CBPF-NF-065/98

Classical and Quantum V-algebras [1]

J. F. Gomes, F. E. Mendonça da Silveira, A. H. Zimerman
Instituto de Física Teórica - UNESP,
Rua Pamplona, 145 , 01405-900, Sao Paulo - SP, Brazil
jfg@axp.ift.unesp.br, eugenio@axp.ift.unesp.br, zimerman@axp.ift.unesp.br

G. M. Sotkov[2]
Departamento de Campos e Partículas - CBPF
Rua Dr. Xavier Sigaud, 150 -Urca 22290-180, Rio de Janeiro - RJ, Brazil
sotkov@cbpfsu1.cat.cbpf.br

ABSTRACT

The problem of the classification of the extensions of the Virasoro algebra is discussed. It is shown that all H-reduced $\hat{\mathcal{G}}_r$-current algebras belong to one of the following basic algebraic structures: local quadratic W-algebras, rational U-algebras, nonlocal V-algebras, nonlocal quadratic WV-algebras and rational nonlocal UV-algebras. The main new features of the quantum V-algebras and their heighest weight representations are demonstrated on the example of the quantum $V_3^{(1,1)}$-algebra.

Key-words: Non-Abelian Toda models; V-algebras; 2-D critical phenomena.

[1] Talk given by G.M. Sotkov at the IV International Conference on Non Associative Algebra and its Applications, July 1998, São Paulo, Brazil

[2] On leave of absence from the Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 1784, Sofia

1 Introduction

The concept of algebras and groups of symmetries (finite and infinite, Lie and non-Lie etc) is, by no means, the key stone of all the field and string theories of unification of the interactions. An impressive example of the computational power of the algebraic methods, however, is provided by the theory of the *second order phase transitions*, in two dimensions. It turns out [1] that the complete nonperturbative description of the critical behaviour of a class of 2-D statistical mechanics models is given by the highest weight ($h.w.$) unitary representations $\{c(m), \Delta_{p,q}(m)\}$ of the Virasoro algebra

$$[L_n, L_l] = (n-l)L_{n+l} + \frac{c}{12}n(n^2-1)\delta_{n+l,0}, \qquad n, l = 0, \pm 1, \pm 2, ... \tag{1}$$

In words, all the physical data of the critical model - the exact values of the critical exponents, the partition function, all the correlation functions etc - are encoded in the representation theory of the algebra (1). The exact formulation of the above statement is as follows:

• **Physical data** (critical $RSOS$ models on 2-D planar lattice [2]): For each fixed $m = 3, 4, 5, ...$, the $m - th$ $RSOS$ model is defined by attaching to each site $\vec{i}$ a height l_i of length $l_i = 1, 2, ..., m$ under the condition that the maximal length difference of the nearest neighbours ($n.n.$) heights is one, i. e., $|l_{\vec{i}} - l_{\vec{i+1}}| = 1$. The only $n.n.$'s interact and the energy of a given configuration is

$$H = -\sum_{<ij>} J_{ij} l_i l_j + h \sum_i l_i.$$

The partition function $\mathcal{Z}(T, h) = Tr_{\mathcal{H}} \exp\left(-\frac{H}{kT}\right)$ ($Tr_{\mathcal{H}}$ denotes a sum over all allowed height configurations), found in [2] shows that, at a certain critical temperature $T = T_c(m)$, the $m - th$ $RSOS$ exhibits a second order phase transition. This means that, at $\tau = \frac{T-T_c}{T_c} \to 0$, all thermodynamical characteristics of the model have *power-like singularities*:

$$C_V \sim \tau^{-\alpha}, \quad M \sim \tau^{-\beta}, \quad X \sim \tau^{-\gamma}, \; ...$$

The critical exponents $\alpha(m)$, $\beta(m)$, $\gamma(m)$, ..., turns out to be certain nonnegative rational numbers [2]. For example, the $m = 3$ odd lattice ($l_i = 1, 3$) model is equivalent to the Ising model and it has $\alpha = 0$, $\beta = \frac{1}{8}$; the $m = 5$ ($l_i = 1, 3, 5$) describes the 3-state Potts model etc.

• **Mathematical data** ([3]): For each fixed c, the $h.$ $w.$ states $|\Delta, c>$ of the Virasoro algebra (1) are defined by requiring

$$L_0|\Delta, c> = \Delta|\Delta, c>, \quad L_n|\Delta, c> = 0, \quad n > 0. \tag{2}$$

The $h.$ $w.$ unitary representations[3] of (1) are given by

$$\Delta_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \qquad c(m) = 1 - \frac{6}{m(m+1)}, \tag{3}$$

[3]The unitary condition was found in ref [6].

where $1 \leq p \leq m-1$, $1 \leq q \leq p$, $m = 3, 4, ...$

• **Identification** ([1], [4], [5]): The *scale invariance* of 2-D statistical models, at the critical point $T = T_c$, is shown to be a part of a larger group of conformal transformations $(z, \bar{z}) \to (f(z), \bar{f}(\bar{z}))$, which governes the critical behaviour of these models in the continuous (thermodynamical) limit. Therefore, the critical $RSOS$ models can be described in terms of certain *conformal invariant 2-D field theories* (CFT's) ($l_{\vec{i}} \equiv l^{(m)}_{i_1 i_2} \to l^{(m)}(z, \bar{z})$). The symmetries of these CFT's are generated by the two components $T(z)$ and $\bar{T}(\bar{z})$ of the conserved traceless stress-tensor $T_{\mu\nu}$. Its short distance operator-product expansion (OPE) is completely determined by the symmetry

$$T(z_1)T(z_2) = \frac{c/2}{z_{12}^4} + \frac{2T(z_2)}{z_{12}^2} + \frac{\partial_2 T(z_2)}{z_{12}} + O(1) \tag{4}$$

and the same for $\bar{T}(\bar{z})$. Introducing the corresponding conserved charges $L_n = \oint T(z) z^{n+1} dz$, where $n = 0, \pm 1, \pm 2, ...$ (and the charge $\bar{L}_n$ for $\bar{T}(\bar{z})$), and substituting them in (4), we realize that the algebra of the charges L_n's (and $\bar{L}_n'$s) contains two (mutualy commuting) *Virasoro algebras* (1). As a consequence, the (Hilbert) space of states of such quantum CFT can be constructed as a tensor product of two *h. w.* representation spaces (2), (with $c(m) = \bar{c}(m)$): $\quad |\Delta, \bar{\Delta}, c >= P(|\Delta, c> \otimes |\bar{\Delta}, c>)$, where P is denoting an appropriate projection on the subspace of states in $\mathcal{H}_{\Delta,c} \otimes \mathcal{H}_{\bar{\Delta},c}$, satisfying certain physical conditions - crossing symmetry, semi-locality etc - see refs [1] and [4]. To each *h. w.* state, one can make, in correspondence, a *primary* field $\phi_{\Delta,\bar{\Delta}}(z, \bar{z})$ of *spin* $s = \Delta - \bar{\Delta}$ and *dimension* $d = \Delta + \bar{\Delta}$ such that $|\Delta, \bar{\Delta}, c >= \phi_{\Delta,\bar{\Delta}}(0,0)|0>$. One of the most important properties of the primary fields $\phi_{\Delta_{p,q}} \equiv \phi_{p,q}$, from the Kac-table (3), is that, together with the standard conformal Ward identities

$$T(z_1)\phi_{p,q}(z_2)|0>= \left(\frac{\Delta_{p,q}}{z_{12}^2}\phi_{p,q}(z_2) + \frac{1}{z_{12}}\partial_2\phi_{p,q}(z_2) + O(1) \right) |0>, \tag{5}$$

it has to satisfy the so-called *null vector conditions*, which say for $p = 2$, $q = 1$ appears to be in the form

$$\left\{ L_{-1}^2 - \frac{2}{3}(1 + \Delta_{21}(m)L_{-2} \right\} |\Delta_{21}, c >= 0. \tag{6}$$

Eqns. (5) and (6) allow us to calculate the structure constants of the OPE's $\phi_{p_1q_1}(z_1)\phi_{p_2q_2}(z_2)$, as well as the exact 4-point (and n-point) correlation functions $< \phi_{p_1q_1}(1)\phi_{p_2q_2}(2) \cdot ... \cdot \phi_{p_nq_n}(n) >$. Finally, the identification with the $RSOS$ models is done by comparing the ABF-exponents $\alpha, \beta, \gamma, ...$[2], with the Kac-dimensions (3). For the Ising model ($m = 3$, $c(m) = \frac{1}{2}$), we have $\alpha = 0, \quad \beta = \frac{1}{8} \quad$ and $\quad \Delta_{21}(3) = \frac{1-\alpha}{2-\alpha} = \frac{1}{2}, \quad \Delta_{22}(3) = \frac{\beta}{2-\alpha} = \frac{1}{16}$.

• **The problem of classification of 2-D universality classes**. The purely algebraic description of the critical $RSOS$ models address the question whether one can find appropriate infinite algebras, which representation theories provide the exact solutions for all known 2-D critical statistical models having second order phase transition. The algebras, we are looking for, have to contain the Virasoro algebra (1) as a subalgebra. The following three examples of extended Virasoro algebras are to ilustrate the main features of the *new algebraic structures* one need to introduce in the description of the universality classes, in two dimensions.

Example 1.1. $N = 1$ **supersymmetric Virasoro algebra**[9]: An infinite Lie superalgebra, containing together with the bosonic Virasoro generators L_n, an infinite set of fermionic ones G_s $(s \in \frac{1}{2}Z)$

$$[L_n, G_s] = \left(\frac{n}{2} - s\right) G_{n+s}, \qquad [G_s, G_t]_+ = 2L_{s+t} + \frac{c}{2}\left(s^2 - \frac{1}{4}\right)\delta_{s+t,0}, \tag{7}$$

where $[\ ,\ ]_+$ denotes the *anticommutator* of G_t's. The *h. w.* representations of (7)[3]

$$c(m) = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right), \qquad \Delta_{pq}(m) = \frac{[(m+2)p - mq]^2 - 2}{12m(m+2)}, \tag{8}$$

where $m = 4, 5, ...$ and $1 \le p \le m-2$, $1 \le q \le p$, give rise to a family of superconformal minimal models [7], [8] which describe the critical behaviour of $k = 2$ generalized $RSOS$ models. The difference with the standard $(k = 1)$ $RSOS$ is that now the allowed *maximal* length difference between *n. n.* heights $l_{\vec{i}}$ and $l_{\vec{i+1}}$ is $k = 2$. As it is evident from (7), the $k = 2$ critical $RSOS$ possess symmetry larger than the conformal one. The stress-tensor $(T, \bar{T})$ and the new spin $\frac{3}{2}$ supercurrent $(G, \bar{G})$, $G_n = \oint z^{n+\frac{1}{2}} G(z)dz$, generate 2-$D$ superconformal transformations.

