2]. There, one works with these space-times because one does not know, in the general case, how to treat the contributions coming from boundary terms at the spatial infinity [2]. However, we may mention few attempts to write the quantum mechanics of gravitational systems with open spatial sections [6].

The work of Louko and Ruback is an example, in quantum cosmology, where one has to compactify the initially open, flat, spatial sections of a set of space-times in order to compute the 'no-boundary' wave-function [7] for certain universes.

An important property of the compactification process, is that it may cause the development of singularities of conical, or even more complicate nature in the resulting space-times [4], [8]. And sometimes, as in the case of the space-times contributing to the 'no-boundary' wave-function [7], one is not interested in the presence of singularities. It means that it would be important to determine what are the conditions for one obtaining a regular space-time with compactified spatial sections.

Recently, a technique aimed at identifying, in a systematic way, the presence of conical singularities in a given space-time has been introduced. It is the so-called holonomy method [9].

In the present paper, we would like to apply the holonomy method to investigate the regularity of space-times with spatial sections which are closed as the result of compactifications. The use of the holonomy method implies that we will treat, here, only the cases where after the compactifications the space-times may have, at most, conical singularities. In particular, we would like to determine if there are conditions on the physical constants describing these space-times, when we require them to be regular, and what these conditions are.

We may divide the space-times to be studied in two sets: the first with negatively curved spatial sections and the second with flat spatial sections. Here, we shall consider only space-times of the first set leaving the others, belonging to the second set, to a future work.

As a matter of simplicity we shall restrict our attention to space-times which spatial sections have a constant negative curvature. In order to facilitate the calculations, we shall work in (2+1)-dimensions and comment, at Sec. V, on the modifications needed in order to treat (3+1)-dimensional space-times.

The next section, Sec. II, is devoted to the introduction of the particular space-time studied here. It is the (2+1)-dimensional anti-deSitter space-time with compact spatial sections. We shall modify the usual anti-deSitter metric by the introduction of an arbitrary and constant lapse function [10], N, which will allow us to treat a certain set of cases.

In Sec. III, after demonstrating that the singularities that may be formed by the compactifications are, at most, conical, we apply the holonomy method to the anti-deSitter space-time, written with an unspecified lapse function. Our final result in this section is a holonomy matrix which elements depend on N, among other quantities.

In Sec. IV, we shall derive the regularity conditions for two elements of the set mentioned above.

The first case is treated in Subsec. IV A. It is the usual expression for the anti-deSitter space-time metric, with N equal to the identity, which is relevant to classical cosmology [3]. The regularity condition, in this case, is a constraint upon the periods of the parameters introduced to describe the compactified directions.

In Subsec. IV B, we treat the second case where N is a given function of few constants present in the chosen space-time. As we shall see, in the Appendix A, this space-time contributes at the semi-classical level to the 'no-boundary' wave-function of a certain universe. It means that this case is relevant to quantum cosmology. Here, the regularity condition turns out to produce a transcendental equation for the only free physical parameter of the model, the cosmological constant Λ. For certain values of the integration constants, this equation has solutions which form a discrete and finite set.

Finally, in Sec. V, we conclude by outlining the main results of the paper and commenting on generalizations of the present analysis to other (2+1) and (3+1)-dimensional anti-deSitter space-times.

II. (2+1)-DIMENSIONAL ANTI-DESITTER SPACE-TIME WITH COMPACT SPATIAL SECTIONS.

In $(2+1)$-dimensions the anti-deSitter space-time has the following metric [11],

$$ds^2 \;=\; -\,N^2\,dt^2 \;+\; a^2(t)\,(\,d\chi^2 \;+\; \sinh^2\chi\,d\theta^2\,), \tag{1}$$

where,

$$0 \,\leq\, t \,\leq\, 1\,; \qquad 0 \,\leq\, \chi \,<\, \infty\,; \qquad 0 \,\leq\, \theta \,\leq\, 2\pi\,; \tag{2}$$

$$a(t) \;=\; \pm\,\alpha\,\sin\left(\frac{Nt}{\alpha}\right)\;; \qquad \text{and} \qquad \alpha^2 \;=\; \frac{1}{|\Lambda|}. \tag{3}$$

Λ, in eq. (3), is the negative cosmological constant.

