Does the Gravitational Waves Travel at Light Velocity?

by

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ABSTRACT

We extend the standard Feynman-Deser approach of field theoretical derivation of Einstein's gravitational theory. We show that it is possible to obtain a theory that incorporates a great part of General Relativity (GR) and can be interpreted in the standard geometrical way like GR, as far as the interaction of matter to gravity is concerned. The most important distinction of the new theory concerns the gravity to gravity interaction. This theory satisfies all standard tests of gravity and lead to new predictions about gravitational propagation. Since there is a large expectation that the detection of gravitational waves will occur in the near future, the question of which theory describes Nature better will probably be settled soon.

Key-words: Gravity; Gravitational waves; Observational.

1 Introduction

1.1 A. Introductory Remarks

There is a general expectation concerning the possibility that we could detect gravitational waves before the next century is born. Such situation is based on the great number of new experimental devices that many laboratories, throughout the world, are constructing. Many scientists are going to become involved into this enterprise. Thus the time is ripe to theoretical re-examination of gravity theory in order to make predictions on the propagation of gravitational disturbances that could be tested in the near future.

Although the great majority of physicists expect that the observation of these waves will confirm the General Relativity prediction[1], it seems worthwhile to remember that such an expectation may just be one of those prejudgements that sometimes spread into the scientific community. On the other hand, we would like to emphasize that if this General Relativity result is proven to be false, it does not destroy the remaining and by far the major part of Einstein's theory that has already been experimentally proved.

The Equivalence Principle - which states that all kinds of matter (including massless particles as photons) interact in an unique and same way with the gravitational field - gave to Einstein the possibility to treat gravitational phenomena as a sort of modification of the spacetime geometry.

Does this universal behavior occur also in the case of gravity-gravity interaction? General Relativity makes an implicit hypothesis by means of which the answer to this question is affirmative. Although there is not even a single observational evidence that this is true, the fact that gravity must carry energy as any other field gave a strong motivation to believe that such an extrapolation concerning gravity-gravity interaction should be valid.

This assumption, that still today remains beyond any real observation, led to the most impressive result of General Relativity, that is, that gravitational processes are nothing but a real universal modification of the spacetime geometry.

However, if we want to keep observation as the true guide of our analysis of Nature, the whole actual situation can be summarized in two statements:

- As far as matter-to-gravity interaction is concerned, the GR scheme of geometrization of gravity seems to be a very good procedure.

- There is not any single direct observational evidence that supports that the self-interaction of gravity can be described in the same way.

Thus, if we limit ourselves to the traditional scientific method of submission of theory to observation, we must say that the universal modification of the geometry proposed by GR is indeed an extrapolation that is not still confirmed by experimental means, as far as the behavior of gravity-gravity coupling is concerned.

We are thus led to take seriously into account the two excludent alternatives concerning such gravity-to-gravity interaction, to wit:

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[1]The recent spectacular success of the description of pulsar behavior, through loss of gravitational energy, by General Relativity, increased enormously the status of this theory.

- Gravity couples to gravity as any other form of energy.

- Gravity couples to gravity in a special way distinct from all different forms of energy.

In the first case, the universal modification of the geometry, as proposed in General Relativity, becomes a natural scenario. Nevertheless, the complete absence of any experimental evidence that could help us to solve this question tells us that the decision must be dictated by other means, e.g., either by theoretical arguments[2] or by some sort of additional requirements imposed on the interaction mechanism. Einstein's hypothesis that gravitational interaction is the same not only when gravity-matter interaction is concerned but also for gravity-gravity processes, appeared to be the most natural way. In those early times, the complete absence of observations postponed the decision on this for the future. Later, in the fifties, the discovery of a simple way to treat gravity in terms of a standard field theory [1], reducing the gap between GR and all other field theories, made Einstein's hypothesis less questionable.

The purpose of the present work is to show a new way to generate a field theoretical description of gravity.

The fundamental property, that rests on the basis of our theory and that distinguishes the present program of investigation from GR, can be synthesized by the assertion that we will examine the possibility of conciliating the universality of matter to gravity interaction but considering that gravity to gravity coupling is somehow distinct. Since, as we shall see, the most dramatic consequence of the theory we propose here deals with the modification of the velocity of the gravitational waves, we limit our analysis here to the exam of this question.

Just in order to support this statement we note that it is a theoretical prejudice to argue that GR has already settled this question: only observation could do this.

The new net result (concerning the propagation of gravitational waves) of our model can be summarized by noting that in this theory gravitational waves propagate in an effective geometry that is not the same as viewed by matter. One can then ask "what is the true geometry of spacetime"? The answer will be: it depends on the instrumentation we use to observe it. All forms of energy, except the gravitational, measure the same universal modification of curved spacetime. However, gravitational waves behave as if they were imbedded in a distinct geometry. What should the origin of this fact be? We postpone this question for latter analysis.

Just to provide a simple example of a theory that displays these ideas we will take as a model a non-linear theory proposed many years ago by Born [4] for spin-one field. We will apply a similar model for the spin-two case. Although in the original proposal of spin-one field such a non-linear theory appeared just as an exotic one, in the case of spin-two that we treat here the non-linearity is mandatory. We shall see in a subsequent section that this situation is just a consequence of the fact that spin-two couples to the energy-momentum tensor and the spin-one field couples to a conserved current. The fact that gravity itself must, for consistency, have energy makes the non-linearity a necessary requirement of any description of gravity.

In the standard geometrical way such self-interaction is described by a non-polinomial Lagrangian. In our re-exam of gravity from the point of view of field theory we will

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[2]At most times such a decision comes from theoretical prejudices.

take (just as an example of constructing a coherent field theoretical model) the Born non-polinomial Electrodynamics as a paradigm for the gravitational interaction.

Finally, let us note that although we concentrate here all our presentation of the new theory only to the gravitational waves, we will analyse the properties of the new theory related to the behavior of the gravitational field in two important situations, to wit:

- The static spherically symmetric field;

- Cosmology.

In this paper we present a solution concerning the first case. We show then that the standard tests of gravitational processes are satisfied by our present theory. The second imortant case of cosmological framework of our theory is left for a subsequent paper.

1.2   B. Synopsis

The presentation of the paper is the following. In Section (2) we present the main definitions and symbols that we use.

In Section (3) we discuss briefly how and why Feynman [1], Deser [2] and others (see [3] for a recent review) were led to describe General Relativity in terms of a field theory only based on the universality of gravitational interaction; we revise the Fierz linear theory and propose a class of non-linear theory of spin-two field. We also study the behaviour of the gravity energy-momentum tensor in the new theory. We present a new derivation of Fierz equation of motion that is worth of generalization in the non-linear case.

In the Section (4) we propose a specific theory and, by analyzing the evolution of the disturbances in this case, we show that the gravitational waves propagate on the null cone of an effective geometry distinct from that one seen by matter.

In Section (5) we present the gravity-matter interaction process and following the standard procedure we show how such an interaction can be described in terms of a modification of the geometry of the spacetime, in the same manner as it occurs in General Relativity.

In section (6.1) we present a solution for the static and spherically symmetric configuration. In section (6.2) we show the effective geometry that is the observed effect of the previous solution as seen by any non-gravitational form of energy.