The critical $k - RSOS$ models for $k = 3, 4, ...$ require *fractional* spin $\frac{l}{k}$ extensions of the Virasoro algebra [10, 11], [12]. Our next example represents the main features of such "parafermionic type" algebras.

Example 1.2. Z_N **Parafermionic algebra** [12], [13]: The Z_N generalizations of the Ising (Z_2) and Potts (Z_3) models are lattice spin models, where each site (i) is occupied by a "spin variable" $\sigma(i)$ that takes values $\sigma^l_{(i)} = \exp\left(\frac{2\pi\imath l}{N}\right)$ in the discrete group Z_N. To describe (multi) critical behaviour of these models, one has to consider, together with $T(z)$, a set of $N-1$ new conserved currents $\psi_l^+(z) = \psi_{N-l}^-(z)$, where $l = 1, 2, ..., N-1$, of spins $s_l = \frac{l(N-l)}{N}$, with OPE's in the form [12], [13]

$$\begin{aligned}
\psi_1^\pm(z_1)\psi_1^\pm(z_2) &= c_{11} z_{12}^{-\frac{2}{N}}(\psi_2^\pm(z_2) + O(z_{12})),\\
\psi_1^+(z_1)\psi_1^-(z_2) &= z_{12}^{\frac{2}{N}}\left(\frac{1}{z_{12}^2} + \frac{N+2}{N}T(z_2) + O(z_{12})\right),
\end{aligned} \tag{9}$$

where $c_{11} = \sqrt{\frac{2(N-1)}{N}}$.Introducing the parafermionic (PF) conserved charges in (9)

$$\begin{aligned}
A^\pm_{\frac{1\mp l}{N}+n}\phi_l(0) &= \oint dz \psi_1^\pm(z) z^{\mp\frac{l}{N}+n}\phi_l(0),\\
\psi_1^\pm(z\exp(2\pi\imath))\phi_l(0) &= \exp(\frac{2\pi\imath l}{N})\psi_1^\pm(z)\phi_l(0),
\end{aligned}$$

we derive the Z_N PF-extension of the Virasoro algebra (2)

$$\sum_{p=0}^{\infty} C^p_{\left(\frac{2}{N}\right)}\left(A^\pm_{\frac{3\mp l}{N}-p+m} A^\pm_{\frac{1\mp l}{N}+p+n} - A^\pm_{\frac{3\mp l}{N}-p+n} A^\pm_{\frac{1\mp l}{N}+p+m}\right) = 0,$$

$$\sum_{p=0}^{\infty} C^p_{\left(-\frac{2}{N}\right)} \left(A^+_{m-\frac{1+l}{N}-p} A^-_{\frac{1+l}{N}+n+p-1} + A^-_{n-\frac{1-l}{N}-p-1} A^+_{\frac{1-l}{N}+p+m} \right)$$
$$= \frac{N+2}{N} L_{m+n-1} + \frac{1}{2}\left(n-1+\frac{l}{N}\right)\left(\frac{l}{N}+m-2\right)\delta_{m+n-1,0} \qquad (10)$$

of central charge $c(N) = 2\frac{N-1}{N+2}$, where $N = 2, 3, ...,$ and structure constants $C^p_{(r)} = \frac{\Gamma(p-r)}{p!\Gamma(-r)}$. The *h. w.* representations of this infinite associative algebra, found in ref [12], are of dimensions $\Delta_l = \frac{l(N-l)}{2N(N+2)}$ and Z_N charge - $l = 1, 2, ..., N-1$ for the (*order parameter*) fields $\sigma_l(z, \bar{z})$; and $\Delta_j = \frac{j(j+1)}{N+2}$ where $j = 1, 2, ... \leq \left[\frac{N}{2}\right]$ are the dimensions for the Z_N neutral (*energy operator*) fields $\epsilon_j(z, \bar{z})$. It is important to note that the *origin* of the fact that in the PF-algebra (10), the Lie commutator $[a, b] = ab - ba$ is replaced by an infinite sum of bilinears $A^{\pm}A^{\mp}$ is in the *branch cut singularities* $z^{\pm\frac{2}{N}}$ $(N \geq 3)$ in the OPE's (9). These types of singularities are a consequence of the *fractal spins* $s_1^{\pm} = 1 - \frac{1}{N}$ of the PF-currents $\psi_1^{\pm}$. Observe that for $N = 2$ (and for half-integer spins, in general), the OPE's have odd poles z_{12}^{-1} (or z_{12}^{-2s-1}) singularities, which lead to anticommutators $[a, b]_+ = ab + ba$. For integer spins $s = 1, 2, 3, ...$, the leading singularities in the OPE's are even poles z_{12}^{-2s} (as in eqn. (4)) and they give rise to the standard Lie commutators.

Example 1.3. **W_3-Zamolodchikov algebra** [15], [14]: The most important property of the spin 3 extension of the Virasoro algebra (generated by $T(z)$ and $W(z)$ of spin $s_W = 3$) is that the commutator of the charges W_n of the spin 3-current $W_n = \oint W(z) z^{n+2} dz$ is quadratic in the Virasoro generators L_n's

$$[W_n, W_l] = (n-l)[d(n,l)L_{n+l} + b\Lambda_{n+l}] + \frac{c}{360} n(n^2-4)(n^2-1)\delta_{n+l,0}, \qquad (11)$$

where

$$\Lambda_n = \sum_{k=-\infty}^{\infty} :L_k L_{n-k}: + \frac{1}{5} f_n L_n, \quad f_{2s} = 1 - s^2, \quad f_{2s+1} = (1-s)(2+s)$$

and

$$d(n,l) = \frac{1}{6}\left[\frac{2}{5}(n+l+2)(n+l+3) - (n+2)(l+2)\right], \qquad b = \frac{16}{22+5c}.$$

The *h. w.* states $|\Delta, w, c>$ of this *non-Lie associative* algebra are defined by

$$L_0|\Delta, w, c> = \Delta|\Delta, w, c>, \qquad W_0|\Delta, w, c> = w|\Delta, w, c>,$$

$$L_n|\Delta, w, c> = W_n|\Delta, w, c> = 0, \quad n > 0.$$

Its *h. w.* unitary representations $c(m) = 2\left(1 - \frac{12}{m(m+1)}\right)$, where $m = 4, 5, ...,$ and $\Delta_{p_i q_i}(m)$, $w_{p_i q_i}(m)$ $(i = 1, 2)$ found in ref [14] give rise to a family of Z_3 symmetric CFT's that provide the exact solutions for a new class of critical statistical models. The simplest representative $m = 4$ of this class is again the critical 3-states Potts model.

The above examples of three different associative extensions of the Virasoro algebra, (7), (10), (11), suggest the following *organization* of the list of all known infinite algebras (and their 2-D CFT's):

(i) *Lie-algebraic extensions*: Conformal current (affine) $\hat{\mathcal{G}}_r$-algebras [16]:

$$[J_n^a, J_l^b] = \imath f^{abc} J_{n+l}^c + kn\delta^{ab}\delta_{n+l,0}, \tag{12}$$

where $n, l = 0, \pm 1, \pm 2, ...$; $a, b = 1, 2, ..., dim\ \mathcal{G}$; f^{abc} are the structure constants of an arbitrary (finite dimensional) semisimple Lie algebra $\mathcal{G}_r$; k is called the level of $\hat{\mathcal{G}}_r$. Its generators are the conserved charges of the spin $s = 1$ chiral current $J^a(z) = \sum_{n=-\infty}^{\infty} z^{-n-1} J_n^a$, which also satisfy $[L_n, J_l^a] = -lJ_{n+l}^a$. The $h.\ w.$ representations of (12) (and its CFT's) were constructed in refs [17, 18].

(ii) *Lie-superalgebraic extensions*: The $N = 1$ superVirasoro algebra (7); $N = 2, 3, 4$ superconformal algebras [19, 20, 21]; the affine $\hat{\mathcal{G}}_r$-superalgebras, where $\mathcal{G}_r$ is an arbitrary rank r finite dimensional superalgebra; $N = 1$ superconformal current $\hat{\mathcal{G}}_r$-algebras [23, 22], with generators J_n^a and ψ_n^a determined by (7), (12) and $[J_n^a, \psi_l^b] = f^{abc}\psi_{n+l}^c$, $[\psi_n^a, \psi_l^b]_+ = k\delta^{ab}\delta_{n+l,0}$.

(iii) *PF-extensions*: The Z_N (and D_{2N})-PF algebra (10) and its (p, M)-generalizations [12], by considering PF currents of spins $s_l = p\frac{l(N-l)}{N} + M_l$; Gepner's $\mathcal{G}_r$-parafermions [24].

(iv) *Quadratic W-algebras*: The W_n-algebras [25], [27], [26] generated by the charges of the spin $s = 2, 3, ..., n$-currents, and the more general $W\mathcal{G}_n$ [25]; the $W_n^{(l)}$-algebras [28], [29], [35]; the supersymmetric W_n-algebras etc.

To complete our table of extended Virasoro algebras, we have to add the family of the recently discovered classical Poisson brackets *nonlocal and nonlinear (quadratic) V-algebras* [31], [32], [33], [34].

(v) *V-algebras*: The simplest example is given by $V_3^{(1,1)} \equiv VA_2^{(1,1)}$-algebra [33], [34], generated by one local spin 2 $T(\sigma)$ (the stress-tensor) and two spin $\frac{3}{2}$-non local currents $V^{\pm}(\sigma)$:

$$\{T(\sigma), V^{\pm}(\sigma')\} = \frac{3}{2}V^{\pm}(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') + \partial_{\sigma'}V^{\pm}(\sigma')\delta(\sigma - \sigma'),$$

$$\{V^{\pm}(\sigma), V^{\mp}(\sigma')\} = \pm\frac{2}{k}\delta''(\sigma - \sigma') \mp \frac{2}{k}T(\sigma')\delta(\sigma - \sigma') + \frac{3}{2k^2}V^{\pm}(\sigma)V^{\mp}(\sigma')\epsilon(\sigma - \sigma'),$$

$$\{V^{\pm}(\sigma), V^{\pm}(\sigma')\} = -\frac{3}{2k^2}V^{\pm}(\sigma)V^{\pm}(\sigma')\epsilon(\sigma - \sigma'). \tag{13}$$

where $\epsilon(\sigma) = sign\ \sigma$. The $V_3^{(1,1)}$ is the first member of the $VA_n^{(1,1)}$-family of V-algebras, spanned by two non-local currents $V_{(n)}^{\pm}$ of spins $s = \frac{n+1}{2}$ and $n-1$ local currents W_{n-l+2} of spins $s_l = n - l + 2$, where $l = 1, 2, ..., n$. The Bilal's VB_2-algebra [31] is quite similar to (13), but $V^{\pm}$-currents have spin $s^{\pm} = 2$ in this case.

