CBPF-NF-066/98

# Ohm's Law on Wave Mechanics, A Simple Deduction

*Luiz C.L. Botelho*

*Departamento de Física*
*Universidade Federal Rural do Rio de Janeiro*
*23851–180 – Itaguaí, RJ, Brasil*

ABSTRACT

We deduce the classical Ohm's Law in a conductor medium from the Schrödinger equation associated to classical electron Caldirola-Kanai action with a damping dissipative anomaly factor.

It is an interesting problem in Dissipative Quantum Mechanics ([1]) to deduce Ohm's law directly from the Schrödinger equation without taking recourse to complicated transport (Maslov) many body formalism.

In this letter, we consider the effective Schrödinger equation associated to the classical Caldirola-Kanai action associated to the motion of a electron flow in a medium under the presence of a constant (unidimensional) Elective field and a damping term phenomenologically associated to electron collisions with ions and deduce in a simple way the classical Ohm's law.

Let us start our analysis with the classical one electron Caldirola-Kanai lagrangean in one dimensional case for the damped electron in the presence of a Electric field $E$

$$\mathcal{L}[x(t)] = \int_0^T d\sigma \left( e^{\nu\sigma} \left[ \frac{1}{2} m_e \left( \frac{dx}{d\sigma} \right)^2 - e\ E \cdot x(\sigma) \right] \right) \tag{1}$$

According to our previous studies ([1],[2]), and "effective"wave function can be associated to the dissipative system with a mass time dependent term involving the damping reaction of the medium on the quantum electron

$$i\hbar \frac{\partial \psi^{ins}(x,t)}{\partial t} = \left( -\frac{1}{2e^{\nu t} m_e} \cdot \frac{d^2}{dx^2} + \frac{e}{m_e}(e^{\nu t} E)x \right) \psi^{ins}(x,t) \tag{2}$$

the initial condition

$$\psi^{ins}(x,0) = \delta(x) \tag{3}$$

is related to the fact that the electron is in a rest-origin situation at $t = 0$.

In order to deduce the Ohm's law by a simple procedure, we should evaluate the electron velocity quantum operator (the electronic current) in the quantum damped state eq. (2)

$$j(t) = \int_{-\infty}^{+\infty} dx\ \psi^{ins}(x,t) (i\hbar \frac{\partial}{\partial x} (\psi^{ins}(x,t))^* \tag{4}$$

Let us now solve exactly eq. (2)-eq. (3), by considering the new "time" variable on those equations

$$\zeta = \frac{1}{m_e \nu}(1 - e^{-\nu t})$$
$$\tilde{\psi}^{ins}(x,\zeta) = \psi^{ins}(x,t) \tag{5}$$

The new Schrödinger equation takes, thus, the form of exactly soluble problem in the new coordinate system $(x, \zeta)$

$$i\hbar\frac{\partial\tilde{\psi}^{ins}(x,\zeta)}{\partial\zeta} = \left[-\frac{1}{2}\frac{d^2}{dx^2} + \frac{e}{m_e}\ E\cdot x\left(\frac{1}{(1-\nu\zeta)^2}\right)\right]\tilde{\psi}^{ins}(x,\zeta) \tag{6}$$

with

$$\tilde{\psi}(x,\zeta)_{\zeta\to 0^+} = \delta(x) \tag{7}$$

The solution of eq. (6) is well-know ([3]) for the initial condition eq. (7)

$$\tilde{\psi}^{ins}(x,\zeta) = \sqrt{\frac{m_e}{2\pi i\hbar\zeta}}$$
$$exp\left\{\frac{i}{\hbar}\frac{m_e}{2\zeta}\left[x^2 - \frac{2x}{m_e}\int_0^\zeta L(s)\cdot s\cdot ds - \frac{2}{m_e^2}\int_0^\zeta ds\ L(s)\int_0^s ds'(L(s')(\zeta-s)\cdot s')\right]\right\} \tag{8}$$

Here we have introducedd the simplified notation

$$L(s) = \frac{e}{m_e}E\cdot\frac{1}{(1-m_e\nu s)^2} \tag{9}$$

It is a straightforward calculation now with $m_e = 1$ to obtain the form of the electronic current per volume in our theoretical model for quantum dissipative system ([1],[2]) after disregarding the current associated to the free case of our analysis. It yields the following result

$$j(t) = 2e\ E\nu\cdot\frac{t}{(1-e^{-\nu t})2} - 4e\ E\cdot artg\frac{(1-e^{-\nu t})}{(1-e^{-\nu t})^2} \tag{10}$$

By taking now, the steady limit (see appendix A) with the dissipative anomaly factor of ref. [1] eq. (10) with $a - b < 0$ (the damping case), namelly

$$\begin{aligned}\bar{j} &= \lim_{T\to\infty}\ \frac{1}{T}\int_0^T\ j(\alpha)e^{-\frac{\nu}{2}|a-b|\alpha}d\alpha \\ &= 2e\ E\nu\left(\lim_{T\to\infty}\ \frac{1}{T}\int_0^{\nu T}d\beta\ \frac{e^{-\frac{\nu}{2}|a-b|\alpha}}{(1-e^{-\beta})^2}\left(\frac{1}{\nu}\right)^2\cdot\beta\cdot d\beta\right) \\ &\simeq\ \frac{\bar{c}}{\nu}\cdot E\end{aligned} \tag{11}$$

where $\bar{c}$ is a $\nu$-*independent* constant (see appendix A), we get the Ohm's law if one identities the medium electrical resistence $R$ as proportional to the damping model constant $\nu$, a physical acceptable parameter model's correspondence.