In section (7) we evaluate the corresponding distribution of gravitational energy within our theory.

Finally, in the last Section (8) we make some comments and state future perspectives of our scenario on gravity.

2   Definitions and Notations

The auxiliary metric $\gamma_{\mu\nu}$ of Minkowski geometry[3] is written in an arbitrary system of coordinates in order to exhibit the general covariance of the theory. We define the corresponding covariant derivative by

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[3]We shall see in next sections that this metric is not observable neither by matter nor by gravitational field.

$$V_{\mu\,;\,\nu} = V_{\mu\,,\,\nu} - \Delta^{\alpha}_{\mu\nu} V_{\alpha} \tag{1}$$

in which

$$\Delta^{\alpha}_{\mu\nu} = \frac{1}{2}\gamma^{\alpha\beta}\left(\gamma_{\beta\mu\,,\nu} + \gamma_{\beta\nu\,,\mu} - \gamma_{\mu\nu\,,\beta}\right). \tag{2}$$

The associated curvature tensor vanishes identically that is

$$R_{\alpha\beta\mu\nu}(\gamma_{\epsilon\lambda}) = 0. \tag{3}$$

We define a three-index tensor $F_{\alpha\beta\mu}$, which we will call the **gravitational field**, in terms of the symmetric standard variable $\varphi_{\mu\nu}$ (which will be treated as the potential) to describe spin-two fields, by the expression

$$F_{\alpha\beta\mu} = \frac{1}{2}(\varphi_{\mu[\alpha;\beta]} + F_{[\alpha}\gamma_{\beta]\mu}) \tag{4}$$

where we are using the anti-symmetrization symbol [ ] like

$$[A,B] \equiv AB - BA. \tag{5}$$

We use an analogous form to the symmetrization symbol ( )

$$(A,B) \equiv AB + BA, \tag{6}$$

The quantity F_{α} is the trace

$$F_{\alpha} = F_{\alpha\mu\nu}\gamma^{\mu\nu}$$

that is,

$$F_{\alpha} = \varphi_{,\alpha} - \varphi_{\alpha\mu;\nu}\gamma^{\mu\nu}$$

From the above definition it follows that this quantity $F_{\alpha\beta\mu}$ is anti-symmetric in the first pair of indices and obeys the cyclic identity, that is

$$F_{\alpha\mu\nu} + F_{\mu\alpha\nu} = 0, \tag{7}$$

$$F_{\alpha\mu\nu} + F_{\mu\nu\alpha} + F_{\nu\alpha\mu} = 0. \tag{8}$$

The quantity κ represents Einstein's constant, written in terms of Newton's constant G_N and the velocity of light c by the definition

$$\kappa = \frac{8\pi}{G_N c^4}.$$

We set $c = 1$.

3 From the Universal Coupling of Matter to Gravity to the Einstein Geometrization Scheme

In this section we will briefly review some points that were used in order to implement the hypothesis of universality of gravity interaction by the modification of the metrical properties of the spacetime. We will be interested here not in its historical birth of Einstein's formulation but, instead in its equivalent field theoretical formulation. In other words, we will follow here the path that led from the linear field theory of gravity to its non-linear processes and consequently to the corresponding geometrization scheme[1], [2].

There is no better and simpler manner to describe this than the one set by Feynman in his lecture notes of 1962. Let us summarize such standard procedure that led from the field theoretical to the geometrical description of gravitational interaction.

The starting point is the linear massless spin-two field theory (Fierz equation), that reads:

$$G^{(L)}_{\mu\nu} = -kT_{\mu\nu} \tag{9}$$

in which

$$G^{(L)}_{\mu\nu} \equiv \Box\phi_{\mu\nu} - \phi^{\alpha}{}_{\mu;\alpha\nu} - \phi^{\alpha}{}_{\nu;\alpha\mu} + \phi^{\alpha}{}_{\alpha;\mu\nu} - \gamma_{\mu\nu}(\Box\phi^{\alpha}{}_{\alpha} - \phi^{\alpha\beta}{}_{;\alpha\beta}). \tag{10}$$

The quantity $G^{(L)}_{\mu\nu}$ is divergence-free. This implies that, for compatibility, one must impose the condition that the energy-momentum tensor $T_{\mu\nu}$ of matter should also be divergenceless. Now, since the gravitational field contributes to the balance of the conservation law through its own energy, this imposition faces a difficulty since the matter energy-momentum tensor cannot be separately conserved.

It is precisely at this point that the hypothesis that gravity-gravity process follows the same type of behavior as matter-gravity interaction acts as a guide to the choice of the gravitational contribution to the source of $G^{(L)}_{\mu\nu}$. This means to add to the energy-momentum tensor of matter, the corresponding energy-momentum tensor for the gravitational field at the right-hand-side of equation (9).

The idea is to proceed step by step. We start by adding to the right-hand-side of equation (10) the tensor $\mathcal{T}^{(1)}_{\mu\nu}$, that is the energy momentum tensor of the linear equation for gravity. As a consequence we must add to the original Lagrangian an additional term of higher order that yields, after variation, the term $\mathcal{T}^{(1)}_{\mu\nu}$ to be added to the equation of motion. This generates a new compatibility condition, which is solved by adding to the right-hand-side of the equation of motion a new term of higher order, $\mathcal{T}^{(2)}_{\mu\nu}$. This will impose, once more, that a new term must be added to the Lagrangian. This process continues indefinitely, since at each step a new term must be added to the Lagrangian in order to achieve the compatibility lost at the precedent lower order. To accomplish the task and to solve completely the compatibility condition, we must deal with a recurrence procedure such that yields an infinite series to appear [1]. It is precisely the summation of such infinite series that can be described by the equivalent geometrical formulation of General Relativity. Our purpose in the present work is to re-analyse this field theoretical description and to show that the above traditional procedure of searching compatibility is not unique.

3.1   Fierz Linear Theory Revisited

We will now show that the linear theory of spin-2 field can be described using the invariants constructed with the gravitational field $F_{\alpha\mu\nu}$. We will perform this simple exercise here just in order to present the motivation for our further non-linear theory.

There are only two invariants that can be constructed with the field.[4] They are:

$$A \equiv F_{\alpha\mu\nu} F^{\alpha\mu\nu},$$

$$B \equiv F_\mu F^\mu.$$

General covariance imposes that the Lagrangian that one can construct to describe the evolution of the gravitational field must be a functional of these invariants, that is

$$L = L(A, B).$$

The linear theory for the spin-2 field is given by the action

$$S^L = \int d^4x \sqrt{-\gamma} \, (A - B). \tag{11}$$

in which γ represents the determinant of $\gamma_{\mu\nu}$.

The proof of this assertion can be made either by a direct inspection on the equation of motion obtained from this Lagrangian or by noting that up to a total divergence we can write

$$S^L = \int d^4x \sqrt{-\gamma} \varphi^{\mu\nu} G^L_{\mu\nu}. \tag{12}$$

Indeed, we have from eq. (12),

$$S^L = - \int d^4x \sqrt{-\gamma} \varphi^{\mu\nu} F^\lambda{}_{(\mu\nu);\lambda} = \int d^4x \sqrt{-\gamma} \varphi^{\mu\nu;\lambda} F_{\lambda(\mu\nu)}. \tag{13}$$

A straightforward manipulation of this expression shows, up to a total divergence, that

$$\varphi^{\mu\nu;\lambda} F_{\lambda(\mu\nu)} = -2(A - B). \tag{14}$$

This demonstrates our assertion. What we have learned from this simple manipulation is that any theory that provides Fierz linear equation of motion in the weak field limit should reduce to the above combination of the invariants A and B. It is tempting then to examine those theories that are functionals **only** of this combination. We will limit thus all our analysis only to this set of theories. Besides, in the present paper we will consider a specific example of dynamics represented by a Lagrangian that is constructed as a non-polinomial functional of the field variables.