Our main purpose, in what follows, is the construction of the *quantum* V_{n+1}-algebras and their minimal conformal models (i. e., their *h. w.* representations). The most important result is that the classical spins $s^{cl} = \frac{n+1}{2}$, of the nonlocal currents $V^{\pm}$, gets renormalized, i. e., $s^{qu} = \frac{n+1}{2}\left(1 - \frac{1}{2k+n+1}\right)$ and their algebra shares the main patterns of the PF-algebras. While the quantum local currents W_{n-l+2} manifest properties similar to the W_n-algebras. Therefore, the quantum $V_{n+1}^{(1,1)}$-algebras represent an appropriate *unification* of the features of the Z_{2k+3} PF-algebra with the W_{n+1}-one.

2 Constrained $\mathcal{G}_r$-current algebras

The list of the five known families of extended Virasoro algebras we have made, however, does *not* solve the problem of the *classification of 2-D universality classes* (i. e., all allowed critical behaviours in two dimensions). We need a method of exhausting all the possible extensions of the Virasoro algebra. The hint is coming from the fact that all the considered algebras[4] - the Virasoro-one, the PF-, the W_n- and V_n-ones - can be obtained by imposing a specific set of constraints on the currents of certain $\hat{\mathcal{G}}_r$-current algebras ($SL(2,R)$, for (1) and (10), and $SL(3,R)$, for (11) and (13) etc):

$$\{J^a(\sigma), J^b(\sigma')\} = \imath f^{abc} J^c(\sigma')\delta(\sigma-\sigma') + k\partial_{\sigma'}\delta(\sigma-\sigma'). \tag{14}$$

It suggests that the desired classification of the extended Virasoro algebras can be reached by the methods of the Hamiltonian reduction [29], [36], [30], i. e., by considering all consistent sets of constraints on the currents $J^a(\sigma) \in \hat{\mathcal{G}}_r$

$$\hat{J}(\sigma) = g^{-1}\partial g = \sum_{\text{allroots}} J_{\{\alpha\}} E_{\{\alpha\}} + \sum_{i=1}^{r} J_i \frac{\vec{\alpha}_i \cdot \vec{H}}{\alpha_i^2}$$

where $E_{\{\alpha\}}$, $h_i = \frac{\vec{\alpha}_i \cdot \vec{H}}{\alpha_i^2}$ are the generators of the finite Lie algebra $\mathcal{G}_r$. Therefore, the question now is *whether and how* one can *classify* all constraints to be imposed on $\hat{J}(\sigma)$.

We start with few selected examples of constrained $SL(n,R)$ ($n = 2,3,4$) algebras, which demonstrate the way the *algebraic structure* of the reduced algebras depends on the specific choice of the constraints.

Example 2.1. $SL(2,R)$ **reductions**.

(1a) $A_1/\mathcal{N}_+ \equiv$ *Virasoro algebra*: Take $J_\alpha = 1$ as a constraint and $J_1(\equiv J_0) = 0$ as its gauge fixing (i. e., J_1 is the canonically conjugated momentum of J_α, since $\{J_\alpha(\sigma), J_1(\sigma')\} = -J_\alpha(\sigma)\delta(\sigma-\sigma') \approx \delta(\sigma-\sigma')$).[5] Under these conditions, the classical Poisson bracket (PB) algebra of the remaining current $J_{-\alpha} \equiv T$ can be derived from eqn. (11), by calculating the corresponding Dirac brackets

$$\{T(\sigma), T(\sigma')\}_D = \frac{k^2}{2}\delta'''(\sigma-\sigma') - 2T(\sigma')\delta'(\sigma-\sigma') + \partial_{\sigma'}T(\sigma')\delta(\sigma-\sigma'), \tag{15}$$

[4] The supersymmetric extensions arise from the constrained superconformal current algebras

[5] An equivalent explanation of the $J_1 = 0$ condition (which is not a constraint) is that, due to the residual gauge transformation $h = \exp(\beta(z))E_{-\alpha}$, $J' = h^{-1}Jh + kh^{-1}\partial h$, which leaves invariant the constraint $J_\alpha = 1$, one can make $J'_1 = 0$, by an appropriate choice of $\beta(z)$.

which is nothing, but the *classical* PB's Virasoro algebra. Another form of the Dirac method, proposed by Polyakov [29] consists in imposing the constraints and gauge fixing conditions on the infinitesimal $\hat{\mathcal{G}}_r$-gauge transformations

$$\delta_\epsilon J^a(\sigma) = f^{abc} J^c(\sigma)\epsilon^b(\sigma) + \frac{k}{2}\partial_\sigma \epsilon^a. \tag{16}$$

Next, we solve the $\delta_\epsilon J_\alpha = \delta J_1 = 0$ equations for the redundant gauge parameters ϵ_1 and ϵ_α

$$\epsilon_1 = \frac{k}{2}\partial\epsilon, \quad \epsilon_\alpha = -\frac{k^2}{2}\partial^2\epsilon + J_{-\alpha}\epsilon, \quad \epsilon \equiv \epsilon_{-\alpha},$$

and substituting them in $\delta J_{-\alpha}$, we find

$$\delta_\epsilon J_{-\alpha} = -\frac{k^2}{2}\partial^3\epsilon + 2J_{-\alpha}\partial\epsilon + \partial J_{-\alpha}\epsilon,$$

i. e., the functional form of eqn. (15).

(1b) $A_1/U(1) \equiv$ *classical PF-algebra*: Take $J_0(\equiv J_1) = 0$ as a constraint (no residual gauge transformations exists). In this case, as a consequence of eqn. (14), we have $\{J_0(\sigma), J_0(\sigma')\} = \partial_{\sigma'}\delta(\sigma - \sigma')$ ($k = 2$). To find the Dirac brackets of the $J'_{\pm\alpha}$s, we have to invert the ∂_σ-operator, i. e., $\partial_\sigma(\partial_{\sigma'}^{-1}) = \delta(\sigma - \sigma')$, hence $\partial_{\sigma'}^{-1} = \frac{1}{2}\epsilon(\sigma - \sigma')$) and thus to introduce *nonlocal* $\epsilon(\sigma)$-terms in the $J_{\pm\alpha} \equiv V^{\pm}$-algebra

$$\begin{aligned} \{V^{\pm}(\sigma), V^{\pm}(\sigma')\}_D &= -V^{\pm}(\sigma)V^{\pm}(\sigma')\epsilon(\sigma - \sigma'), \\ \{V^{+}(\sigma), V^{-}(\sigma')\}_D &= \partial_{\sigma'}\delta(\sigma - \sigma') + V^{+}(\sigma)V^{-}(\sigma')\epsilon(\sigma - \sigma'). \end{aligned} \tag{17}$$

Following the Polyakov method, we get

$$\epsilon_0(\sigma) = -\int \epsilon(\sigma - \sigma')(V^{+}(\sigma')\epsilon^{-}(\sigma') - V^{-}(\sigma')\epsilon^{+}(\sigma'))d\sigma',$$

and pluging it back in the $\delta_{\epsilon^\pm}V^{\pm}$-transformations, we arrive at eqn. (17). The reason to call this *nonlocal PB* algebra as a *classical parafermionic* one is that an appropriate $N \to \infty$ limit of the (quantum) PF OPE' (9) reproduces exactly eqn. (17), as we will demonstrate, in detail, in the next section.

Example 2.2. Constrained $SL(3, R)$-algebras.

(2a) $A_2/\mathcal{N}_+ \equiv$ *W_3-algebra*: In this case, $\mathcal{N}_+ = \{E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_1+\alpha_2}\}$ and $J_{\alpha_i} = 1$, $J_{\alpha_1+\alpha_2} = 0$ are the constraints; $J_i = J_{-\alpha_1} = 0$ are the gauge fixing conditions, in Drinfeld-Sokolov gauge. The classical W_3-algebra, generated by one spin 2 $T(z) \equiv J_{-\alpha_2}(z)$ and one spin 3 $W_3(z) = J_{-\alpha_1-\alpha_2} - \frac{1}{2}\partial J_{-\alpha_2}$ currents, has the form ($k = 2$)

$$\{T(\sigma), W_3(\sigma')\} = 3W_3(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') + 2\partial_{\sigma'}W_3(\sigma')\delta(\sigma - \sigma'),$$

$$\begin{aligned} \{W_3(\sigma), W_3(\sigma')\} &= -4\delta^{(v)}(\sigma - \sigma') + 5T(\sigma')\delta'''(\sigma - \sigma') - \frac{15}{2}\partial_{\sigma'}T(\sigma')\delta''(\sigma - \sigma') \\ &- \left(T^2(\sigma') - \frac{9}{2}\partial^2_{\sigma'}T(\sigma')\right)\delta'(\sigma - \sigma') + \partial_{\sigma'}\left(\frac{1}{2}T^2(\sigma') - \partial^2_{\sigma'}T(\sigma')\right)\delta(\sigma - \sigma'). \end{aligned} \tag{18}$$

(2b) $A_2/\mathcal{N}^1_+ \otimes U(1) \equiv V_3^{(1,1)}$*-algebra*: Take $\mathcal{N}^{(1)}_+ = \{E_{\alpha_2}, E_{\alpha_1+\alpha_2}\}$, $U(1) = \{\vec{\lambda}_1 \cdot \vec{H}\}$; $J_{\alpha_2} = 1$, $J_{\alpha_1+\alpha_2} = 0$, $\sum_{i=1}^2 \lambda_1^{(i)} J_i = 0$ are the constraints and $J_{-\alpha_1} = \sum_{i=1}^2 \alpha_2^i J_i = 0$ are the gauge fixing conditions. The remaining currents $V^+ = J_{\alpha_1}$, $V^- = J_{-\alpha_1-\alpha_2}$, of spin $\frac{3}{2}$ (nonlocal), and $T = J_{-\alpha_2}$, of spin 2 (local), generate the following nonlocal $V_3^{(1,1)}$-algebra ($k = 2$) (13). We have to mention that if one *relaxes* the $U(1)$-constraint $J = \vec{\lambda}_1 \cdot \vec{J} = 0$, then the *local* currents $V^\pm$ ($s = \frac{3}{2}$), T ($s = 2$) and J ($s = 1$) span the well known local quadratic (in J) Bershadsky-Polyakov $A_2/\mathcal{N}^{(1)}_+ \equiv W_3^{(2)}$-algebra [28].