The presence of a constant and arbitrary lapse function in the metric (1), allow us to treat a set of different expressions of the anti-deSitter space-time, each differing from the others by the value of N. Let us call this set ADS. The physical significance of the possible choices of N can be appreciated in Ref. [10].

¿From the literature [12], we know that for fixed values of t the spatial sections of the anti-deSitter space-time (1) are two-dimensional one-sheeted hyperboloids. The separation between the origin of the coordinate system and the pole of the one-sheeted hyperboloid is given by the scale factor, $a(t)$, at that instant. It means that the spatial sections are open. Therefore, since we are interested in the case where they are closed we must compactify them.

In what follows, with the aid of few mathematical results we shall introduce the relevant compact spatial sections.

The one-sheeted hyperboloid mentioned above, is a representation of the two-dimensional hyperbolic space H^2. In two dimensions, every closed orientable surface of genus at least two allows a geometric structure modeled on the H^2 (see the review article Ref. [13] for more details on this result due to

Poincaré, among others). Based upon this result and in order to keep our treatment as simple as possible let us choose our closed orientable surface to be a double torus. Then, the one-sheeted hyperboloid, which describes the spatial section of the space-time (1), will be compactified into a double torus.

One way to proceed is working in a projection of the H^2 onto a plane, which results in a disc called the Poincaré Disc (P.D.) [14]. The conformal transformations which takes one from the coordinates (χ, θ) to the polar coordinates (r, θ) on the P.D., are,

$$r = \sqrt{\alpha^2 - v_0^2} \tanh(\chi/2) ; \qquad \theta \quad \text{is unchanged} , \tag{4}$$

where

$$v_0 = \sqrt{\alpha^2 - a^2(t)}. \tag{5}$$

This mapping from the H^2 to the P.D. induces a non-trivial line element on the disc, given by,

$$ds^2_{PD} = 4 (\alpha^2 - v_0^2)^2 \frac{(dr^2 + r^2 d\theta^2)}{(\alpha^2 - v_0^2 - r^2)^2} . \tag{6}$$

The next step in the compactification process, is the selection of a fundamental region (in general a polygon) which tessellates [15] the P.D., and may be transformed into a double torus. Then, we shall take this fundamental region and construct the double torus, or in other words, the quotient surface H^2/Γ, where Γ is a certain symmetry transformation of the H^2.

In two dimensions, the unique symmetry transformations of the H^2 capable of generating compact surfaces are the hyperbolic ones [16], in other words, Lorentzian boosts. Our compact surface will be fabricated by the identification of the sides, through the relevant boost transformations, of the above selected polygon.

The simplest fundamental region which gives rise to a double torus is the regular octagon [14]. If we now identify the opposite sides of this polygon, with the aid of four different boost transformations (one for each relevant direction), we obtain a representation of the desired compact surface.

Therefore, the spatial sections of our space-time will be given by the time evolution of a double torus with line element (6).

III. THE HOLONOMY MATRIX.

We start by writing down the line element of the space-time described by the evolution of the double torus, in Cartesian coordinates, with the aid of eqs. (1), (2) and (6),

$$ds^2 \;=\; -\,N^2 dt^2 \,+\, \frac{4\,(\alpha^2 - v_0^2)^2\,(dx^2 + dy^2)}{(\alpha^2 - v_0^2 - x^2 - y^2)^2}\,. \tag{7}$$

Where the ranges of x and y will depend on the octagon dimensions; the opposite sides of the octagon are identified; and t has the range given in eq. (2).