Before going into the exam of such a non-linear theory for the gravitational field, let us make a very short résumé of a typical example of a class of non-linear spin-one theory.

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[4]We could use instead of invariant A the one construted with $C_{\alpha\mu\nu}$, the traceless part of $F_{\alpha\mu\nu}$ which employs its irreducible parts. For simplicity of comparison to the traditional Fierz theory we decided here to make the above choice.

We shall see that many properties of this example will have a deep analogy to the spin-two case. This will be useful since it will act as a guide for the analysis of the more complex case of the gravitational field.

3.2   Non-Linear Spin-One Theory

The dynamics is provided by an action[5]

$$S = \int d^4x \sqrt{-\gamma} L(F) \tag{15}$$

where the Lagrangian L that depends non-linearly on the invariant F constructed with the field $F_{\mu\nu}$ by the product[6]

$$F \equiv F_{\mu\nu} F^{\mu\nu}$$

The corresponding equation of motion that follows is[7]

$$\{L_F F^{\mu\nu}\}_{;\nu} = \frac{1}{4} J^\mu \tag{16}$$

where L_F represents the functional derivative of the Lagrangian with respect to the invariant. Maxwell theory is the case in which this derivative is the constant $-\frac{1}{4}$.

We will limit our analysis here to the theory that represents a non-linear electrodynamics, which was suggested by Born and developed by Infeld many years ago. The Lagrangian is given by

$$L = -\frac{1}{4} \left\{ \sqrt{b^4 + 2b^2 F} - b^2 \right\}, \tag{17}$$

in which the constant b has the meaning of the maximum possible value of the field. There are two important properties of such theory that interest us, to wit:

- The theory is non-linear.

- The propagation of the electromagnetic waves can be described as if the metrical properties of the spacetime were changed by the presence of the non-linear electromagnetic field.

These two qualities of this type of theory will be explored in the sequence in order to construct a non-linear spin-two field theory.

Using the general form of expressing the energy-momentum tensor $T_{\mu\nu}$, presented in a previous section, we obtain from the Lagrangian (17) the following:

$$T_{\mu\nu} = -L\gamma_{\mu\nu} - 4L_F F_{\mu\alpha} F^\alpha{}_\nu \tag{18}$$

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[5] The attentive reader should notice that in this section the quantity $F_{\mu\nu}$ represents the Electromagnetic field.

[6] We consider here, just for simplicity the particular form of the theory which does not contain the invariant constructed with the dual of the field.

[7] We remind the reader that although we deal here with Minkowski background metric, we are using covariant derivatives just in order to exhibit the general covariance of physics.

We note that this quantity has basically all algebraic properties and symmetries that appear in the linear Maxwell case. For our purposes here the interesting property in the non-linear case is the fact that the interaction of the field with external currents remains the same. This can be seen most easily by a direct evaluation of the exchange of field's energy with its sources. Since we are dealing in the present paper with a most complex situation for the case of spin-two field, let us spend some time here and show this simple result in detail in the simpler case of spin-one field, just in order to get some insight of what should be expected for the spin-two case.

Thus, our task now is to evaluate the ratio of exchange of energy of the field, that is, the divergence of the symmetric energy-momentum tensor $T_{\mu\nu}$ of the class of non-linear spin-one field. From the equation of motion (16) contracted with $F_{\alpha\mu}$ we obtain

$$\{L_F F_{\alpha\mu} F^{\mu\nu}\}_{;\nu} - L_F F_{\alpha\mu;\nu} F^{\mu\nu} = \frac{1}{4} F_{\alpha\mu} J^{\mu}$$

Using the expression of the tensor $\mathcal{T}_{\mu\nu}$ we re-write this under the form

$$T_{\alpha}{}^{\mu}{}_{;\mu} + L_F F_{,\alpha} + 4 L_F F^{\mu\nu} F_{\alpha\mu;\nu} = -F_{\alpha\mu} J^{\mu}$$

and thus finally

$$T_{\alpha}{}^{\mu}{}_{;\mu} = -F_{\alpha\mu} J^{\mu} \tag{19}$$

The remarkable fact that follows from this expression is the well-known result that the balance of forces trough the exchange of energy of the field and the currents is independent of the form of the dependence of the Lagrangian on the invariant F. This is the lesson we learn from this simple analysis. Let us pass now to the gravitational field.

3.3   A Class of Non-linear Spin-Two Theory

As we saw in the previous section, when passing from the linear theory of gravity to the general case the standard procedure is to add to the energy-momentum tensor of matter, the corresponding energy tensor for the gravitational field. Proceeding step by step, the first non-linear term contains $\mathcal{T}_{\mu\nu}^{(1)}$, which is the energy-momentum tensor obtained from the linear part. This procedure is based on the implicit hypothesis that one should treat the gravitational energy in the same foot as any other form of energy. In other words, the gravitational field generated by the gravitational energy is not distinct from the field generated by any other form of energy. This is a further extrapolation of the Equivalence Principle, applied to gravitational energy. At this point we take a path which is different from the one followed by Feynman, Deser [1, 2], and others. Instead of adding to the source of the field the successive energy-momentum tensors of the gravitational field for each order of non-linearity, we make the hypothesis that these terms that represent gravity-to-gravity interaction must be constructed as a functional of the two invariants A and B. We thus set our action for the free gravitational field to be given by

$$S = \int d^4 x \sqrt{-\gamma} L(A, B) \tag{20}$$

Variation of the potential $\varphi_{\mu\nu}$ yields

$$\delta S = \int d^4 x \sqrt{-\gamma} \, \Theta^{\lambda}{}_{\mu\nu;\lambda} \, \delta\varphi^{\mu\nu} \tag{21}$$

giving the equation of motion

$$\Theta^{\lambda}{}_{\mu\nu;\lambda} = 0 \tag{22}$$

and $\Theta^{\lambda}{}_{\mu\nu}$ is

$$\Theta_{\lambda\mu\nu} \equiv L_A F_{\lambda(\mu\nu)} - (L_A + L_B)[2F_\lambda \gamma_{\mu\nu} - F_\mu \gamma_{\nu\lambda} - F_\nu \gamma_{\mu\lambda}] \tag{23}$$

in which we have used the definitions $L_A \equiv \delta L / \delta A$. In the special linear case in which $L_A = -1/2$ and $L_B = 1/2$ this expression reduces to the Fierz case:

$$\Theta^{(L)}_{\lambda\mu\nu} = -F_{\lambda(\mu\nu)}. \tag{24}$$

in which the upperscript (L) stands for the linear case. Using the property above (see Section 2) it follows that the equation of motion reduces to

$$F^{\lambda}{}_{(\mu\nu);\lambda} = -G^{(L)}_{\mu\nu} = 0. \tag{25}$$

and $G^{(L)}_{\mu\nu}$ is the Fierz linear operator (see eq. (9)).