(2c) $A_2/\mathcal{N}^{(2)}_+ \otimes U(1) \otimes U(1) \equiv V_3$ *-algebra*: In this case, $\mathcal{N}^{(2)}_+ = \{E_{\alpha_1+\alpha_2}\}$, the constraints are $J_i = 0$, $J_{\alpha_1+\alpha_2} = 0$ and the gauge fixing (*g. f.*) conditions are $J_{-\alpha_1-\alpha_2} = 0$, $J_i = 0$. The *nonlocal* V_3-algebra, of the four spin 1 currents $V_i^\pm = J_{\pm\alpha_i}$ ($i = 1, 2$), has the form [37]

$$\{V_i^\pm(\sigma), V_j^\pm(\sigma')\} = \frac{1}{2k^2}[V_i^\pm(\sigma)V_j^\pm(\sigma') + V_i^\pm(\sigma')V_j^\pm(\sigma)]\epsilon(\sigma-\sigma'),$$

$$\begin{aligned}\{V_i^+(\sigma), V_j^-(\sigma')\} &= \delta_{ij}\partial_{\sigma'}\delta(\sigma-\sigma') - \frac{1}{2k^2}[V_i^+(\sigma)V_j^-(\sigma') \\ &+ \delta_{ij}\sum_{s=1}^2 V_s^-(\sigma)V_s^+(\sigma')]\epsilon(\sigma-\sigma'). \end{aligned} \qquad (19)$$

The stress-tensor $T(\sigma) = \frac{1}{2}\sum_{s=1}^2 V_s^+(\sigma)V_s^-(\sigma)$ satisfies the standard Virasoro algebra (15), but without a central term.

(2d) $A_2/\mathcal{N}^{(2)}_+ \otimes U(1) \equiv V_3^{(2)}$ *-algebra*: $\mathcal{N}^{(2)}_+$ is the same as in (2c), $U(1) = (\lambda_1 - \lambda_2)^i J_i$; the constraints are $J_{\alpha_1+\alpha_2} = 1$, $\sum_{i=1}^2 (\lambda_1 - \lambda_2)^i J_i = 0$ and the *g. f.* conditions are $(\alpha_1 + \alpha_2)^i J_i = 0$. The $V_3^{(2)}$-algebra, of the local spin 2 stress-tensor $T = J_{-\alpha_1-\alpha_2} - \frac{1}{2}(J_{\alpha_1}J_{-\alpha_1} + J_{\alpha_2}J_{-\alpha_2})$, and four nonlocal currents $V_1^+ = J_{\alpha_1}$ ($s_1^+ = \frac{1}{2}$), $V_1^- = J_{-\alpha_1} - 2\partial J_{\alpha_2}$ ($s_1^- = \frac{3}{2}$), $V_2^- = J_{\alpha_2}$ ($s_2^- = \frac{1}{2}$) and $V_2^+ = J_{-\alpha_2} + 2\partial J_{\alpha_1}$ ($s_2^+ = \frac{3}{2}$), takes the form [37]

$$\{V_i^\pm(\sigma), V_j^\pm(\sigma')\} = (i-j)[V^\pm_{\frac{1}{2}(i+j\mp 1)}(\sigma)]^2\delta(\sigma-\sigma') + \frac{3}{8}V_i^\pm(\sigma)V_j^\pm(\sigma')\epsilon(\sigma-\sigma'),$$

$$\{V_i^-(\sigma), V_i^+(\sigma')\} = 2V_1^-(\sigma)V_2^+(\sigma)\delta(\sigma-\sigma') - \frac{3}{8}V_i^-(\sigma)V_i^+(\sigma')\epsilon(\sigma-\sigma'),$$

$$\begin{aligned}\{V_1^-(\sigma), V_2^+(\sigma')\} &= -4\partial^2_{\sigma'}\delta(\sigma-\sigma') - \frac{3}{8}V_1^-(\sigma)V_2^+(\sigma')\epsilon(\sigma-\sigma') \\ &+ 3[V_1^+(\sigma)V_2^-(\sigma') + V_1^+(\sigma')V_2^-(\sigma)]\partial_{\sigma'}\delta(\sigma-\sigma') \\ &+ \{T(\sigma) + \frac{3}{2}[V_1^+(\sigma)V_1^-(\sigma) + V_2^+(\sigma)V_2^-(\sigma)]\}\delta(\sigma-\sigma'),\end{aligned}$$

$$\{V_2^-(\sigma), V_1^+(\sigma')\} = \delta(\sigma-\sigma') - \frac{3}{8}V_2^-(\sigma)V_1^+(\sigma')\epsilon(\sigma-\sigma'). \qquad (20)$$

Thus, $V_3^{(2)}$ is an example of *nonlocal quadratic* (non-Lie) algebra.

(2e) $A_2/\mathcal{N}_+^{(1)} \equiv W_3^{(1,1)}$*-algebra*: In this case, the constraints are $J_{\alpha_2} = J_{\alpha_1+\alpha_2} = 1$ and the *g. f.* conditions are $\alpha_2^i J_i = (\alpha_1+\alpha_2)^i J_i = 0$. The algebra $W_3^{(1,1)}$, of the local currents $J^\pm = J_{\pm\alpha_1}$ $(s^\pm = 1)$ and $T_2 = J_{-\alpha_2}$ $(s_2 = 2)$, $T_{12} = J_{-\alpha_1-\alpha_2}$ $(s_{12} = 2)$, appears to be a local quadratic algebra [37], of W-type.

Example 2.3. **Constrained** $SL(4,R)$**-algebras**.

(3a) $A_3/\mathcal{N}_+ \equiv W_4$*-algebra*: $\mathcal{N}_+ = \{E_{[\alpha]} : [\alpha]$are all positive roots$\}$, the constraints are $J_{\alpha_i} = 1$ $(i = 1,2,3)$, $J_{\alpha_1+\alpha_2} = J_{\alpha_2+\alpha_3} = 0$, $J_{\alpha_1+\alpha_2+\alpha_3} = 0$ and the *g. f.* conditions are $J_i = 0$, $J_{-\alpha_1} = 0$, $J_{-\alpha_2} = 0$, $J_{-\alpha_1-\alpha_2} = 0$. The algebra of the remaining currents $T = J_{-\alpha_3}$, $W_3 = J_{-\alpha_2-\alpha_3}$ and $W_4 = J_{-\alpha_1-\alpha_2-\alpha_3}$ is the standard quadratic W_4-algebra [25], [26], [27].

(3b) $A_3/\mathcal{N}_+^{(1,1)} \otimes U(1) \equiv V_4^{(1,1)}$ *-algebra*: $\mathcal{N}_+^{(1,1)} = \{E_{[\alpha]_1}\}$, where $[\alpha]_1$ are all positive roots, but α_1, $U(1) = \lambda_1 \cdot H$, the constraints are $J_{\alpha_2} = J_{\alpha_3} = 1$, $J_{\alpha_1+\alpha_2} = J_{\alpha_2+\alpha_3} = 0$, $J_{\alpha_1+\alpha_2+\alpha_3} = 0$, $\lambda_1^i J_i = 0$ and the *g. f.* conditions are $\alpha_2^i J_i = \alpha_3^i J_i = 0$, $J_{-\alpha_2} = J_{-\alpha_1-\alpha_2} = 0$, $J_{-\alpha_1} = 0$. The spin 2 currents $V^+ = J_{\alpha_1}$ and $V^- = J_{-\alpha_1-\alpha_2-\alpha_3}$ are nonlocal and $W_3 = J_{-\alpha_2-\alpha_3}$ $(s_3 = 3)$, $T = J_{-\alpha_3}$ $(s_T = 2)$ are local ones. Their algebra is a nonlocal extension of the W_3-one (18)

$$
\begin{aligned}
\{V^+(\sigma), V^-(\sigma')\} &= \partial_{\sigma'}^3\delta(\sigma-\sigma') - T(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') - W_3(\sigma')\delta(\sigma-\sigma') \\
&+ \frac{1}{3}V^+(\sigma)V^-(\sigma')\epsilon(\sigma-\sigma'),
\end{aligned}
$$

$$
\begin{aligned}
\{W_3(\sigma), V^\pm(\sigma')\} &= \mp\frac{10}{3}V^\pm(\sigma')\partial_{\sigma'}^2\delta(\sigma-\sigma') \mp 5\partial_{\sigma'}V^\pm(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') \\
&\pm \frac{1}{3}[T(\sigma')V^\pm(\sigma') - 6\partial_{\sigma'}^2 V^\pm(\sigma')]\delta(\sigma-\sigma'),
\end{aligned}
$$

$$
\{V^\pm(\sigma), V^\pm(\sigma')\} = -\frac{1}{3}V^\pm(\sigma)V^\pm(\sigma')\epsilon(\sigma-\sigma'), \tag{21}
$$

and the remaining PB $\{W_3(\sigma), W_3(\sigma')\}$ has the same form as in (18), but T^2, in the quadratic terms, is replaced by $T^2 + 6V^+V^-$.

(3c) $A_3/\mathcal{N}_+^{(1,2)} \equiv U_4^{(1,2)}$ *-algebra*: $\mathcal{N}_+^{(1,2)} = \{E_{[\alpha]_2}\}$, where $[\alpha]_2$ are all positive roots, but α_2, the constraints are $J_{\alpha_1} = J_{\alpha_3} = 1$, $J_{-\alpha_1-\alpha_2} = J_{-\alpha_2-\alpha_3} = 0$, $J_{-\alpha_1-\alpha_2-\alpha_3} = 0$ and the *g. f.* conditions are $\alpha_1^i J_i = 0$, $\alpha_3^i J_i = 0$, $J_{\alpha_1+\alpha_2} = J_{\alpha_2} = 0$, $J_{\alpha_2+\alpha_3} = 0$. The *nonlocal quadratic* $U_4^{(1,2)}$-algebra is generated by one spin 1 current $J = \lambda_2^i J_i$, three local spin 2 currents $V^+ = J_{\alpha_2}$, $V^- = J_{-\alpha_1-\alpha_2-\alpha_3}$, $T = J_{-\alpha_1} + J_{-\alpha_3} + 4J^2$ and one nonlocal spin 2 current $U = J_{-\alpha_3} - J_{-\alpha_1}$ [37]

$$
\{U(\sigma), J(\sigma')\} = \{V^\pm(\sigma), V^\pm(\sigma')\} = 0, \qquad \{U(\sigma), V^\pm(\sigma')\} = \frac{1}{2}V^\pm(\sigma)U(\sigma')\epsilon(\sigma-\sigma'),
$$

$$
\{J(\sigma), V^\pm(\sigma')\} = \mp\frac{1}{4}V^\pm(\sigma)\delta(\sigma-\sigma'), \qquad \{J(\sigma), J(\sigma')\} = \frac{1}{8}\partial_\sigma\delta(\sigma-\sigma'),
$$

$$\{V^{\pm}(\sigma), V^{\mp}(\sigma')\} = -\frac{1}{2}\partial_\sigma^3\delta(\sigma-\sigma') + \mathcal{T}(\sigma)\partial_\sigma\delta(\sigma-\sigma') + \frac{1}{2}\partial_\sigma\mathcal{T}(\sigma)\delta(\sigma-\sigma') - \frac{1}{4}U(\sigma)U(\sigma')\epsilon(\sigma-\sigma'),$$

$$\{U(\sigma), U(\sigma')\} = -\partial_\sigma^3\delta(\sigma-\sigma') + 2\mathcal{T}(\sigma)\partial_\sigma\delta(\sigma-\sigma') + \partial_\sigma\mathcal{T}(\sigma)\delta(\sigma-\sigma') - [V^+(\sigma)V^-(\sigma') + V^+(\sigma')V^-(\sigma)]\epsilon(\sigma-\sigma'), \tag{22}$$

where $\mathcal{T} = T - 4J^2$. As it is shown in ref [37], by choosing another set of gauge fixing conditions, the nonlocal $U_4^{(1,2)}$-algebra takes the form of the *rational* ($\frac{1}{U}$-terms) local algebras of ref [35]. If one further imposes $J = \lambda_2^i J_i = 0$ as a new constraint, the J-reduced $U_4^{(1,2)}$-algebra (i. e., $U_4^{(1,2)}/U(1)$) coincides with the nonlocal $V_4^{(1,2)}$-algebra (see sec. 7 of ref [33]). The main difference with $U_4^{(1,2)}$ is that the spin 2 currents $V^{\pm}$ become *nonlocal*, in the $V_4^{(1,2)}$-case.