Observing eqs. (1) and (3), we notice that the anti-deSitter space-time has a singularity at $t = 0$. This singularity can, at most, be conical. One can see it by noting that for Robertson-Walker space-times all the scalars, formed out of contractions of the Riemann tensor, are proportional to appropriate powers of the curvature scalar (R). And from direct calculations one finds that our space-time has an everywhere constant curvature scalar,

$$R \;=\; -\,6\,|\Lambda|\,. \tag{8}$$

In fact, it is known that this singularity is a coordinate singularity [12].

The same conclusion does not directly apply to the space-time with line element (7) because at the event $t = 0$ ($v_0^2 = \alpha^2$), due to the identifications mentioned after eq. (7), a conical singularity may have formed.

Let us, then, apply the holonomy method in order to investigate the presence, or not, of a conical singularity in the space-time with line element (7), at the event $t = 0$.

Following the instructions of the holonomy method [9], we start by writing down the parallel transport equations for a fiducial vector v, along a closed loop (to be specified), in the space-time represented by eq. (7).

The easiest way to derive them, from eq.(7), is by working on the orthonormal basis, defined by the transformations,

$$w^t \;=\; dt\,; \qquad w^i \;=\; \frac{2\,(\alpha^2 - v_0^2)}{\alpha^2 - v_0^2 - x^2 - y^2}\,dx_i\,, \tag{9}$$

where $i = x, y$ and $x_i = x, y$.

The non-vanishing connection coefficient components in this basis are,

$$\Gamma_{txx} = \Gamma_{tyy} = \frac{-2\dot{v}_0 v_0 (x^2+y^2)}{(\alpha^2 - v_0^2)(\alpha^2 - v_0^2 - x^2 - y^2)} = -\Gamma_{xtx} = -\Gamma_{yty}\,, \tag{10}$$

$$\Gamma_{ijj} = \frac{-\,x_i}{\alpha^2 - v_0^2} = -\,\Gamma_{iji}, \tag{11}$$

where $j = x, y$; $i \neq j$ and i and x_i vary as in eq. (9).

The parallel transport equation for the fidutial vector v, is given by [9],

$$\frac{dv^\alpha}{d\lambda} + v^\beta\, \Gamma^\alpha_{\beta\gamma}\, \Omega^\gamma_\delta \frac{dx^\delta}{d\lambda} = 0, \tag{12}$$

where λ is an affine parameter describing the curve; the Greek indices vary over all the coordinates; and the matrix Ω relates the non-coordinate basis with the coordinate one.

In order to obtain the explicit expression of eq. (12), we have to introduce the closed loop around which we shall parallel transport v.

We start by noting that for a given instant t there are four independent, closed directions, one for each pair of opposite, identified, sides of the octagon. Each one of the four independent, close, direction gives rise to a set of closed loops, which we shall call S_i $(i = 1, ..., 4)$. All closed loops which may be constructed, taking in account the identifications, are made up of at least one element of the four S_i. Therefore, since the space-time event under investigation is in the t direction, the simplest closed loops we can choose are the elements of the four S_i at constant and different values of t. These loops will collapse to $t = 0$ when we take the limit $t \rightarrow 0$.

Then, the results derived by the use of the elements of the four S_i will be enough to draw conclusions about the regularity of our space-time.

In order to apply the holonomy method to one of the four S_i, we must choose an element of this set. Let us say the closed loop C formed by joining the middle points of each opposite sides. Now, we orient our axes so that the x axis coincide with C. In terms of our coordinates, the parametric equation of C, for a certain instant of t, is

$$t = \text{constant}\,; \qquad y = 0\,; \qquad x = g(\lambda), \tag{13}$$

where λ is a periodic parameter varying in the range $[\lambda_0, \lambda_f], \lambda_f \equiv \lambda_0 + p_\lambda$, and p_λ is the period of λ; $g(\lambda)$ is a function of λ, to be specified, which describes x in terms of λ and varies between the extreme values of x, $[-x_0(t), x_0(t)]$. The extreme values of x are the following time dependent functions from eq. (4),

$$x_0(t) \; = \; \sqrt{\alpha^2 - v_0^2}\, \tanh(\chi_0/2)\, ; \qquad \theta \; = \; 0\, ,\, \pi\, , \tag{14}$$

where χ_0 is one fixed value of χ.