Since we would like to impose that our theory should provide the good weak field limit, that is, Fierz linear equation, we will restrict our analysis in this paper to those Lagrangians whose dependence on the invariants obeys the relationship:

$$L_B = -L_A. \tag{26}$$

Under this hypothesis the equation of motion for the free gravitational field within our scheme, equation (22) takes the form

$$\left\{ L_A F^{\lambda}{}_{(\mu\nu)} \right\}_{;\lambda} = 0. \tag{27}$$

Using the properties of $F^{\lambda}{}_{(\mu\nu)}$ we can re-write this expression in a more convenient form:

$$G^{(L)}_{\mu\nu} = L_A{}^{-1} \left\{ L_{A;\lambda} F^{\lambda}{}_{(\mu\nu)} \right\} \tag{28}$$

Note that this is an exact equation, that is, it does not contain any sort of approximation term. We have just isolated the linear Fierz operator on the left-hand side and set all non-linearity terms to the right-hand side. Besides, under this form one can see directly that the source of the non-linearity, contrary to the case of GR, is not expressed in terms of the energy-momentum tensor of the gravitational field. We will make this point clearer when we treat the case of interaction with matter.

3.4   The Gravitational Energy-Momentum Tensor

The standard definition of the energy-momentum tensor for any field is provided by the variation of the Lagrangian with respect to the underlying metric, through the expression

$$\mathcal{T}_{\mu\nu}^{(g)} = \frac{2}{\sqrt{-\gamma}} \frac{\delta L \sqrt{-\gamma}}{\delta \gamma^{\mu\nu}} \tag{29}$$

For the Lagrangians that we examine here (which obey the condition set up trough equation (26)) we have

$$\mathcal{T}_{\mu\nu}^{(g)} = -L\,\gamma_{\mu\nu} + 2L_A \left\{ \frac{\delta A}{\delta \gamma^{\mu\nu}} - \frac{\delta B}{\delta \gamma^{\mu\nu}} \right\} \tag{30}$$

After a rather long although direct calculation we obtain the form:

$$\begin{aligned}
\mathcal{T}_{\mu\nu}^{(g)} &= -L\gamma_{\mu\nu} + 2L_A \left\{ 2F_{\mu\alpha\beta}\,F_\nu{}^{\alpha\beta} + F_{\alpha\beta\mu}F^{\alpha\beta}{}_\nu - F^\alpha F_{\alpha(\mu\nu)} \right. \\
&\quad \left. -\; F_\mu F_\nu \right\},
\end{aligned} \tag{31}$$

in which the symbol g stands for **gravity**.

This is the form of the gravitational energy-momentum tensor of the gravitational field, obtained from the Lagrangian L.

Take the trace of the above form to arrive at

$$\mathcal{T}^g = -4L + 6L_A \left\{ A - B \right\} \tag{32}$$

It is a direct exercise, left to the reader, to show that in the particular case of the linear action, (equation (11)), this expression reduces to Gupta energy-momentum tensor.

The tensor $\mathcal{T}_{\mu\nu}^{(g)}$ differs from the one obtained from Noether's theorem by a total divergence. For future references it is useful to write the Noether energy-momentum tensor:

$$N_{\mu\nu} = -L\,\gamma_{\mu\nu} - \frac{1}{2}\varphi_{\alpha\beta;\nu}\,L_A F_\mu^{(\alpha\beta)} \tag{33}$$

Under this form the examination of the balance of energy between the gravitational field and its sources assumes a very simple expression

$$N^\mu{}_{\nu;\mu} = -T_{\alpha\beta}\,\varphi^{\alpha\beta}{}_{;\nu} \tag{34}$$

This can be shown either using an analogous trick as in the spin-one case as described above or just by direct calculation. Let us remark that, like in the previous case of spin-one, the balance of energy between the gravitational field and its sources is independent of the form of the Lagrangian one takes to represent the gravitational field.

In the next section we will consider a specific simple case of gravitational theory by a choice of the Lagrangian in this scheme.

4   A Suggestive Model to Gravitation

We have dealt in the precedent sections with the general scenario for our construction of a theory of gravity. The aim of this section is to produce a specific characterization of the gravitational equations of motion by searching a Lagrangian that satisfies the requirements set up in the precedent sections. This means that we will undertake now the task to produce a specific example of a field theory for the gravitational field that fulfills both conditions:

- obeys the requirements set up in the precedents section (including that it satisfies the equation (27)).

- agrees with the observed tests of the gravitational field.

Just to simplify our explanation of the properties of our proposal we will take as a model of our scenario the non-linear theory of electromagnetic field proposed by Born and Infeld that we presented above. We do not intend to solve completely our task of searching a field theory for the gravitational field with such a naive model, but only to present our general scheme to deal with this question.

Taking this theory as an example, let us assume as a toy model for the self-interacting gravitational field the Lagrangian

$$L^g = \frac{1}{\kappa} \left\{ \sqrt{b^4 + b^2(-A + B)} - b^2 \right\} \tag{35}$$

The quantity b has the dimension of $(lenght)^{-1}$. At this level of the theory it is a free parameter that can be choose either by some speculative consideration (e.g. by setting it equal to Planck's lenght, for instance) or by future observational requirements. By the time being we will left it free. Let us remark that, for the class of Lagrangians (eq. (35)), this quantity does not have the same meaning as in the original Born proposal of the maximum value possible for the field.

In the present presentation we will concentrate our interest on the consequences of such theory in the propagation of the gravitational waves. We leave the exam of the consequences of this model to a subsequent paper[8].

4.1   Gravitational Waves

The main purpose of this section is to examine the behavior of the gravitational waves in this theory. In order to do this we will analyse the evolution of discontinuities of the equation of motion through a characteristic surface Σ. This analysis give us the velocity of the gravitational wave that will be the key point to distinguish this kind of theory from General Relativity. For pedagogical reasons we start by examining the non-linear spin-one case.

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[8]See Conclusions for other remarks related to the properties of this theory.

4.1.1   Propagation of the Discontinuities: The Non-Linear Spin-1 Case

Let Σ be the surface of discontinuity. We set (using Haddamard's condition)

$$[F_{\mu\nu}]_\Sigma = 0, \tag{36}$$

and

$$[F_{\mu\nu,\lambda}]_\Sigma = f_{\mu\nu}\mathbf{k}_\lambda. \tag{37}$$

in which the symbol $[J]_\Sigma$ represents the discontinuity of the function J through the surface Σ.

Applying this into the equation of motion (16) we obtain

$$L_F f^{\mu\nu}\mathbf{k}_\nu + 2L_{FF}\eta F^{\mu\nu}\mathbf{k}_\nu = 0, \tag{38}$$

where η is

$$F^{\alpha\beta}f_{\alpha\beta} \equiv \eta.$$

After some algebraic manipulations the equation of propagation of the disturbances is provided by

$$\left\{\gamma_{\mu\nu}\left(\frac{L_F^2}{L_{FF}} + L\right) + \mathcal{T}_{\mu\nu}\right\}k^\mu k^\nu = 0 \tag{39}$$

In the case of the Born theory this expression reduces to

$$\left\{\gamma_{\mu\nu} + \frac{4}{b^2}\mathcal{T}_{\mu\nu}\right\}k^\mu k^\nu = 0 \tag{40}$$

In this particular non-linear non-polynomial theory we see that the disturbances propagate in the modified geometry, changing the background geometry $\gamma_{\mu\nu}$ into an effective one $g_{\mu\nu}$, which depends on the energy distribution of the field. We shall see later that the same structure occurs for spin-2.