The analysis of the above examples of H-reduced $SL(n,R)$-current algebras allows to conclude that they all fit into the following *basic algebraic structures*:

(A) W-algebras (quadratic): **(2a)**, **(2e)**, **(3a)** and $W_n^{(2)}$ of ref [28], W_S^G of ref [35];

(B) U-algebras (rational or nonlocal): **(3c)**;

(C) V-albegras (nonlocal or PF-type): **(1b)**, **(2b)**, **(2c)**;

and the following mixtures of **(A)** and **(B)** with **(C)**:

(D) WV-algebras (nonlocal (PF) quadratic): **(2d)** and **(3b)** (and all $V_{n+1}^{(1,1)}$-algebras of sec. 3 of ref [33]);

(E) UV-algebras (nonlocal (PF) rational): $V_4^{(1,2)} = U_4^{(1,2)}/U(1)$.

This observation adresses *the question* about the algebraic conditions that a given set of constraints (and gauge fixing conditions) $\{H\} \in \mathcal{G}$ should satisfy in order to lead to one of the above mentioned algebraic structures (U, W, V, UV, UW). To answer this question, as well as whether other families of algebras can exist, we need an *efficient method* for describing all unequivalent (and irreducible) sets of first class constraints one can impose on the currents of a given affine algebra $\hat{\mathcal{G}}$. Given a Lie algebra $\mathcal{G}_r$, by introducing a grading operator[6] $Q_r^{(s)} = \sum_{n=1}^r s_n \frac{2\vec{\lambda}_n\cdot\vec{H}}{\alpha_n^2}$ we provide it with a specific graded structure[7]

$$\mathcal{G}_r = \oplus_i \mathcal{G}_{\pm i}^{(s)}, \quad [Q_r^{(s)}, \mathcal{G}_{\pm i}^{(s)}] = \pm i\mathcal{G}_{\pm i}^{(s)}, \quad [\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}.$$

For each fixed $l = 1, 2, ..., r$ (and $Q_r^{(s)}$), define the nilpotent subalgebra $\mathcal{N}_+^{(l,s)} = \oplus_{i=l}\mathcal{G}_i$ and choose a generic element $\epsilon_+^{(l)} \in \mathcal{G}_l$, i. e., $\epsilon_+^{(l)} = \sum_{\alpha\in[\alpha]_l}\mu_\alpha E_{[\alpha]_l}$ where $E_{[\alpha]_l}$ are all the step operators of grade l and μ_α are arbitrary constants. Next, we consider the $\epsilon_+^{(l)}$-invariant subalgebras of $\mathcal{G}_0^{(l)} = \oplus_{i=0}^{l-1}\mathcal{G}_i$ and $\mathcal{G}_-^{(l)} = \oplus_{i=l}^{r}\mathcal{G}_{-i}$

[6] $\vec{\lambda}_n$ are the fundamental weights of $\mathcal{G}_r$, $\vec{\alpha}_n$ its simple roots, $\vec{H}$ its Cartan subalgebra and s_n are nonnegative integers.

[7] The nonequivalent graded structures $\mathcal{G}_r$ can have (i. e., the set of the allowed $Q_r^{(s)}$), are given by the Kac theorem [38]; this method was introduced in ref [39], in the construction of the conformal non-Abelian Toda models.

$$\mathcal{K}^0_\epsilon(s,l) = \mathrm{kerad}\epsilon^{(l)}_+ \cap \mathcal{G}^{(l)}_0 = \{g^0_0 \in \mathcal{G}^{(l)}_0 : [\epsilon^{(l)}_+, g^0_0] = 0\},$$

$$\mathcal{K}^-_\epsilon(s,l) = \mathrm{kerad}\epsilon^{(l)}_+ \cap \mathcal{G}^{(l)}_- = \{g^- \in \mathcal{G}^{(l)}_- : [\epsilon^{(l)}_+, g^-] = 0\}.$$

Finally, we define the "constraint" subalgebra as $\mathcal{H}^{(l,s)}_c(\epsilon^{(l)}_+) = \mathcal{N}^{(l)}_+(\epsilon) \oplus \mathcal{H}^{(l)}_0$, where $\mathcal{H}^{(l)}_0 \subset \mathcal{K}^0_\epsilon(s,l)$ denotes those subalgebras of $\mathcal{K}^0_\epsilon$, which elements (i. e., the currents belonging to $\mathcal{H}^{(l)}_0$) are constrained to zero; $\mathcal{N}^{(l)}_+(\epsilon)$ caries the information about the constraints we are imposing on the currents from $\mathcal{N}^{(l)}_+$, namely all the elements of the subalgebra $\mathcal{N}^{(l+1)}_+ \subset \mathcal{N}^{(l)}_+$ are zero and the elements of $\mathcal{G}_l$, which are constrained to be constants $\mu_\alpha \neq 0$, are collected in $\epsilon^{(l)}_+ = \sum_\alpha \mu_\alpha E_\alpha$ (all the remaining $\mathcal{G}_l$ elements are zero). In this language, *the problem of the classification* of the allowed set of constraints $\mathcal{H}^{(l,s)}_c(\epsilon)$ reads as follows: for each fixed $Q^{(s)}_r$ and fixed grade l (say $l = 1$), to make a list of all the nonequivalent choises of the $\epsilon^{(l)\prime}_+$s. One can further organizes the different sets of $\epsilon^{(l)\prime}_+$s (l and $Q^{(s)}_r$ fixed) in *families* ($\mathcal{K}^0_\epsilon(s,l)$, $\mathcal{K}^-_\epsilon(s,l)$), according to their *invariant subalgebras.* For example, the family ($\lambda_i \cdot H$, Ω) is characterized by the conditions: (a) $\mu_\alpha = 0$, for the $\vec{\alpha^i} \cdot \vec{\lambda_i} \neq 0$ ($\alpha \in [\alpha]_l$), and (b) $i = 1$ or $i = r$ (for the $l = 1$ case), in order to have $\mathcal{K}^-_\epsilon(s,1) = \Omega$. We call *equivalent* the sets of constraints (and gauge fixing conditions) which can be obtained from each other by certain discrete transformations from the Weyl group of $\mathcal{G}_r$. As it is shown in sec. 8 of ref [33] (for the grade $l = 1$), they give rise to the same $\mathcal{H}^{(l,s)}_c(\epsilon)$-reduced $\mathcal{G}_r$-algebra. Therefore, it is sufficient to consider only one representative of such "Weyl families" of constraints. The problem of the *irreducibility* is more delicate. Depending on our choice of $\epsilon^{(l)\prime}_+$s, it might happens that the $\mathcal{G}_r/\mathcal{H}^{(l,s)}_c(\epsilon)$-algebra splits into two (or more) mutualy commuting algebras [37]. This is the case when one takes, for example $\mu_\alpha = 0$, for all E_α that contains the simple root α_i (i. e., E_{α_i}, $E_{\alpha_i+\alpha_{i+1}}$, $E_{\alpha_{i-1}+\alpha_i}$, $E_{\alpha_{i-1}+\alpha_i+\alpha_{i+1}}$ etc).

The organization of the constraints in the families ($\mathcal{K}^0_\epsilon(s,l)$, $\mathcal{K}^-_\epsilon(s,l)$) simplifies the derivation of the $\mathcal{G}^{(l,s)}_r(\epsilon, \mathcal{H}^{(l)}_0) = \mathcal{G}_r/\mathcal{H}^{(l,s)}_c(\epsilon)$-algebras (i. e., the calculation of the corresponding Dirac brackets). Depending on the algebraic data $\{\mathcal{G}_r, Q^{(s)}, l, \epsilon^{(l)}_+\}$, which defines $\mathcal{H}^{(l,s)}_c(\epsilon)$, one can *classify* all the $\mathcal{G}^{(l,s)}_r(\epsilon, \mathcal{H}^{(l)}_0)$-algebras in the following $\{\mathcal{H}^{(l)}_0, \mathcal{K}^-_\epsilon(s,l)\}$-families of algebras:

THEOREM. Given $\mathcal{G}_r$ and the graded structure ($Q^{(s)}_r$, l, $\epsilon^{(l)}_+$), which define the constraints subalgebra $\mathcal{H}^{(l,s)}_c(\epsilon) \subset \mathcal{G}_r$. Each $\mathcal{H}^{(l,s)}_c(\epsilon)$-reduced $\mathcal{G}_r$-current algebra $\mathcal{G}^{(l,s)}_r(\epsilon, \mathcal{H}^{(l)}_0)$ belongs to one of the following five types of extended Virasoro algebras:

(1) W-algebras, when $\mathcal{H}^{(l)}_0 = \Omega$ (or $\mathcal{H}^{(l)}_0 \neq \Omega$ but $[\mathcal{H}^{(l)}_0, \mathcal{G}^\pm_0] = 0$) and $\mathcal{K}^-_\epsilon = \Omega$ ($\mathcal{G}^\pm_0$ are the $\pm$ step operators of grade 0);

(2) U-algebras, when $\mathcal{H}^{(l)}_0 = \Omega$ (or $\mathcal{H}^{(l)}_0 \neq \Omega$ but $[\mathcal{H}^{(l)}_0, \mathcal{G}^\pm_0] = 0$) and $\mathcal{K}^-_\epsilon \neq \Omega$; dim $\mathcal{K}^-_\epsilon$ is the number of the nonlocal currents (or of the "rational currents" of ref [35]);

(3) V-algebras, when $\mathcal{H}^{(l)}_0 \neq \Omega$ and $\mathcal{H}^{(l)}_0 = U(1)^r$ or $U(1)^{r-1} = \{\oplus^r_{i=2}\lambda_i \cdot H\}$ or $\{\oplus^{r-1}_{i=1}\lambda_i \cdot H\}$, $\mathcal{K}^-_\epsilon = \Omega$; the case $\epsilon^{(l)}_+ = 0$, $\mathcal{H}^{(l)}_0 = U(1)^r$, which also lead to V-algebras, has to be treated separately (see ref [37]);

(4) VW-algebras, when $\mathcal{H}^{(l)}_0 \neq \{\Omega,\ U(1)^r,\ U(1)^{r-1}\}$ and $\mathcal{K}^-_\epsilon = \Omega$; the case $[\mathcal{H}^{(l)}_0, \mathcal{G}^\pm_0] = 0$ ($\mathcal{G}^\pm_0$ are the $\pm$ step operators of grade zero) has to be excluded, since it gives rise to W-algebras;

(5) VU-algebras, when $\mathcal{H}_0^{(l)} \neq \{\Omega,\ U(1)^r,\ \oplus_{i=2}^{r}\lambda_i \cdot H,\ \oplus_{i=1}^{r-1}\lambda_i \cdot H\}$, and $\mathcal{K}_\epsilon^- \neq \Omega$; in this case, again $[\mathcal{H}_0^l, \mathcal{G}_0^\pm] \neq 0$.