So, with the aid of the relevant Ω components, we may write the parallel transport equations (12), for C eq. (13),

$$\frac{dv^y}{d\lambda} \; = \; 0, \tag{15}$$

$$\frac{dv^t}{d\lambda} \; + \; \frac{4\gamma f^2 (df/d\lambda)}{(1-f^2)^2}\, v^x \; = \; 0, \tag{16}$$

$$\frac{dv^x}{d\lambda} \; + \; \frac{4\gamma f^2 (df/d\lambda)}{(1-f^2)^2}\, v^t \; = \; 0, \tag{17}$$

where,

$$f(\lambda) \; \equiv \; \frac{g(\lambda)}{\sqrt{\alpha^2 - v_0^2}} \qquad \text{and} \qquad \gamma \; \equiv \; \frac{\dot{v}_0 v_0}{\sqrt{\alpha^2 - v_0^2}}\, . \tag{18}$$

Observing eq. (14), we note that the main motivation for the introduction of the new parametrization function $f(\lambda)$, eq. (18), is to restrict all the time dependence of our subsequent results to be concentrated in γ, eq. (18).

The solution of eq. (15) is given by,

$$v^y \; = \; v_0^y\, , \tag{19}$$

where v_0^y is the value of $v^y(\lambda)$ for $\lambda = \lambda_0$. The two remaining equations (16) and (17), form a coupled system of first order differential equations. In order to solve it we shall have to introduce the explicit value of $f(\lambda)$.

One of the simplest choices for $f(\lambda)$, which gives periodic solutions for the system (16)-(17), is obtained by demanding that,

$$\frac{4\, f^2\, (df/d\lambda)}{(1\, -\, f^2)^2} \; = \; \cot(\gamma\lambda). \tag{20}$$

Which gives, after an integration, the following implicit equation for f as a function of λ,

$$\left(\frac{1-f}{1+f}\right)^{\gamma} \exp\left(\frac{2f\gamma}{1-f^2}\right) = A \sin(\gamma\lambda)\,, \tag{21}$$

where A is an integration constant.

Now, introducing our parametrization choice eq. (20), in the system (16)-(17) and solving it, we find the general solutions,

$$v^t(\lambda) = \frac{D}{2}\left[\frac{B^2+\sin^2(\gamma\lambda)}{B\sin(\gamma\lambda)}\right] \quad \text{and} \quad v^x(\lambda) = \frac{D}{2}\left[\frac{B^2-\sin^2(\gamma\lambda)}{B\sin(\gamma\lambda)}\right], \tag{22}$$

where B and D are integration constants to be determined by the initial conditions.

Observing the solutions (22), we note that (for non-vanishing C and D) they are singular whenever $\lambda = 2n\pi/\gamma$, n being an integer. This property comes directly from our choice for the parametrizing function $f(\lambda)$, eq. (20). This singularity of the solutions do not prevent us from use them in the holonomy method, because the holonomy matrix is computed by comparing the fiducial vector v at the same point [9]. Before and after the parallel transport of v around the closed loop. Therefore, we must only avoid choosing as the initial point of the loop, one of the singular points of the solutions (22).

For the initial conditions,

$$v^j(\lambda=\lambda_0) = v_0^j\,, \tag{23}$$

where $j = t, x$, and the solutions (19) and (22), we obtain the following holonomy matrix M,

$$\begin{pmatrix} v^t(\lambda_f) \\ v^x(\lambda_f) \\ v^y(\lambda_f) \end{pmatrix} = \begin{pmatrix} M_{tt} & M_{tx} & 0 \\ M_{xt} & M_{xx} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_0^t \\ v_0^x \\ v_0^y \end{pmatrix}, \tag{24}$$

where,

$$M_{tt} = M_{xx} = \frac{\sin^2(\gamma\lambda_0) + \sin^2(\gamma\lambda_f)}{2\sin(\gamma\lambda_0)\sin(\gamma\lambda_f)} \equiv M_+\,, \tag{25}$$

$$M_{tx} = M_{xt} = \frac{\sin^2(\gamma\lambda_0) - \sin^2(\gamma\lambda_f)}{2\sin(\gamma\lambda_0)\sin(\gamma\lambda_f)} \equiv M_-\,. \tag{26}$$

IV. REGULARITY CONDITIONS.

The next step in the holonomy method is the determination of the limit of the holonomy matrix elements as the closed loops collapse to the event $t = 0$. It means that we must take the limit of the non-trivial elements of M, given in eqs. (25) and (26), as $t \to 0$.