4.1.2   Propagation of the Discontinuities: The Non-Linear Spin-2 Case

In an analogous way we set for the spin-2 field the discontinuity conditions

$$[F_{\mu\nu\alpha}]_\Sigma = 0 \tag{41}$$

and

$$[F_{\mu\nu\alpha;\lambda}]_\Sigma = f_{\mu\nu\alpha}\mathbf{k}_\lambda \tag{42}$$

Taking the discontinuity of the equation of motion of the gravitational field we obtain

$$f_{\mu(\alpha\beta)}\mathbf{k}^\mu = -2\frac{L_{AA}}{L_A}(\eta - \psi)F_{\mu(\alpha\beta)}\mathbf{k}^\mu = 0 \tag{43}$$

in which the quantities η and Λ are defined by

$$\eta \equiv F_{\alpha\beta\mu} f^{\alpha\beta\mu}$$

$$\Lambda \equiv F_\mu f^\mu$$

Considering the discontinuity relation and using the cyclic identity (see the Appendix for some details) we have

$$f_{\alpha\beta}{}^\mu \mathbf{k}_\lambda + f_{\beta\lambda}{}^\mu \mathbf{k}_\alpha + f_{\lambda\alpha}{}^\mu \mathbf{k}_\beta = \frac{1}{2}\delta^\mu_\alpha f_{[\lambda}\mathbf{k}_{\beta]} + \frac{1}{2}\delta^\mu_\beta f_{[\alpha}\mathbf{k}_{\lambda]} + \frac{1}{2}\delta^\mu_\lambda f_{[\beta}\mathbf{k}_{\alpha]}$$

Multiplying this equation by $F^{\alpha\beta\mu}K^\lambda$ we obtain

$$(\eta - \psi)K^2 - 2F^{\alpha\beta\mu}f_{\lambda\beta\mu}K^\lambda K_\alpha + f_\mu K^\mu F_\lambda K^\lambda + F_{\alpha\beta\lambda}F^\alpha K^\beta K^\lambda = 0. \tag{44}$$

Or, after some algebraic manipulations

$$K^\mu K^\nu [\gamma_{\mu\nu} + \Lambda_{\mu\nu}] = 0 \tag{45}$$

in which the quantity $\Lambda_{\mu\nu}$ is written in terms of the gravitational field as:

$$\Lambda_{\mu\nu} \equiv 2\frac{L_{AA}}{L_A}[F_\mu{}^{\alpha\beta}F_{\nu(\alpha\beta)} - F_\mu F_\nu]$$

We note that in our theory we face a very similar behavior as in the previous spin-1 case. Indeed, the disturbances propagate in a modified geometry, changing the background geometry $\gamma_{\mu\nu}$, into an effective one $g_{\mu\nu}$ which depends on the energy distribution of the field $F_{\alpha\beta\mu}$. This fact shows that such a property stems from the structural form of the Lagrangian.

From the above calculations we conclude that, differently from General Relativity, in the present theory the caracteristic surfaces of the gravitational waves propagate on the null cone of an effective geometry distinct of that observed by **all** other forms of energy and matter. This result gives a possibility to choose between these two theories just by observations of the gravitational waves. This is a challenge that is expected to be solved in the near future.

5 The Gravity-Matter Interaction

As we pointed out in Section (1) there are strong evidences that matter couples universally with gravitation in such a way that its net effect can be described as if matter is imbedded in a Riemannian geometry produced by the gravitational field. Many authors [2, 3] have shown that this geometry can be written in terms of an unobservable background geometry $\gamma_{\mu\nu}$ and the gravitational field through the potential $\varphi_{\mu\nu}$ as[9]

$$g_{\mu\nu} \equiv \gamma_{\mu\nu} + \varphi_{\mu\nu} \tag{46}$$

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[9]For some practical simplifications some authors do not deal with such a definition, but instead with pseudo-tensors obtained by multiplication of these tensors by the square-root of the corresponding determinants. See, for instance, [3].

This means that in order to know how matter couples to gravity one must just make the substitution of the background geometry $\gamma_{\mu\nu}$ by an effective geometry $g_{\mu\nu}$ and change accordingly the derivatives by the covariant derivatives constructed with $g_{\mu\nu}$. Since such procedure is already a standard one and there are very good reviews on this in the scientific literature we will not enter here in more details on this field theoretical equivalence of description of General Relativity. The reader that is not familiar with this should consult the review done in [3].

Let us just take here as a very simple example, the case of a scalar field Ψ. The free field equation of motion is provided by the action

$$S = \int d^4x \sqrt{-\gamma}\, \Psi_{,\mu} \Psi_{,\nu} \gamma^{\mu\nu} \tag{47}$$

In order to couple the scalar matter field with gravity we make the substitution of the metric $\gamma_{\mu\nu}$ into the effective one defined as above. The action takes then the form

$$S = \int d^4x \sqrt{-g}\, \Psi_{,\mu} \Psi_{,\nu} g^{\mu\nu} \tag{48}$$

It seems worth to remark that in this expression we have set the determinant of the effective metric:

$$g \equiv det\, g_{\mu\nu} = det(\gamma_{\mu\nu} + \varphi_{\mu\nu}).$$

Note that the quantity $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and constitutes an infinite series in terms of the tensors $\gamma_{\mu\nu}$ and $\varphi_{\mu\nu}$, that is

$$g^{\mu\nu} = \gamma^{\mu\nu} - \varphi^{\mu\nu} + \varphi^{\mu\alpha}\varphi_\alpha{}^\nu - \varphi^{\mu\alpha}\varphi_{\alpha\beta}\varphi^{\beta\nu} + ... \tag{49}$$

Then the equation of motion of the scalar field becomes

$$\Box\Psi \equiv \frac{1}{\sqrt{-g}} \left\{ \sqrt{-g}\, \Psi_{,\mu} g^{\mu\nu} \right\}_{,\nu} = 0. \tag{50}$$

We can thus obtain the modification of the equation of motion of the gravitational field in presence of matter. In the above hypothesis (borrowed from General Relativity) that the matter feels the gravitational field only by the combination $g_{\mu\nu} \equiv \gamma_{\mu\nu} + \varphi_{\mu\nu}$ it follows that

$$\frac{\delta L}{\delta\gamma_{\mu\nu}} = \frac{\delta L}{\delta g_{\mu\nu}}$$

Thus in the theory developed here the general equation of motion of the gravitational field containing source terms takes the form

$$\left\{ L_A F^\lambda{}_{(\mu\nu)} \right\}_{;\lambda} = -T_{\mu\nu}, \tag{51}$$

or the equivalent one

$$G^{(L)}_{\mu\nu} = \frac{1}{2} L_A{}^{-1} \left\{ L_{A;\lambda} F^\lambda{}_{(\mu\nu)} + T_{\mu\nu} \right\} \tag{52}$$

in which $T_{\mu\nu}$ represents the energy-momentum tensor of matter.