The algebraic conditions that separate W- from the U-algebras are given in ref [35]. The equivalence of the *rational* U-algebras, to certain *nonlocal* algebras, and the explicit form of the gauge transformations, from the "rational" gauge fixing conditions to "nonlocal" gauge fixing conditions, is demonstrated in ref [37]. The proof of this theorem, for the generic $Q_r^{(s)}$ grade one ($l = 1$) case [37], is based on the analysis of the properties of the inverse matrix Δ_{ij}^{-1} of the constraints and the gauge fixing conditions. The *origin of the nonlocal terms* in the V- , VW- and VU-algebras, are the $\mathcal{H}_0^{(l)}$ constraints and their gauge fixing's. Their PB's are always in the form $\{J_i(\sigma), J_j(\sigma')\} = k\delta_{ij}\partial_\sigma\delta(\sigma-\sigma')$ or $\{J_{-\alpha_i}(\sigma), J_{\alpha_i}(\sigma')\} = k\partial_\sigma\delta(\sigma-\sigma')$. Their contributions to Δ_{ij}^{-1} are the nonlocal $\epsilon(\sigma-\sigma')$-terms.

The explicit form of each $\mathcal{G}_r^{(l,s)}(\epsilon, \mathcal{H}_0^{(l)})$- algebra (from a given class U, V, VW etc) indeed depends on the algebra $\mathcal{G}_r$ and on the choice of $\epsilon_+^{(l)}$ and $\mathcal{H}_0^{(l)}$ as one can see from our Examples 1, 2 and 3. The full algebraic structure (all explicit PB's) is known in the case of the W_n-algebras [25] and of the simplest A_r-family of VW-algebras ($V_{r+1}^{(1,1)}$-algebras [33]) defined by $Q = \sum_{i=l}^{r}\lambda_i \cdot H$, $l = 1$, $\epsilon_+^{(1)} = \sum_{i=2}^{r} E_{\alpha_i}$, $\mathcal{H}_0^{(1)} = \{\lambda_1 \cdot H\}$. Various examples of the U- V- and VU-algebras ($V_{(n,m)}$) have been constructed by Bilal [32], by calculating the second Gelfand-Dikii brackets, associated with certain matrix differential operators.

3 Quantum V-algebras

The classification of all the classical (PB's) extensions of the Virasoro algebra is an important step forward the classification of the universality classes in two dimensions. The complete solution of this problem requires, however, the knowledge of the exact critical exponents, i. e., we need to know the *h. w.* representation of the corresponding *quantum* W-, U-, V- and VW-, VU-algebras. The quantization of the classical W-algebras is a rather well understood problem. It consists in replacing the currents functions T, W_n by currents operators $\hat{T}$, $\hat{W}_n$, acting on some Hilbert space, and their PB's $\{a, b\}$ by the commutators $-\frac{i}{\hbar}[a, b]$. The only changes that occur in this procedure are the new (*quantum corrections*) coefficients in front of the central term $\delta^{(s)}(\sigma)$ and those of the quadratic terms. Another option is to start with the quantum current algebra $\mathcal{G}_r$ and to implement the operators constraints $\mathcal{H}_c^{(l,s)}(\epsilon)$ on it, following the methods of the *quantum Hamiltonian reduction* [36]. The advantage of this method is that it provides a simple way of deriving the W-algebra *h. w.* representations from the *h. w.* representations of the $\mathcal{G}_r$-current algebra. The specific nonlocal terms $V^+(\sigma)V^-(\sigma')\epsilon(\sigma-\sigma')$ that appears in the V- (and VW-, VU-) algebras, as well as the nonlocal nature of the part of the currents ($V_i^\pm$), are the main obstacle to the construction of the corresponding *quantum* V-algebras. It turns out [33], [34] that their quantization require deep changes in the classical algebraic structure (13), (19), (20), namely: (a) renormalization of the bare spins of the nonlocal currents (say for $V_{n+1}^{(1,1)}$, $s_{cl}^\pm = \frac{n+1}{2}$ goes to $s_q^\pm = \frac{n+1}{2}\left(1 - \frac{1}{2k+n+1}\right)$); (b) the quantum counterpart of the PB's of the $V^{\pm\prime}$s charges appears to be specific PF-type commutators, similar to eqn. (10);

(c) breaking of the global $U(1)$ symmetry, to some discrete group Z_{2k+n}.

The fact that all the complications in the quantization of the classical V- and VW-algebras are coming from the $\mathcal{H}_0^{(l)}$-constraints suggests the following *strategy*: relax the $\mathcal{H}_0^{(l)}$-constraints (i. e., leave the currents $\lambda_a^i J_i \in \mathcal{H}_0^{(l)}$ unconstrained) and consider the corresponding local "*intermidiate*" W-algebra, generated by the V- (or VW-) algebra currents (which are all local now) and the additional spin one $\mathcal{H}_0^{(l)}$-currents. Since all the currents are local, the quantization of this algebra is similar to the one of the W_{n+1}- or $W_{n+1}^{(l)}$-algebras [25], [36]. The problem we address here is the following: *Given the quantum W-algebra and its h. w. representations, to derive the quantum $V = W/\mathcal{H}_0^{(l)}$-algebra and its h. w. representations by implementing the (operator) constraint $\mathcal{H}_0^{(l)} \approx 0$.* The method we are going to use is an appropriate generalization of the derivation of the Z_N parafermionic algebra [12] from the affine $SU(2)$-one (or $SL(2,R)$, for the noncompact PF's), by imposing the constraint $J_3(z) \approx 0$.

Example 3.1. **Quantization of the PF-algebra**. Following the arguments of ref [12], we define the quantum (compact) V_2-algebras as $V_2 = \{SU(2)_k, J_3(z) = 0\}$. Therefore, the V_2-generators $\psi^\pm$ have to represent the $J_3 = \sqrt{\frac{k}{2}}\partial\phi$ -independent part of the $\hat{SU}(2)_k$-ones, i. e.,

$$J^\pm = \psi^\pm \exp(\mp\alpha\phi), \quad T = T_V + \frac{1}{2}(\partial\phi)^2, \quad J_3(z_1)\psi^\pm(z_2) = O(z_{12}),$$

$$\phi(z_1)\phi(z_2) = -\ln(z_{12}) + O(z_{12}). \tag{23}$$

Taking into account the $SU(2)$ OPE's

$$J_3(z_1)J^\pm(z_2) = \pm\frac{\imath}{z_{12}}J^\pm(z_2) + O(z_{12}) \tag{24}$$

and eqn. (23), we find $\alpha = \imath\sqrt{\frac{2}{k}}$ and, as a consequence, the spins of $\psi^\pm$ are $s^\pm = 1 - \frac{1}{k}$ (we have used that $s_{J^\pm} = 1$). Finally, eqns. (23) and (24) lead to the following V_2-algebra OPE's

$$\begin{aligned} \psi^\pm(z_1)\psi^\pm(z_2) &= z_{12}^{-\frac{2}{k}}\psi^\pm_{(2)}(z_2) + O(z_{12}), \\ \psi^+(z_1)\psi^-(z_2) &= z_{12}^{\frac{2}{k}}\left(\frac{k}{z_{12}^2} + (k+2)T_V + O(z_{12})\right), \end{aligned} \tag{25}$$

which are nothing, but the PF-algebra OPE's (9), with $k = N$ and $\psi_1^\pm = \frac{1}{\sqrt{k}}\psi^\pm$. Although the V_2-algebra (25) is, by construction, the quantum version of the classical PB's PF-algebra (17), the discrepancy between the spins $s^\pm = 1 - \frac{1}{k}$ and $s_{V^\pm} = 1$ requires a more precise definition of the relation of algebras (25) and (17). The exact *statement* is as follows: let $V^\pm = \frac{1}{k}\psi^\pm$ and the $V^\pm$ PB's are defined as a certain limit of the OPE's (25):

$$\{V^a(z_1), V^b(z_2)\} = \lim_{k\to\infty}\frac{k}{2\pi\imath}[V^a(z_1)V^b(z_2) - V^a(z_2)V^b(z_1)] \tag{26}$$

$(a,b=\pm)$. Then, the $k\to\infty$ limit of the OPE's (25) reproduces the PB's (17). The proof is straightforward. Applying twice the OPE's (25), we find

$$z_{12}^{\frac{2}{k}}\{V^{\pm}(z_1)V^{\pm}(z_2)-V^{\pm}(z_2)V^{\pm}(z_1)e^{-\frac{2\pi\imath}{k}\epsilon}(z_{12})\} \;=\; \frac{1}{k^2}O(z_{12}),$$

$$\begin{aligned} z_{12}^{-\frac{2}{k}}\{V^{-}(z_1)V^{+}(z_2) &- V^{+}(z_2)V^{-}(z_1)e^{\frac{2\pi\imath}{k}\epsilon}(z_{12})\} = \frac{k+2}{k^2}O(z_{12}) \\ &+ \frac{1}{k}\left(\frac{1}{z_{12}^2+\imath 0}-\frac{1}{z_{21}^2+\imath 0}\right), \end{aligned} \tag{27}$$

where the identity $\imath\pi\epsilon(z_{12})\equiv\ln\frac{z_{12}+\imath 0}{z_{21}+\imath 0}$ has been used. The $k\to\infty$ limit of eqns. (27) reproduces exactly the classical PF-algebra[8] (17). The conclusion is that the nonlocal PB's-algebra (17) is a semiclassical limit $(k\to\infty)$ of the PF's OPE's (25). As we have seen, the quantization requires renormalization of the spins $s_q=s_{cl}-\frac{1}{k}$ of the nonlocal currents $V^{\pm}$. Therefore, the PB's (17) have to be replaced by the PF-commutators (10) and for k-positive integers, the classical global $Z_2\otimes U(1)$-symmetry is broken to $Z_2\times Z_k$, in the quantum theory.

The structure of the classical $V_3^{(1,1)}$ PB's algebra (13) is quite similar to the PF-one (17). An importante difference is that in its derivation from the classical $SL(3,R)$ (see our example 2b), one has to impose, together with the $\mathcal{H}_0^1$-type (PF) constraint $\vec{\lambda}_1\cdot\vec{J}=0$, two more constraints, on the nilpotent subalgebra $\mathcal{N}_+^{(1)}$: $J_{\alpha_2}=1$ and $J_{\alpha_1+\alpha_2}=0$. In order to demonstrate how this type of (purely W-) constraints are treated, in the frameworks of the quantum Hamiltonian reduction, we consider the simplest example of such reduction: the $\mathcal{N}_+$-reduced $SL(2,R)$ $(J_\alpha=1)$ which gives rise to the Virasoro algebra (example 1a).