The limits of M_+, eq. (25), and M_-, eq. (26), as $t \to 0$ are easy to determine because they depend on t only through γ. With the aid of eqs. (5) and (18) we compute the limit of γ as,

$$\lim_{t\to 0} \gamma \; = \; -N \, . \tag{27}$$

¿From eq. (27), the desired limits of $M_\pm$ are,

$$\lim_{t\to 0} M_\pm \; = \; \frac{\sin^2(N\lambda_0) \, \pm \, \sin^2(N\lambda_f)}{2\sin(N\lambda_0)\sin(N\lambda_f)} \, . \tag{28}$$

where the $+$ and $-$ signs in the right hand side of eq. (28) are associated, respectively, with the limits of M_+ and M_-.

Since we would like to obtain the conditions to have regular space-times, following the holonomy method, we have to impose that the limits of M_+ and M_- are,

$$\lim_{t\to 0} M_+ \; = \; 1 \qquad \text{and} \qquad \lim_{t\to 0} M_- \; = \; 0 \, . \tag{29}$$

Before we impose that the limits of M_+ and M_- have the value given by eq. (29), we must rewrite them in terms of the other free parameter (besides N) of the model. It is the period p_λ of the parameter λ which describes the closed loop C, eq. (13).

Using the expression which relates λ_f with λ_0 and p_λ, shown just after eq. (13), we may introduce p_λ in the M_+ and M_- limits eq. (28). If, after doing that, we impose that the resulting expressions have the values given in eq. (29), we obtain the following independent equations for generic values of λ_0,

$$\sin^2(N p_\lambda) \; = \; 0 \, , \tag{30}$$

$$\sin(N p_\lambda) \, \cos(N p_\lambda) \; = \; 0 \, , \tag{31}$$

$$\sin(N p_\lambda) \, [\, \cos(N p_\lambda) \, - \, 1 \,] \; = \; 0 \, , \tag{32}$$

$$\cos^2(Np_\lambda) - 2\cos(Np_\lambda) + 1 = 0\,. \tag{33}$$

The most general solution to this system of equations (30)-(33), in the variables N and p_λ, is

$$N\,p_\lambda = 2\pi\,n\,, \tag{34}$$

where n is a non-zero, positive, integer.

Let us now consider what conditions eq. (34) will introduce for two elements of the ADS, eq. (7). They will have definite values of the lapse function N.

A. The case relevant to classical cosmology.

For this case, as mentioned in the introduction, the lapse function is equal to the identity,

$$N = 1\,. \tag{35}$$

It means that eq. (34) is reduced to the below condition upon p_λ,

$$p_\lambda = 2\pi\,n\,. \tag{36}$$

¿From the discussion on Ref. [9], we know that the closed, or angular, direction must be defined in the range $[0, 2\pi]$. Therefore, we can only accept the solution (36) for eq. (34), such that,

$$n = 1\,. \tag{37}$$

B. The case relevant to quantum cosmology.

Here, as demonstrated in the Appendix A, the lapse function has the following expression,

$$N = \alpha\,[\pi/2 \pm \arccos(a_1/\alpha)]\,, \tag{38}$$

where a_1 is positive, and is the value of the scale factor, $a(t)$, for $t = 1$.

In the present case we do not need to restrict n to be given by eq. (37), because from eq. (34) with N written above eq. (38), we may still have the closed direction in the desired range for a generic value of n.