Finally, we would like to emphasize that, as we have pointed out before, we can thus conclude that, in what concerns the matter to gravity interaction the theory proposed here is indistinguishable of the General Relativity.

In order to show this, in the case of standard tests of gravity we will present now some explicit consequences of our theory.

6   The Static Gravitational Field of a Spherically Symmetric Body

In this section we will improve the exam of the above theory of gravity that satisfies the Einstein Equivalence Principle (EEP) for any kind of matter/energy, except for the gravitational energy. This is part of a research program that intends to re-examine the standard Feynman-Deser approach of field theoretical derivation of Einstein's General Relativity. The hypothesis implicit in such precedent derivations [1, 2] concerns the universality of gravity interaction. Although there is a strong observational basis supporting the universality of matter to gravity interaction, there is not an equivalent situation that supports the hypothesis that gravity interacts with gravity as any other form of non-gravitational energy. We analyse here a kind of gravity-gravity interaction distinct from GR but, as we shall see, that conforms with the actual status of observation. We exhibit the gravitational field produced by a spherically symmetric static configuration as described in our field theory of gravity. The values that we obtain for the standard PPN parameters ($\alpha = \beta = \gamma = 1$), supported by observations, coincide with those of General Relativity. Thus, as we pointed out above, the main different aspect of our theory and GR concerns the velocity of the gravitational waves. Since there is a large expectation that the detection of gravitational waves will occur in the near future, the question of which theory describes Nature better will probably be settled soon.

The starting point of our analysis rests on the well-known fact that Einstein's Equivalence Principle (EEP) is supported by experiment by at least[10] one part in 10^{11}. This statement should be understood in the very limited sense that it concerns observed effects of gravity restrained to the coupling of matter (that is, any non-gravitational form of energy) to the gravitational field. However, many theories of gravity, starting from General Relativity, go far beyond such limited domain of observation and assume a generalized form which states that the EEP comprises absolutely all forms of energy (including gravitational). This, of course, is a speculation that still nowadays remains beyond any experimental test. The recent general excitation among the scientific community concerning the possible detection of gravitational waves led to the belief that the observation of these waves could act as a crucial test on gravity-gravity interaction and to provide for a decision concerning which, among the rivals gravity theories, is the best one.

In the precedent sections an alternative model for gravity was presented. Our purpose here is to proceed with such an investigation on the consequences of our theory. We will describe a solution of our set of equations of the gravitational field produced by a

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[10]See the status of the STEP (Satellite Test of the Equivalence Principle), an international tentative of improving this result to one part in 10^{17}.

spherically symmetric and static configuration. We shall show that as far as the post-Newtonian approximation is concerned, the behavior of matter (i.e. any form of non-gravitational energy), in our theory is indistinguishable from General Relativity (GR).

From the experimental standard tests we can state the following results[11]

- $\alpha = 1$

- $\gamma = 1.000 \pm 0.002$

- $\frac{1}{3}(2\gamma + 2 - \beta) = 1.00 \pm 0.02$

- $\chi < 10^{-3}$

- $\alpha_2 < 4 \times 10^{-4}$

- $\frac{2}{3}\alpha_2 - \alpha_1 < 0.02$

- $49\alpha_1 - \alpha_2 - 2.2\chi < 0.1$

We shall prove in this paper that, for the new theory of gravity that we are examining, like in the very same manner as General Relativity, the unique non vanishing parameters are α, β and γ. The extra parameters, present in others theories describe strange non-usual properties of the gravitational interaction[12].

This result supports a remark made previously which states that the crucial distinction of ours gravity theory and GR should be made in evidence from the detection of the gravitational waves.

6.1   Field Theory of Gravity: A Solution

Let us make a pause and see what we have done. We deal here with a field theoretical model of gravitational interaction. In the standard traditional way this field is represented by a symmetric second order tensor that we denote by $\varphi_{\mu\nu}$. This field is supposed to propagate in an auxiliary background Minkowski geometry[13] and obeys the equation of motion

$$G^{(L)}_{\mu\nu} = \frac{1}{2}L_A^{-1}\left\{L_{A;\lambda}F^{\lambda}{}_{(\mu\nu)} + T_{\mu\nu}\right\} \tag{53}$$

in which we have added the source term represented by the energy-momentum tensor of matter.

In the theory of General Relativity the corresponding equation of motion of the gravitational field in the geometrical representation takes the form:

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[11]We are following the standard notation for the PPN parameters. The reader not aware of this formulation should consult the Appendix and the references.

[12]Like, for instance, possible deviation from Lorentz symmetry, effects of prefered frames, non conservation of energy-momentum at the post-Newtonian limit and an eventual spatial anisotropy of three-body interaction.

[13]We shall see that this metric is not observable neither by matter nor by the gravitational field.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -T_{\mu\nu}$$

in which the curvature of the associated Riemannian metric appears explicitly. However there is an equivalent way to describe the equation of motion in GR that seems worth to mention here just in order to exhibit the similarity and the distinction between both theories. Indeed, many authors (see, for instance [2]) have shown that it is possible to re-write Einstein's equation of motion by isolating the linear operator $G_{\mu\nu}^{(L)}$ from the non-linear terms, showing explicitly the form of the self-interacting terms. It is given by:

$$G_{\mu\nu}^{(L)} = -T_{\mu\nu} - \mathcal{T}_{\mu\nu} \tag{54}$$

in which $\mathcal{T}_{\mu\nu}$ is a complicated non-linear expression constructed with the metric tensor and its derivative[14] A simple inspection on both theories thus exhibits very clearly the particular characterization of the self-interacting terms in each theory. Let us emphasize that in both cases of equations (53) and (54) they represent the full theory. There is no approximation of any sort in these expressions.

Let us just make one more comment on the properties of these rivals theories of gravity. Since both theories satisfy the restricted EEP, the behavior of matter (or any form of non-gravitational energy) is precisely the same in both theories. Matter follows geodesics in an effective Riemannian geometry. Only the gravity-gravity processes are distinct. Since we are not dealing in the present paper with gravitational waves we will not consider any longer such distinction here.

So much for these general considerations. Let us now look for a solution of our equation (53), in the absence of matter.

We set for the auxiliary metric of the background the form

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{55}$$

This means that all operations of raising and lowering indices are made by this Minkowski metric $\gamma_{\mu\nu}$[15].

We search for the simple solution in such a way that the only non-identically null gravitational potential components $\varphi^{\mu\nu}$ are only

$$\varphi_{00} = \varphi^{00} = \mu(r)$$

and

$$\varphi_{11} = \varphi^{11} = -\nu(r).$$

The trace φ is then given by

$$\varphi = \mu + \nu.$$

From this we obtain the gravitational field $F_{\alpha\beta\mu}$. The only non-null terms are

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[14]The reader not acquainted with this formulation should consult [3] for a very didactic presentation of it.

[15]The reader should be attentive to the fact that matter (massive or massless particles –photons, for instance – that is, any form of non-gravitational energy) feels a modified geometry. See the previous sections.

$$F_{100} = -\frac{\nu}{r}$$

$$F_{122} = \frac{1}{2}\nu r - \frac{1}{2}\mu' r^2$$

$$F_{133} = \sin^2\theta F_{122}.$$

in which we have used a prime $'$ to symbolise the derivative with respect to the radial variable r.