## Acknowledgements

Luiz Carlos C.L. Botelho is thankfull to CNPq (Brazil Science Government Agency) for a grant.


## References


[1] Luiz C.L. Botelho and Edson P. Silva, *"Feynman Path-Integral for the Damped Caldirola-Kanai Action"*, Phys. Rev. E, vol. 58, $n^{\underline{0}}$ 1 (1998), 1141.

[2] Luiz C.L. Botelho and Edson P. Silva, Modern Physics Letters B, vol. 12, $n^{\underline{0}}$ 14 e 15 (1998), 569–573.

[3] R.P. Feynman and A.R. Hibbs, *"Quantum Mechanics and Path Integrals"* (McGraw-Hill, NY), 1965.

# Appendix A

In this appendix we give a proof of the following theorem of ours

Theorem: Let $f(\alpha)$ be a integrable function in any internal of the form $[0, \nu T]$ with $T \in R^+$, $0 < a \leq \nu \leq b$ and such that 1) $\lim_{\alpha\to\infty} f(\infty) - f(\infty) < 0$ and the ergodic mean time average exists

$$\lim_{T\to\infty} \frac{1}{T} \int_0^{\nu T} f(\alpha) d\alpha = I(\nu) \tag{A.1}$$

Then the function $I(\nu)$ is a linear function of the form $I(\nu) = f(\infty)\nu$ for $a \leq \nu \leq b$.

In order to show this mathematical result, let us consider the derivative definition at a point $\bar{\nu} \in (a, b)$

$$I(\bar{\nu}) = \lim_{h\to 0} \frac{I(\bar{\nu}+h) - I(\bar{\nu})}{h} \tag{A.2}$$

$$= \lim_{h\to 0} \left( \frac{1}{h} \left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^{(\bar{\nu}+h)T} f(\alpha)d\alpha - \lim_{T\to\infty} \frac{1}{T} \int_0^{\bar{\nu}T} f(\alpha)d\alpha \right\} \right)$$

$$\lim_{h\to 0} \left( \frac{1}{h} \left\{ \lim_{T\to\infty} \frac{1}{T} \int_{\bar{\nu}T}^{(\bar{\nu}+h)T} f(\alpha)d\alpha \right\} \right)$$

$$\lim_{h\to 0} \left( \left\{ \lim_{T\to\infty} \frac{1}{Th} (f((\bar{\nu}+\varepsilon)T)) \cdot Th \right\} \right) = f(\infty) \tag{A.3}$$

here we have applied the mean value theorem with $0 \leq \varepsilon \leq h$ in the last line of our sequence of equations.

In our case $f(\infty) = 0$, and, thus, leading to the result that for any $\nu > 0$.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^{\nu T} \frac{e^{-\frac{|a-b|}{4}\cdot\beta} \beta d\beta}{(1-e^{-\beta})^2} = \lim_{T\to\infty} \frac{1}{T} \int_0^{T} d\beta \, \frac{e^{-\frac{|a-b|}{4}\cdot\beta} \beta d\beta}{(1-e^{-\beta})^2} \tag{A.4}$$

# Appendix B – Ohm's law from Langevin equation

In the classical framework, the Lagrangian equation describing the (one-dimensional) electronic flux in a Ohm's resistive medium is given by the following equation for the electrical current $I(t)$

$$\frac{\partial I(t)}{\partial t} = \delta\left(\frac{e}{m_e}\right) E - \nu I - \delta \frac{f(t)}{m_e} \tag{B.1}$$

where $\sigma$ is an area flux parameter and $f(t)$ is the stochastic white noise component of the "Brownian" resistive medium with the following pure Gaussian path integral representing its characteristic functional

$$Z[k(t)] = \frac{\int D^F[f(s)] e^{-\frac{1}{\gamma}\int_0^\infty ds[f(s)]^2} e^{i\int_0^\infty ds\ k(s)f(s)}}{\int D^F[f(s)] e^{-\frac{1}{\gamma}\int_0^\infty ds[f(s)]^2}} \tag{B.2}$$

After substituting eq. (B.1) into eq. (B.2) and by taking into account that $det\left[\frac{d}{dt} - \nu I\right] = 1$, one gets the characteristic functional for the electrical current

$$\begin{aligned} Z[J(t)] &= \frac{x1}{Z(0)} \int D[I(s)] \cdot det\left[\frac{d}{dt} - \nu I\right] \\ &\times \exp\left\{-\frac{1}{2\gamma\sigma^2}\int_0^\infty ds \left[\dot{I} - \left(\frac{e}{m_e}\right)\sigma E + \nu I\right]^2 (s)\right\} \\ &\times \exp\left\{i \int_0^\infty ds I(s) J(s)\right\} \end{aligned} \tag{B.3}$$

Now it is a straightforward calculation to evaluate the current average and obtain the Ohm's law in this classical situation (compare with eq. (11) in the text)

$$\begin{aligned} \bar{I} = \langle I(\bar{t})\rangle = \frac{FZ[J(t)]}{FJ(\bar{t})}|_{J(t)=0} &= \nu\left(\frac{e}{m_e}\right)\sigma\ E \int_0^\infty ds \left[\frac{e^{-\nu(s-t)} \cdot \theta(s-\bar{t})}{\nu}\right] \\ &= \frac{1}{\nu}\left(\frac{e}{m_e}\right) \cdot \sigma \cdot E \end{aligned} \tag{B.4}$$