If we introduce N, eq. (38), in eq. (34), we find the following equation for the variables p_λ and x,

$$a_1\, x \;=\; \sin\left(\frac{2\pi n}{p_\lambda}\, x\right)\,, \tag{39}$$

where x is related with the cosmological constant by,

$$x \;=\; \sqrt{|\Lambda|}\,, \tag{40}$$

and varies in the range $(0, 1/a_1)$.

For given values of a_1 and n, eq. (39) furnishes a relationship between p_λ and the cosmological constant. It is clear, then, that for a proper choice of p_λ one may derive from eq. (39) the allowed values of Λ, such that, the space-time is regular.

Suppose we adopt the same value of p_λ as in the Subsection IV A, and for simplicity choose n to be given by eq. (37). With these choices eq. (39) reduces to,

$$a_1\, x \;=\; \sin x\,. \tag{41}$$

Equation (39) has solutions for $a_1 < 2/\pi$, and they form a discrete set. The number of solutions will depend on a_1 due to its presence in the superior limit of x.

We may write an approximate rule to the solutions of eq. (39), if we separate them in two sets. The first set has the even order solutions and the second the odd order solutions.

Calling m the order of the solutions, we have the following approximate rules for the two sets of solutions,

$$\text{Even order}\;:\quad |\Lambda|_k \;\approx\; (1-a_1)^2\,(2k-1)^2\,\pi^2\,,\quad k = 1, 2, 3, \ldots, \tag{42}$$

$$\text{Odd order}\;:\quad |\Lambda|_l \;\approx\; (1+a_1)^2\,(2l\pi)^2\,,\quad l = 1, 2, 3, \ldots, \tag{43}$$

where $k = m/2$; $l = (m-1)/2$; and we have used eq. (40). For values of n other than 1 we shall also have equations and solutions similar to eqs. (41) and (42)-(43), respectively.

As we have discussed in Sec. III, the above conditions eqs. (36), (37), (42) and (43), guarantee the regularity of the two expressions of the anti-deSitter space-time, as investigated by using closed loops of one of the four S_i. Their regularity can only be assured if we repeat the above procedure to the other three S_i.

We start by choosing as the representative of each of other three S_i, the closed loops which join the middle points of each remaining pairs of opposite identified sides, as we did for the S_i studied in Sec. III. In the next step, we simply rotate our (x, y) axes, so that, the x axis successively coincide with these three closed loops just introduced. Finally, we separately repeat the analysis above, Secs. III and IV, for each of the three cases.

The choice of the regular octagon as our fundamental region, means that all four independent closed loops have the same length at any time. That property implies that after repeating the analysis above, we shall obtain the same regularity conditions eqs. (36), (37), (42) and (43), for each of the three independent loops above.

Therefore, we may conclude that those conditions guarantee the regularity of the two examples of different expressions of the anti-deSitter space-time studied in this paper.

V. CONCLUSIONS.

In the present paper we have applied the holonomy method to investigate the regularity of space-times which spatial sections are closed as the result of compactifications. In particular we were interested in the conditions to obtain regular space-times.

As a matter of definiteness and simplicity we have studied a (2+1)-dimensional set of different expressions of the anti-deSitter space-times (ADS). Each element of the set differing from the others by the value of a constant, lapse function.

The holonomy matrix of that set was constructed in Sec. III. The regularity conditions for two particular elements of the ADS were derived in Sec. IV.

The first element of the ADS, relevant to classical cosmology [3], will be regular only if the periods of the compactified directions are equal to 2π, Subsec. IV A.

The second element of the ADS, relevant to quantum cosmology as shown in the Appendix A, was studied in Subsec. IV B. Here, the space-time will be regular if, for given values of the initial conditions and two other free parameters, the negative cosmological constant present in the model has the discrete values of eqs. (42) and (43).

It is important to add that the discrete spectrum for the cosmological constant has been derived just by demanding that the space-time be regular. This requirement, following Hartle and Hawking [7], can be considered a first principle.