Thus the unique component of the trace that remains is F_1 which is given by:

$$F_1 = \mu' - 2\frac{\nu}{r}$$

From these we can evaluate the associated quantities

- The invariant A.

- The invariant B.

We obtain directly the following values:

$$A = -3\frac{\nu^2}{r^2} - \mu'^2 + 2\frac{\nu\mu'}{r} \tag{56}$$

$$B = -\frac{1}{r^2}\left(\mu' r - 2\nu\right)^2 \tag{57}$$

Under the above hypothesis of spherically symmetry of the solution, there remain only two non-trivially satisfied equations, which are given by

$$\left\{L_A \frac{\nu}{r}\right\}' + 2L_A \frac{\nu}{r^2} = 0. \tag{58}$$

and

$$\mu' r - \nu = 0. \tag{59}$$

It seems worth to remark that the above set of equations (58) and (59) are the same for any theory that satisfies the fundamental condition, that is, for functionals only of the quantity Z defined by

$$Z \equiv \frac{1}{b^2}(-A + B).$$

Let us now especialize this for our theory characterized by equation (53).
We have

$$L_A = -\frac{1}{2}\left(1 + Z\right)^{-\frac{1}{2}} \tag{60}$$

Substituting the above tentative form of a solution in the expressions of the invariants we find for the quantity Z:

$$Z = \frac{1}{b^2}\frac{\nu^2}{r^2}. \tag{61}$$

Using this value in the equation (58) we find the solution

$$\nu = \frac{2M}{r} \left\{ 1 - (\frac{r_c}{r})^4 \right\}^{-\frac{1}{2}}.$$
(62)

in which the constant r_c is given by

$$r_c{}^2 \equiv \frac{2M}{b}.$$

The remaining function μ is given in terms of the elliptic function $F(\alpha, \frac{\sqrt{2}}{2})$ by

$$\mu = \frac{1}{2} \sqrt{bM} \left\{ F(\alpha, \frac{\sqrt{2}}{2}) + \mu_0 \right\}$$
(63)

in which the constant μ_0 must be chosen to yield the correct assymptotic limit. The quantity α is given by

$$\alpha \equiv \arcsin \left\{ \frac{\cosh x - 1}{\cosh x} \right\}^{\frac{1}{2}}$$

and x is defined by

$$\cosh x \equiv \left\{ \frac{r}{r_c} \right\}^2.$$

At this point let us make a preliminary comment concerning the value of the constant b. At the value $r = r_c$ the field has apparently a singularity. We shall see in a subsequent section that we deal here with a true singularity once the associated gravitational energy is singular at point r_c. Although the quantity b is a free parameter of the theory, its value must be such that it provides that the domain $r < r_c$ should be hidden inside all known celestial bodies. A typical example associates b to the inverse of Planck length. This provides a good possible value that satisfies the above requirement. We leave the actual value of b, for the time being, as an open parameter, the true value of which is to be decided later on.

6.2   The Effective Geometry

In this section we will turn our attention to a geometric representation of the present theory. Following the standard procedure [1, 2] we define a Riemannian metric tensor in terms of the gravitational potential as[16]

$$g_{\mu\nu} \equiv \gamma_{\mu\nu} + \varphi_{\mu\nu}$$
(64)

It seems worth to call attention to the reader that this definition has a deep meaning, once for all forms of non-gravitational energy the net effect of the gravitational field is felt precisely as if gravity was responsible of changing the metrical properties of the spacetime

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[16]The reader should note that one can make different forms of geometric representation. For instance one can uses the contravariant representation to set $g^{\mu\nu} = \gamma^{\mu\nu} + \varphi^{\mu\nu}$; or use pseudo quantities by means of the determinant γ in the definition of the geometry. Each one of these choices provides non-equivalent representations.

Table 1: PPN Parameters

	GR	LN
α	1	1
β	1	1
γ	1	1

from the flatness structure to a curved one that is related to the gravitational field precisely by the above expression. This means that any material body (or photons) follows along the geodesics (null, in the case of photons) as if the metric tensor of spacetime was given by the above expression.

From the previous calculation we find for the effective geometry the form

$$ds^2 = g_{00}dt^2 - g_{11}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{65}$$

with

$$g_{00} = 1 + \mu.$$

and

$$g_{11} = -(1 + \nu).$$

Expanding for $r_c << r$ we have

$$g_{00} \approx 1 - \frac{2M}{r} - \frac{1}{10}\left\{\frac{r_c}{r}\right\}^4 + ...$$

and

$$g_{11} \approx -1 - \frac{2M}{r} - \frac{M}{r}\left\{\frac{r_c}{r}\right\}^4 + ...$$

We see that the modification of the flatness of spacetime, as seen by matter, beyond the order $O(\frac{M}{r})$, occurs only at order $O(\frac{M^3}{r^5})$. This is radically different from the results of General Relativity and should provide the basis for a future test.

Before this, however let us look into the corresponding post-Newtonian parameter of our theory. Using the isotropic coordinate system (see the Appendix) we can re-write our effective geometry under the form

$$ds^2 = (1 - 2\alpha\frac{M}{\rho} + 2\beta\frac{M^2}{\rho^2} + ...)dt^2 - (1 + 2\gamma\frac{M}{\rho})\left\{d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)\right\}$$

It then follows from a direct inspection on this form of the effective geometry the values of the PPN parameters corresponding to our solution. We can use the associated table in order to compare the values of our model and those obtained from General Relativity.

A direct inspection on this table led us to conclude that, at the PPN level of observation, both theories provide the same answer for the efects of the gravitational field on matter.

7   Gravitational Energy

The fact that we are dealing with a field theory in the conventional way provide us directly as a by-product, with a well-defined definition of the energy. We can arrive at the expression of the energy-momentum tensor either by a direct variation of the associated background metric or by means of Noether's theorem[17].

Using the definition

$$\mathcal{T}_{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta L \sqrt{-\gamma}}{\delta \gamma^{\mu\nu}} \tag{66}$$

we find, from the above Lagrangian,

$$\mathcal{T}_{\mu\nu} = -L\gamma_{\mu\nu} + 2L_A \left\{ 2F_{\mu\alpha\beta}F_\nu{}^{\alpha\beta} + F_{\alpha\beta\mu}F^{\alpha\beta}{}_\nu - F^\alpha F_{\alpha(\mu\nu)} - 2F_\mu F_\nu \right\} \tag{67}$$

Let us specialize this for the case of a static and spherically symmetric configuration. Since in our theory the Lagrangian is given by eq. (35) we have, under the conditions of the above solution,

$$L = \frac{b^2}{k} \left\{ \left(1 - (\frac{r_c}{r})^4\right)^{-\frac{1}{2}} - 1 \right\}. \tag{68}$$

Note that it then follows that

$$L > 0.$$

Finally we obtain, for the density of energy, the expression

$$\frac{k}{b^2} T^g{}_{00} = 1 - \frac{1 - 2(\frac{r_c}{r})^4}{\sqrt{1 - \left(\frac{r_c}{r}\right)^4}} \tag{69}$$

8   Conclusion

Einstein's theory of General Relativity is one of the most beautiful, comprehensive and deeply important achievements of classical field theory. Not only it contains and exhibits such a simplicity and internal coherence but, more than this, it provides a sound step towards the understanding of gravitational processes which has had no rival since the early times of Newton hypothesis concerning the existence of the universal gravitational atraction. Thus, any theory that dares to propose even a small modification on its scenario faces not only the challenge to present a sound new argument -but besides, it has to deal with the enormous difficulty of intending the substitution of a successful paradigm.