Example 3.2. Virasoro algebra *h. w.* representation from the $SL(2,R)$-ones [36]. The implementation of the constraint $J_\alpha=1$ as an operator identity on the $SL(2,R)_k$-space of states $\mathcal{H}_{A_1}^{(k)}$ requires an introduction of a pair of fermionic ghosts ($b(z)$, $c(z)$) of spins (0,1) and of the larger space of states $\mathcal{H}_{A_1}^{(k)}\otimes\mathcal{H}_{b,c}$. The reduced representation space of the constrained system $\{A_1/\mathcal{N}_+\}$ can be defined by means of the $BRST$ operator

$$Q_{BRST}=\oint[J_\alpha(z)-1]c(z)dz,\quad Q_{BRST}^2=0,$$

as Q_{BRST}-invariant states-$|\psi>\in\mathcal{H}_{A_1}^{(k)}\otimes\mathcal{H}_{b,c}$ $(Q_{BRST}|\psi>=0)$, which are not Q_{BRST}-exact, i. e., $|\psi>\neq Q_{BRST}|*>$. The statement is that this $BRST$-cohomology $H_{Q_{BRST}}(\mathcal{H}_{A_1}^{(k)}\otimes\mathcal{H}_{b,c})=\mathrm{ker}Q/\mathrm{Im}Q$ is isomorphic to the irreducible representation space $\mathcal{H}_{Vir}^{(k)}(\equiv\mathcal{H}_{A_1}/\mathcal{N}_+)$ of the Virasoro algebra [36]. To make the constraints condition $J_\alpha=1$ consistent with the conformal invariance, we have to improve the $SL(2,R)$-Sugawara stress-tensor, in such a way that $s_{imp}(J_\alpha)=0$

$$T_{impr}=\frac{1}{k+2}:J^a(z)J^a(z):+\partial J_3.$$

[8] The noncompact case $SL(2,R)/U(1)$ corresponds to the change $\phi\to\imath\phi$, which turns out to be equivalent to the $k\to -k$ one, in the OPE's, spins etc.

Therefore, the new central charge is $c_{impr} = \frac{3k}{k+2} - 6k$. Taking into account the contribution $c_{gh} = -2$, of the ghost stress-tensor $T_{bc} = (\partial b)c$, we find that the total central charge is $c_{tot} = 13 - 6\left(\frac{1}{k+2} + k + 2\right)$. Since the dimension of the $\hat{SL}(2,R)_k$ representation of weight $\vec{\Lambda}$ is

$$\Delta_\Lambda = \frac{1}{2(k+2)}\vec{\Lambda}\cdot(\vec{\Lambda}+2\vec{\alpha})$$

the improved dimensions are found to be

$$\Delta_\Lambda^{impr} = \frac{1}{2(k+2)}\vec{\Lambda}\cdot(\vec{\Lambda}+2\vec{\alpha}) - \vec{\alpha}\cdot\vec{\Lambda}. \tag{28}$$

An important observation of ref [36] is that the $h.\ w.$ states of the reduced space $\mathcal{H}_{Vir}^{(k)}$ are of levels $k+2 = \frac{m}{m+1}$, where $m = 3, 4, ...$, and weights

$$\vec{\Lambda} = [(1-p)(k+2) - (1-q)]\vec{\alpha},$$

with $1 \le p \le m-1$, $1 \le q \le p$. Therefore, $c_{tot} = 1 - \frac{6}{m(m+1)}$ and $\Delta_\Lambda^{impr} = \Delta_{p,q}$, i. e., the $\mathcal{H}_{Vir}^{(k)} = H_{Q_{BRST}}(\mathcal{H}_{A_1}^{(k)} \otimes \mathcal{H}_{b,c})$ (for the above values of the levels, and the A_1-weight $\vec{\Lambda}$) coincides with the space of the $h.\ w.$ unitary representations (3) of the Virasoro algebra.

Example 3.3. **Quantum $V_3^{(1,1)}$-algebra**. As it was pointed out in ref [33], [34], the intermidiate $W_3^{(1,1)}$-algebra is Weyl equivalent (w_{α_1}) to the Bershadsky-Polyakov algebra $W_3^{(2)}$. The improved stress-tensor is given by

$$T^{impr} = \frac{1}{k+3} : J^q(z)J^q(z) : -(\lambda_2 - \frac{1}{2}\lambda_1)^i\partial J_i \tag{29}$$

and we have to introduce the following two pair of ghosts: (b,c) and (ϕ,ϕ^+), of spins (0,1) and $(\frac{1}{2},\frac{1}{2})$. Constructions, similar to the ones in Example 3.2, allow to derive the $W_3^{(1,1)}$ central charge

$$c_{W_3^{(1,1)}} = \frac{8k}{k+3} - 6k - 1$$

and the dimensions $\Delta_{\vec{r},\vec{s}}$ and $U(1)$ charges $q_{r,s}$ of its $h.\ w.$ representations (NS-sector) [28] read as

$$\begin{aligned} \Delta_{\vec{r},\vec{s}}^W &= \frac{1}{2(k+3)}\vec{\Lambda}_{r,s}\cdot(\vec{\Lambda}_{r,s}+2\vec{\beta}) - \vec{\beta}\cdot\vec{\Lambda}_{r,s}, \\ q_{\vec{r},\vec{s}} &= \frac{1}{3}\left[2\frac{p}{q}(r_1-r_2) - (s_1-s_2)\right], \end{aligned} \tag{30}$$

where $\beta = \lambda_2 - \frac{1}{2}\lambda_1$, with $\vec{\Lambda}_{r,s}$ representing the weights of the following specific level $k+3 = 2\frac{p}{q}$ representations of $SL(3,R)_k$

$$\vec{\Lambda}_{r,s} = \sum_{i=1}^{2}\vec{\lambda}_i[(1-r_i)(k+3) - (1-s_i)],$$

where $1 \leq s_i \leq 2p-1$, $1 \leq r_i \leq q$. The quantum $W_3^{(1,1)}$-algebra is generated by one spin $s=1$ $J(z)$, two spin $s=\frac{3}{2}$ $G^{\pm}(z)$ and one spin $s=2$ $T(z)$ currents, with OPE's [28]

$$J(z_1)J(z_2) = \frac{2k+3}{3z_{12}^2} + O(z_{12}), \qquad J(z_1)G^{\pm}(z_2) = \pm\frac{1}{z_{12}}G^{\pm}(z_2) + O(z_{12}),$$

$$\begin{aligned} G^+(z_1)G^-(z_2) &= \frac{(k+1)(2k+3)}{z_{12}^3} + 3\frac{k+1}{z_{12}^2}J(z_2) \\ &+ \frac{1}{z_{12}}\left[3J^2(z_2) - (k+3)T(z_2) + 3\frac{k+1}{2}\partial J(z_2)\right] + O(z_{12}). \end{aligned}$$

$$G^{\pm}(z_1)G^{\pm}(z_2) = O(z_{12}), \tag{31}$$

According to the definition of the $V_3^{(1,1)}$-algebra $V_3^{(1,1)} = \{W_3^{(1,1)}; J=0\}$, its generators $V^{\pm}(z)$ and T_V have to commute with $J(z)$, i. e.,

$$J(z_1)V^{\pm}(z_2) = J(z_1)T_V(z_2) = O(z_{12}). \tag{32}$$

Therefore, $V^{\pm}$, T_V are related to the $J = \sqrt{\frac{2k+3}{3}}\partial\phi$-independent parts of the $W_3^{(1,1)}$-currents

$$G^{\pm} = V^{\pm}\exp(\pm a\phi), \; T_W = T_V + \frac{1}{2}(\partial\phi)^2, \; \phi(z_1)\phi(z_2) = \ln(z_{12}) + O(z_{12}). \tag{33}$$

As a consequence of eqns. (32) and (33), we get $a = \sqrt{\frac{3}{2k+3}}$, and for the spins of the quantum currents $V^{\pm}$ $\left(s_{cl} = \frac{3}{2}\right)$, we obtain $s_q^{\pm} = \frac{3}{2}\left(1 - \frac{1}{2k+3}\right)$. The $W_3^{(1,1)} - OPE'$s (31), and eqn. (33), lead to the following OPE's for $V^{\pm}$ and T_V $(k \neq -3, -\frac{3}{2}, -1)$

$$\begin{aligned} V^+(z_1)V^-(z_2) &= z_{12}^{\frac{3}{2k+3}}\left[\frac{(2k+3)(k+1)}{z_{12}^3} - \frac{k+3}{z_{12}}T_V(z_2) + O(z_{12})\right], \\ T_V(z_1)V^{\pm}(z_2) &= \frac{s_q^{\pm}}{z_{12}^2}V^{\pm}(z_2) + \frac{1}{z_{12}}\partial V^{\pm}(z_2) + O(z_{12}), \\ V^{\pm}(z_1)V^{\pm}(z_2) &= z_{12}^{-\frac{3}{2k+3}}V_{(2)}^{\pm}(z_2) + O(z_{12}), \end{aligned} \tag{34}$$

which define the *quantum* $V_3^{(1,1)}$-algebra. The $T_V(1)T_V(2)$ OPE has the standard form (4), of the Virasoro OPE's, with central charge $c_V = -6\frac{(k+1)^2}{k+3}$. The $V_3^{(1,1)}$-algebra (34) has a structure similar to the PF-one (9), and for $L = 2k+3$ positive integers $(L > 3)$, the OPE's (34) involves more currents $V_l^{\pm}$ $(l = 1, 2, ..., L-1)$ of spins $s_l^{\pm} = \frac{3l}{2L}(L-l)$. Introducing the (Laurent) mode expansion for the currents[9] $V^{\pm}$

$$V^{\pm}(z)\phi_s^{\eta}(0) = \sum_{m=-\infty}^{\infty} z^{\pm\frac{3s}{2L}+m-1\mp\eta}V^{\pm}_{-m\pm\eta-\frac{1}{2}+3\frac{1\mp s}{2L}}\phi_s^{\eta}(0),$$

[9] $\phi_s^{\eta}(0)$ denotes certain Ramond ($\eta = \frac{1}{2}$, s-odd) and Neveu-Schwartz ($\eta = 0$, s-even) fields, where $s = 1, 2, ..., L-1$.