Our study, in the present paper, may be generalized to other cases. The most obvious generalization is to consider elements of the ADS (7), with values for N different from the two studied here.

One may also apply the above procedure to investigate the regularity of (2+1)-dimensional anti-deSitter space-times which closed spatial sections are negative curved surfaces of genus greater than two. It means that the fundamental regions are not octagons anymore. If the new polygons are regular, the analysis will be a repetition of what we did here for a greater number of independent closed loops. If, on the other hand, the polygons are not regular we shall have to consider the regularity conditions derived for each independent closed loops, separately, and demand that they all agree.

Finally, the use of the holonomy method to examine the regularity of (3+1)-dimensional anti-deSitter space-times with closed, negative curved, spatial sections, follows directly from our work. The main differences are the possible uses of differents projections of the H^3, and the presence of polyhedra, or complexes, as the fundamental regions [8]. It is not difficult to show that although we have the differences, mentioned above, between the (2+1) and the (3+1)-dimensional cases, the line element (7) changes only by the straightforward introduction of an extra dimension, and the modification of the value of α [17]. Therefore, everything we did here may be generalized to this new situation.

ACKNOWLEDGMENTS

I am grateful to I. G. Moss for suggestive discussions in the course of this work. I would also like to thank J. Barrett, S. Nag, P. Parker and I. Raptis for helpful discussions or correspondences, and CAPES and FAPERJ of Brazil for the invaluable financial support.

APPENDIX A

In this appendix, we would like to show that the anti-deSitter space-time with line element (7), lapse function (38), compact spatial sections introduced in Sec. II, and regular as assured by the conditions (42) and (43), contributes at the semi-classical level to the 'no-boundary' wave-function of a certain universe.

The 'no-boundary' conditions proposes that only regular, Euclidean spacetimes, with one single spacelike boundary, should contribute to the path integral leading to the desired wave-function. It means that in the semi-classical approximation the wave-function is given by the following expression [18]:

$$\Psi_{nb}[h_{ij}^b, \varphi_b] \;=\; N_0 \sum_k A_k \exp\left[-I_k\right] \tag{A1}$$

where: N_0 is a normalization constant; I_k is the action of the k-th solution to the Euclidean-Einstein's field equations; and the sum is over these solutions. These solutions have a unique boundary which is compact, spacelike, and has the given three-metric h_{ij}^b and matter or gauge fields configuration φ_b on this boundary. The prefactors A_k are given by determinants of small fluctuations about the classical solutions.

We want to compute the 'no-boundary' wave-function of a (2+1) - dimensional, homogeneous, isotropic, and constant negatively curved universe. For simplicity the only source of stress-energy is a negative cosmological constant. We shall restrict our attention to the semi-classical limit.

We start by proposing the Euclidean F-R-W metric ansatz, with the compact spatial sections introduced in Sec. II . In terms of the ADM formalism [10], it is

$$ds^2 \;=\; N^2(t)\, dt^2 \;+\; a^2(t)\, (\, d\chi^2 \;+\; \sinh^2\chi\, d\theta^2\,)\,, \tag{A2}$$

where $N(t)$ is the lapse function; $a(t)$ is the scale factor; t and θ vary as in eq.(2), Sec. II, and χ has a finite superior limit due to the compactifications.

Introducing eq. (A2) in the general expression for the Euclidean action [19], we obtain the following result,

$$I[N, a(t)] \;=\; -\mathcal{A} \int [\, \dot{a}^2(t) \;-\; N^2 \;-\; a^2(t) N^2 \Lambda\,]\, \frac{1}{N} dt\,, \tag{A3}$$

where: $\mathcal{A}$ is a finite defined number proportional to the volume of the compact, spatial sections; and we chose the gauge where N is a constant.

The variation of I, eq. (A3), in terms of $a(t)$ and N, will produce the Euclidean-Einstein's equations. We shall solve these equations under the 'no-boundary' conditions, suitably adapted for minisuperspaces [7]. For our case these conditions are,

(1) There must be an instant when the scale factor vanishes;

(2) The solution must have the scale factor specified at the boundary;

(3) The space-time must be regular.