Thanks to the work of many physicists as Feynman, Deser and others, we have learned that Einstein's geometric vision of General Relativity admits an equivalent field theoretical presentation. In such a formulation, the phenomenon of gravity becomes more similar to the rest of Physics and, besides (and by far the more important) we obtain a deeper

[17]The reader should note that many differents forms of the energy-momentum tensor of the gravitational field have been described in the literature. Most of then suffer from the disease of being pseudo quantities and not true tensor. Some others, like the one described in [3], are true tensors but have a hidden gauge symmetry. We note that our expression does not seem to suffer of any of these difficulties.

comprehension on the mechanism of self-interaction of the gravitational field. It is precisely the analysis of this mechanism that we think is worth to re-examine. We can thus realize that there are two reasons for this:

- The theoretical challenge of providing another coherent manner to describe gravitational self-interaction process.

- The perspective of test alternative theories by the observation, in the near future, of gravitational waves.

In the present paper we describe a new program of analysis of gravitational interaction that has a very similar geometrical interpretation as General Relativity as far as matter to gravity processes are concerned. However, it becomes distinct from GR in the description of gravity to gravity interaction. We have exhibited a specific example of a theory in order to show how these ideas can be implemented. We have found the exact solution of a compact spherically symmetric and static configuration. We have shown that, although the general system of equations is highly non linear, the system reduces, for this symmetry, to a very simple one that allows a direct integration to be done. An analysis of the properties of our solution shows many points in common with GR. The PPN parameters do not allow a distinction between these theories. In order to exhibit a crucial observable difference between them, we should look for the propagation of gravitational disturbances. Indeed, (see the Appendix) the velocity of the gravitational waves is not the same in these two theories. We have evaluated the propagation of gravitational waves within such a theory and shown that the velocity of the waves does not coincide with the GR prediction. Once there is a large expectative that these waves will be detected in the next few years, we could test this aspect of these theories soon.

Finally, we should mention that we have found a cosmological solution of the theory. We can anticipate that such solution is in good agreement to actual observations. This is matter for a future work.

9   Appendix

9.1   The Velocity of Gravitational Waves

We have shown that the gravitational waves travel in different cones than the photons. This means that the gravitational waves feel a geometry of the spacetime that is not the same as the one felt by matter. The propagation of the gravitational disturbances follow along trajectories k^μ that satisfy the equation

$$\left[\gamma_{\alpha\beta} + \frac{2L_{AA}}{L_A}\left(2F_{\alpha\mu\nu}F_\beta{}^{\mu\nu} + F_{\alpha\mu\nu}F^{\mu\nu}{}_\beta - F_\alpha F_\beta\right)\right] k^\alpha k^\beta = 0.$$

This is one of the most important distinctions between the present field theory of gravity and General Relativity. The consequences of this, in respect to future observations, will be discussed elsewhere.

9.2   PPN Formalism

The standard form of the metric in the PPN form is given by

$$g_{00} = 1 - 2\alpha\frac{M}{\rho} + 2\beta\left(\frac{M}{\rho}\right)^2.$$

$$g_{ij} = (-1 - 2\gamma\frac{M}{\rho} - \lambda\frac{M^2}{\rho^2})\delta_{ij}.$$

We make a coordinate transformation to pass from this isotropic system $(t, \rho, \theta, \varphi)$ to the spherical one (t, r, θ, φ):

$$\rho = r\left\{1 - \gamma\frac{M}{r} - \frac{1}{2}(\lambda - \gamma^2)\frac{M^2}{r^2}\right\}$$

It then follows that in the new system the metric takes the form

$$
\begin{aligned}
ds^2 &= \left\{1 - 2\alpha\frac{M}{r} - 2(\beta - \alpha\gamma)\frac{M^2}{r^2}\right\}dt^2 - \left\{1 + 2\gamma\frac{M}{r} + (2\lambda + \gamma^2)\frac{M^2}{r^2}\right\}dr^2 \\
&\quad - r^2(d\theta^2 + \sin^2\theta d\varphi^2).
\end{aligned}
$$

9.3   Some Useful Formulae

9.3.1   The Traceless Part of the Gravitational Field

The traceless tensor $C_{\alpha\mu\nu}$ is defined by

$$C_{\alpha\mu\nu} \equiv F_{\alpha\mu\nu} - \frac{1}{3}F_\alpha\gamma_{\mu\nu} + \frac{1}{3}F_\mu\gamma_{\alpha\nu}$$

The invariant constructed with this quantity C is given by

$$C \equiv C_{\alpha\mu\nu}C^{\alpha\mu\nu}$$

It then follows that between this quantity and the invariants dealt with in the paper (A and B)there is the following relation

$$C = A - \frac{2}{3}B.$$

We could then use the irreducible parts of the gravitational field to describe its dynamics. In this case the generic form of the Lagrangian should be

$$L[B, C].$$

.

9.3.2 Variational Formulae

The evaluation of the energy-momentum of the gravitational field needs some technicalities concerning the dependence of $F^{\alpha\mu\nu}$ on the background metric $\gamma_{\mu\nu}$. There are some expressions that simplify this manipulation. Let us present here some of these.

We have

$$\frac{\delta A}{\delta\gamma^{\mu\nu}} = 2F^{\alpha}\frac{\delta F_{\alpha}}{\delta\gamma^{\mu\nu}} - F^{\alpha}F_{\alpha(\mu\nu)} + 2F_{\mu\alpha\beta}F_{\nu}{}^{\alpha\beta} + F_{\alpha\beta\mu}F^{\alpha\beta}{}_{\nu},$$

in which we used the relation,

$$F^{\alpha\beta\rho}\frac{\delta F_{\alpha\beta\rho}}{\delta\gamma^{\mu\nu}} = F^{\alpha}\frac{\delta F_{\alpha}}{\delta\gamma^{\mu\nu}} - \frac{F^{\alpha}F_{\alpha(\mu\nu)}}{2}.$$

And, for the invariant B we obtain

$$\frac{\delta B}{\delta\gamma^{\mu\nu}} = 2F^{\alpha}\frac{\delta F_{\alpha}}{\delta\gamma^{\mu\nu}} + F_{\mu}F_{\nu}.$$

References

[1] R.P.Feynman in Lectures On Gravitation, California Institute of Technology (1962), unpublished.

[2] S.Deser, J. Gen. Rel. Grav. **1**, 9, (1970)

[3] L.P.Grischuck, A.N.Petrov and A.D.Popova, Commun. Math. Phys. **94**, 379 (1984)

[4] M.Born, Nature, **132**, 282 (1933); Proc. Roy. Soc. **A 143**, 410 (1934); M.Born and L.Infeld, Nature **133**, 63 (1934)

[5] S.L.S. Duque, Ph.D Thesis (1994), unpublished.[18]

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