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Pedagogical Remarks on Free Field Relativistic Wave Equations and their Geometrical Nature. II

by

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In a preceding note, I have proposed a technique of regarding the line element of space-time as an operator acting on a free field variable and this allows one to derive the geometrical form of the free field relativistic wave equations.

In this way we obtained the wave equations for free spin 1/2 fields, spin 1 fields including Proca's equation, spin 3/2 fields and Einstein equation for weak gravitational fields.

In this note, I would like to show that the geometrical equation associated to Dirac's equation leads in a natural way to the Feynman-Wheeler interpretation of anti-particles as fermions with negative energy particles propagating backward in time.

Indeed, Dirac's equation

$$(i\gamma^\mu \hbar \partial_\mu - m_0 c)\psi(x) = 0 \tag{1}$$

leads to homogeneous algebraic equations, as is so well known, of the form:

$$\begin{aligned}
(E - m_0c^2)u_1 - c(p_1 - ip_2)u_4 - cp_3u_3 = 0 \\
(E - m_0c^2)u_2 - c(p_1 + ip_2)u_3 + cp_3u_4 = 0 \\
(E + m_0c^2)u_3 - c(p_1 - ip_2)u_2 - cp_3u_1 = 0 \\
(E + m_0c^2)u_4 - c(p_1 + ip_2)u_1 + cp_3u_2 = 0
\end{aligned} \tag{2}$$

when one puts:

$$\psi(x) = u(\vec{p}, E)e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})} \tag{3}$$

and uses the representation of Dirac matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad , \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & v \end{pmatrix} \tag{4}$$

The condition for the existence of solutions of equations (2) is that the determinant of their coefficients vanish which gives.

$$\left((p^0)^2 - c^2(\vec{p})^2 - (m_0 c^2)^2\right)^2 = 0$$

and so the roots:

$$p^0 = \pm\ c(\vec{p}\ ^2 + m_0^2 c^2)^{1/2} = \pm E \tag{5}$$

are double.There are thus two solutions with positive energy and two other solutions with negative energy. The Feynmann-Wheeler interpretation, namely negative energy electrons travel backward in time, follows immediately from the geometrical equation

$$\gamma_\alpha dx^\alpha \psi(x) = ds\psi(x)\ . \tag{6}$$

If one chooses the representation (4) for the gammas, the equation (6) will give the homogeneous equations similar to equations (2):

$$\begin{aligned}
(dx^0 - ds)\psi_1 - (dx^1 - idx^2)\psi_4 - dx^3\psi_3 = 0 \\
(dx^0 - ds)\psi_2 - (dx^1 + idx^2)\psi_3 + dx^3\psi_4 = 0 \\
(dx^0 + ds)\psi_3 - (dx^1 - idx^2)\psi_2 - dx^3\psi_1 = 0 \\
(dx^0 + ds)\psi_4 - (dx^1 + idx^2)\psi_1 + dx^3\psi_2 = 0
\end{aligned}$$

The determinant of the coefficients of these equations must vanish which gives:

$$\left((dx^0)^2 - ds^2 - (d\vec{x})^2\right)^2 = 0$$

and so the roots

$$dx^0 = \pm\left(ds^2 + (d\vec{x})^2\right)^{1/2}$$

are double. The time intervals are either positive or negative. It is natural to associate positive energy solutions to time intervals which are always positive. Then the negative energy solutions will have negative time intervals – they represent negative energy fermions running always backward in time (the usual interpretation was elaborated by analogy with the classical equation of motion of the electron).