We obtain two solutions satisfying the three conditions above.

The space-time which is important to the present study is the Lorentzian solution,

$$N \;=\; iN_I\,, \qquad \text{where} \qquad N_I \;=\; \alpha\frac{\pi}{2} \pm \arccos(\frac{a_1}{\alpha})\,; \tag{A4}$$

$$a(t) \;=\; \pm\alpha\,\sin(N_I\,\frac{t}{\alpha})\,; \qquad \alpha^2 \;=\; \frac{1}{|\Lambda|}\,; \tag{A5}$$

valid for $a_1 < \alpha$.

The solution eqs. (A4) and (A5) satisfy all the three conditions above. More specifically it satisfies condition:

(1) because $a(t)$ vanishes for $t = 0$;

(2) because $a(t)$ is equal to the given value a_1 at the boundary, which is characterized by $t = 1$;

(3) when the conditions (42) and (43), Subsec. IV B, are valid.

Therefore, we conclude that the anti-deSitter space-time with the line element (A2), lapse function (A4), scale factor (A5), compact spatial sections given in Sec. II, and for which conditions (42) and (43), Subsec. IV B, are valid, contributes to the 'no-boundary' wave-function (A1) of this universe.

In order to complete our proof, we must identify the above solution with the anti-deSitter space-time introduced in Subsec. IV B.

We start by renaming N_I in eqs. (A4) and (A5), calling it N. It leads us to eq. (38), Subsec. IV B. This transformation also makes the scale factor eq. (A5) identical to the one in eq. (3), Sec. II. Then, we apply the coordinate transformations (4), Sec. II, to the line element (A2). Finally, we write the resulting line element in Cartesian coordinates which produces eq. (7), Sec. III, concluding our demonstration.

REFERENCES

[1] G. F. R. Ellis, Gen. Rel. Grav. **2**, 7 (1971).

[2] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).

[3] H. V. Fagundes, Ap. J. **291**, 450 (1985); Astrophys. J. **338**, 618 (1989).

[4] J. Louko and P. J. Ruback, Class. Quantum Grav. **8**, 91 (1991).

[5] G. Burbidge, Ann. N Y Acad. Sci. **375**, 123 (1981).

[6] T. Regge and C. Teitelboim, Ann. of Phys. **88**, 286 (1974); and K. V. Kuchař, Phys. Rev. D **50**, 3961 (1994).

[7] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

[8] W. P. Thurston, *The Geometry and Topology of Three-Manifolds* (Princeton University Press, Princeton, 1982), pp. 3.6-3.9.

[9] G. Oliveira-Neto, J. Math. Phys. **37**, 4716 (1996).

[10] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation* (W. H. Freeman, New York, 1973).

[11] S. Deser and R. Jackiw, Ann. Phys. (NY) **153**, 405 (1984).

[12] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time* (Cambridge University Press, Cambridge, 1973), pp. 124-134.

[13] P. Scott, Bull. London Math. Soc. **15**, 401 (1983).

[14] N. L. Balazs and A. Voros, Phys. Rep. **143**, 109 (1986).

[15] C. Series, in *Dynamical Chaos*, Proceedings of a Royal Society Discussion Meeting, edited by M. V. Berry, I. C. Percival and N. O. Weiss (Princeton University Press, Princeton, 1987), p. 171.

[16] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces* (Wiley and Sons, New York, 1988); and R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry* (Universitext) (Springer-Verlag,

New York, 1992), p. 65.

[17] G. Oliveira-Neto, *Pre-print University of Newcastle-Upon-Tyne - U. K.* **NCL-93/TP-5**, (1993).

[18] S. W. Hawking, in *Relativity, Groups and Topology II*, Les Houches 1983, Session XL, edited by B. S. DeWitt and R. Stora (North Holland, Amsterdam, 1984), p. 333.

[19] S. W. Hawking, in *General Relativity*, An Einstein Centenary Survey, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979), p. 746.