we derive, from (34), the following "commutation relations" (of PF-type) for the $V_3^{(1,1)}(L)$-algebra ($|L|>3$)

$$\sum_{p=0}^{\infty} C^p_{(-\frac{3}{L})}\left(V^+_{-3\frac{s+1}{2L}+m-p-\eta+\frac{1}{2}} V^-_{3\frac{s+1}{2L}+n+p+\eta-\frac{1}{2}} + V^-_{-3\frac{1-s}{2L}+n-p+\eta-\frac{1}{2}} V^+_{3\frac{1-s}{2L}+m+p-\eta+\frac{1}{2}}\right)$$

$$= \frac{1}{2}(L+3)\left[-L_{m+n} + \frac{(L-1)L}{2(L+3)}\left(\frac{3s}{2L}+n+\eta\right)\left(\frac{3s}{2L}+n+\eta-1\right)\delta_{m+n,0}\right] \qquad (35)$$

where $C^p_{(M)} = \frac{\Gamma(p-M)}{p!\Gamma(-M)}$, with $m,n = 0, \pm1, \pm2, ...,$ and

$$\sum_{p=0}^{\infty} C^p_{(\frac{3}{L})}\left(V^{\pm}_{3\frac{3\mp s}{2L}-p+m+\eta-\frac{1}{2}} V^{\pm}_{3\frac{1\mp s}{2L}+p+n+\eta-\frac{1}{2}} - V^{\pm}_{3\frac{3\mp s}{2L}-p+n+\eta-\frac{1}{2}} V^{\pm}_{3\frac{1\mp s}{2L}+p+m+\eta-\frac{1}{2}}\right) = 0. \qquad (36)$$

In the particular cases, when $L = 2,3$, the OPE's $V^{\pm}V^{\pm}$ have also a pole, which makes eqn. (36) nonvalid. The simplest example of such $V_3^{(1,1)}$-algebra, for $L=2$, is spanned by $V^{\pm}$ of $s^{\pm} = \frac{3}{4}$ and T_V, only. The relations (36) are now replaced by

$$\sum_{p=0}^{\infty} C^p_{(\frac{1}{2})}(V^-_{-p+m+\eta-\frac{3}{4}} V^-_{p+n+\eta-\frac{5}{4}} + V^-_{-p+n+\eta-\frac{3}{4}} V^-_{p+m+\eta-\frac{5}{4}}) = \delta_{m+n+2\eta,0},$$

and by the similar one, for V^+V^+. As in the PF-case, one can easily verify that certain limits of the OPE's (34) reproduces the classical PB's $V_3^{(1,1)}$-algebra (13).

The relations (33), between $W_3^{(1,1)}$ and $V_3^{(1,1)}$ currents, lead to the following form for the $W_3^{(1,1)}$-vertex operators $\phi^W_{(r_i,s_i)}(z)$ in terms of the $V_3^{(1,1)}$-ones $\phi^V_{(r_i,s_i)}(z)$ and the free field ϕ

$$\phi^W_{(r_i,s_i)} = \phi^V_{(r_i,s_i)} \exp\left[q_{(r_i,s_i)}\sqrt{\frac{3}{L}}\phi\right]. \qquad (37)$$

The construction (37) is a consequence of eqns. (33), of the following OPE's

$$\begin{aligned} T^W(z_1)\phi^W(z_1) &= \frac{\Delta^W_{r,s}}{z_{12}^2}\phi^W_{(r,s)}(z_2) + \frac{1}{z_{12}}\partial\phi^W_{(r,s)}(z_2) + O(z_{12}), \\ J(z_1)\phi^W(z_2) &= \frac{q_{r,s}}{z_{12}}\phi^W_{(r,s)}(z_2) + O(z_{12}), \end{aligned}$$

and of the fact that $\phi^V_{(r,s)}$ are J-neutral, i. e., $J(z_1)\phi^V_{(r,s)}(z_2) = O(z_{12})$. Finally, we realize that the dimensions $\Delta^V_{(r,s)}$ of the $V_3^{(1,1)}$ primary fields $\phi^V_{(r,s)}$ are related to the $\phi^W_{(r,s)}$ dimensions and charges, given by eqns. (30), as follows

$$\Delta^V_{(r,s)} = \Delta^W_{(r,s)} - \frac{3}{2L}q^2_{(r,s)}. \qquad (38)$$

Taking into account the explicit values of $\Delta^W_{(r,s)}$ and $q_{(r,s)}$ (30), for the class of "completely degenerate" *h. w.* representations of $W_3^{(1,1)}$ $(L+3=4\frac{p}{q})$, we derive the dimensions of the *h. w.* representations of $V_3^{(1,1)}$.

The main purpose of our discussion about the quantization of the classical $(PB's)$ nonlocal $V_3^{(1,1)}$-algebra (13) is to point out the *differences* with the quantization of the W_3- and $W_3^{(1,1)}$-algebras, and the *similarities* with the PF-algebra. The origin of all these complications is the renormalization of the spins of the nonlocal currents $V^{\pm}$, $s_q^{\pm}=s_{cl}^{\pm}-\frac{3}{2L}$, which makes the singularities of the $V_3^{(1,1)}-OPE's$ (34) L-dependent. For certain values of L, this requires to introduce new currents $V_l^{\pm}$ and $W_p^{\pm}$ (see ref [33]), in order to close the OPE-algebra. The typical PF-feature is the replacing of the Lie commutators, with an infinite sum of bilinears of generators, as in eqns. (35) and (36). One might wonder whether the $V_{n+1}^{(1,1)}$-algebras (defined in ref [33]), exhibit similar features. Our preliminary results show that the renormalization of the spins of the nonlocal currents $V_{(n+1)}^{\pm}$ is a commun property of all $V_{n+1}^{(1,1)\prime}$s

$$s_n^{\pm}(q)=\frac{n+1}{2}\left(1-\frac{1}{2k+n+1}\right).$$

As usual, the spins of the local currents W_{l+1} remain unchanged. All this indicates that quantum $V_{n+1}^{(1,1)}$-algebras share many properties of $V_3^{(1,1)}$. The construction of the *h. w.* representations of these algebras, as well as the quantization of the V- and WV-algebras of other types, say as in (19) and in (20), is an interesting open problem. The same is valid for the simplest $U_4^{(1,2)}$-algebra, for the UV-algebra $V_4^{(1,2)}$, of ref [33], and for the various explicit examples of U- and UV-algebras, given in ref [37].

It is important to note, in conclusion, that the classification of the *classical extensions* of the Virasoro algebra, described in this paper, *does not solve* the problem of the classification of universality classes in two dimensions. The complete solution of this challenge problem requires the construction of the *h. w.* representations of the corresponding quantum W-, U-, V (and WV-, UV-)-algebras. We consider the above discussed quantization of the $V_3^{(1,1)}$-algebras as a demonstration that *relatively simple tools*, for the realization of this program, *do exist.*

AcknowledgmentsF. E. Mendonça da Silveira thanks FAPESP, for financial support. G. M. Sotkov thanks FAPESP, IFT-UNESP and DCP-CBPF, for financial support and hospitalily. This work has been partially supported by CNPq.

References

[1] A. Belavin, A. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333;

[2] G. E. Andrews, R. J. Baxter and P. J. Forrester, J. Stat. Phys. **35** (1984) 193;

[3] V. G. Kac, Lecture Notes in Physics **94** (1979) 441;

[4] Vl. S. Dotsenko, Nucl. Phys. **B235** (1984) 54;

[5] D. A. Huse, Phys. Rev. **B30** (1984) 3908;

[6] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52** (1984) 1575; Comm. Math. Phys. **107** (1986) 535;

[7] D. Friedan, Z. Qiu and S. Shenker, Phys. Lett. **B151** (1985) 37;

[8] M. A. Bershadsky, V. G. Knizhnik and M. G. Teitelman, Phys. Lett. **B151** (1985) 31;

[9] A. Neveu and J. H. Schwarz, Nucl. Phys. **B31** (1971) 86;

[10] D. Kastor, E. Martinec and Z. Qiu, Phys. Lett. **B200** (1988) 434;

[11] F. Ravanini, Mod. Phys. Lett. **A3** (1988) 397;

[12] A. B. Zamolodchikov and V. A. Fateev, Sov. Phys. JETP **62** (1985) 215;

[13] A. B. Zamolodchikov and V. A. Fateev, Sov. Phys. JETP **63** (1985) 913;

[14] A. B. Zamolodchikov and V. A. Fateev, Nucl. Phys. **B280** (1987) 644;

[15] A. B. Zamolodchikov, Theor. Math. Phys. **65** (1986) 1205;

[16] V. Kac, Funct. Anal. Appl. **1** (1967) 328; R. V. Moody, Bull. Amer. Math. Soc. **73** (1967) 217;

[17] V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. **B247** (1984) 83;

[18] D. Gepner and E. Witten, Nucl. Phys. **B278** (1986) 493;

[19] M. Ademollo et all, Phys. Lett. **B62** (1976) 105; Nucl. Phys. **B111** (1976) 77;

[20] W. Bucher, D. Friedan and A. Kent, Phys. Lett. **B172** (1986) 316;

[21] P. Di Vecchia, J. L. Peterson and M. Yu, Phys. Lett. **B172** (1986) 211;

[22] P. Di Vecchia, V. G. Knizhnik, J. L. Peterson and P. Rossi, Nucl. Phys. **B253** (1985) 701;

[23] V. G. Kac and I. Todorov, Comm. Math. Phys. **102** (1985) 175;

[24] D. Gepner, Nucl. Phys. **B290** (1987) 10;

[25] V. A. Fateev and S. L. Lukyanov, Int. J. Mod. Phys. **A3** (1988) 507;

[26] A. Bilal and J. L. Gervais, Nucl. Phys. **B314** (1989) 646;

[27] F. A. Bais, P. Bouwkneght, K. Schoutens and M. Surridge, Nucl. Phys. **B304** (1988) 371;

[28] M. Bershadsky, Comm. Math. Phys. **139** (1991) 71;

[29] A. Polyakov, Int. J. Mod. Phys. **A5** (1990) 833;

[30] J. Balog, L. Fehér, L. O′Raifeartaigh, P. Forgács and A. Wipf, Ann. of Phys. **203** (1990) 76;

[31] A. Bilal, Nucl. Phys. **B422** (1994) 258;

[32] A. Bilal, Comm. Math. Phys. **170** (1995) 117;

[33] J. F. Gomes, G. M. Sotkov and A. H. Zimerman, Phys. Lett. **435B** (1998) 49;

[34] J. F. Gomes, G. M. Sotkov and A. H. Zimerman, "Parafermionic Reductions of WZW Model", hep-th/9803234;

[35] L. Fehér, L. O′Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. **222 N°1** (1992) 1;

[36] M. Bershadsky and H. Ooguri, Comm. Math. Phys. **126** (1989) 49;

[37] J. F. Gomes, F. E. Mendonça da Silveira, G. M. Sotkov and A. H. Zimerman, "Symmetries of Non-Abelian Toda Models", in preparation;

[38] V. G. Kac, Funct. Anal. Appl. **3** (1969) 252;

[39] A. N. Leznov and M. V. Saveliev, Comm. Math. Phys. **74** (1980) 111; Comm. Math. Phys. **89** (1983) 59.