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GEODESICS IN GÖDEL-TYPE SPACE-TIMES

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ABSTRACT

We investigate the geodesic curves of the homogeneous Gödel-type space-times, which constitute a two-parameter (ℓ and Ω) class of solutions presented to several theories of gravitation (general relativity, Einstein-Cartan and higher derivative). To this end, we first examine the qualitative properties of those curves by means of the introduction of an effective potential and then accomplish the analytical integration of the equations of motion. We show that some of the qualitative features of the free motion in Gödel's universe ($\ell^2 = 2\Omega^2$) are preserved in all space-times, namely: (a) the projections of the geodesics onto the 2-surface (r,φ) are simple closed curves, and (b) the geodesics for which the ratio of azymuthal angular momentum to total energy, γ, is equal to zero always cross the origin $r = 0$. However, two new cases appear: (i) radially unbounded geodesics with γ assuming any (real) value, which may occur only for the causal space-times ($\ell^2 \geq 4\Omega^2$), and (ii) geodesics with γ bounded both below and above, which always occur for the circular family ($\ell^2 < 0$) of space-times.

Key Words: Gödel-type space-times; geodesics; effective potential

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1. INTRODUCTION

Despite the early discovery by Lanczos[1] in 1924 of the first exact model of a universe endowed with rotating matter in the framework of General Relativity Theory (GRT), it was only with the solution, replete with exotic features (absence of cosmic time, violation of the causality condition, violation of Mach's principle), produced by Gödel[2] in 1949 that the cosmologists' attention was brought to these rotating models. The peculiar properties cited above caused the emergence of a series of papers concerned with the study of the geodesics in that model.[3-8] At the same time, there also came to light a plethora of rotating solutions in GRT analogous to Gödel's one.[9-15] These were studied with regard to their geometrical equivalence and corresponding sources in a comprehensive paper by Rebouças and Tiomno.[16] Soon after, Oliveira, Teixeira and Tiomno[17] extended these solutions to Einstein-Cartan-Sciama-Kibble-Hehl Theory (ECT). More recently, Accioly and Gonçalves[18] have provided solutions of this kind in a higher-derivative gravity theory. From now on, these solutions will be referred to as (homogeneous) Gödel-type models.

The Gödel-type space-times admit the following line element[16]

$$ds^2 = [dt + (4\Omega/\ell^2)\sinh^2(\ell r/2)d\varphi]^2 - (1/\ell^2)\sinh^2(\ell r)d\varphi^2 - dr^2 - dz^2, \tag{1.1}$$

where Ω and ℓ are constants, with $-\infty < \ell^2 < +\infty$. Without any loss of generality, we will assume that Ω is non-negative.

As suggested by (1.1), we can divide our two-parameter class of space-times into three families: (i) *the hyperbolic family* ($\ell^2 > 0$), which includes Gödel's space-time[2] ($\ell^2 = 2\Omega^2$) as a special case; (ii) *the Som-Raychaudhury space-time*[10] ($\ell^2 \to 0$), and (iii) *the circular family* ($\ell^2 < 0$). We note that, for $\ell^2 < 0$, the hyperbolic functions in (1.1) transform into circular functions.

A possible natural choice for the range of the coordinates covering all manifolds in question is $-\infty < t,z < +\infty$, $0 \leq r < +\infty$, $0 \leq \varphi < 2\pi$, for $\ell^2 \geq 0$; and $-\infty < t,z < +\infty$, $0 \leq |\ell| r \leq \pi$, $0 \leq \varphi < 2\pi$, for $\ell^2 < 0$. In any case, t will be called the temporal coordinate, φ the azymuthal one and z the axial one; as concerns the coordinate r, it is naturally interpreted as a linear magnitude for $\ell^2 \geq 0$ and will be called acordingly the radial coordinate, whereas for $\ell^2 < 0$, it is naturally interpreted as an angular magnitude and so will be called the zenithal coordinate. Thus, in this last case ($\ell^2 < 0$), the coordinates r, φ are defined on a (topological) 2-sphere, with $|\ell| r = 0, \pi$ corresponding to the north and south poles, respectively.[19]

For all these space-times, we will take the t-coordinate lines as the natural congruence of observers associated to the coordinate system (t, r, φ, z), since they are the only ones which are everywhere time-like. This congruence defines the unit time-like vector field (frame of reference)

$$u^\mu := \delta^\mu_0. \tag{1.2}$$

The motion of the particles (observers) comoving with this frame is determined by its kinematic parameters[20]

$$a^\mu = \theta = \sigma^{\mu\nu} = 0, \tag{1.3}$$

$$\omega^\mu = \Omega \delta^\mu_3. \tag{1.4}$$

This means that our reference particles (observers) are all in free fall and rotate rigidly and uniformly with respect to one another.

Tiomno *et al.*[16,17] have studied possible sources for these models both in GRT and ECT. A resumé of their results is now given. In GRT, if we restrict the matter content to a perfect fluid, an electromagnetic field and a long range (massless) scalar field, we can only yield line elements with $-\infty < \ell^2 \leq 4\Omega^2$ as solutions of the appropriate coupled field equations (Einstein-Maxwell-Klein-Gordon equations), whereas in ECT it is possible to generate all line elements $(-\infty < \ell^2 < +\infty)$ taking only a Weyssenhoff perfect fluid[21] for matter content. In any case, whenever a perfect (massive) fluid was present, its four-velocity v^μ coincided with u^μ as defined by (1.2).

The present paper is concerned with the investigation of the geodesic motion of free test particles in these Gödel-type space-times. For that purpose, we will take advantage of the symmetries of the models and reduce the motion in the coordinate r to a one-dimensional problem, thereby allowing an easy and clarifying qualitative analysis by means of the introduction of an effective potential. In this regard, this work may be viewed as an extension of a previous article by Novello, Soares and Tiomno,[8] in which an analogous study was carried out for Gödel's space-time. This method has already been applied to the Schwarzschild and Kerr gravitational fields in order to explore the qualitative features of their geodesics.[22,23] Recently, Paiva, Rebouças and Teixeira[24] have examined, *inter alia*, the geodesics of the Som-Raychaudhuri space-time. However, they have not applied the effective potential method to perform a previous analysis of the main qualitative features of the geodesic motion.

The structure of the paper is as follows. In section 2, we set up the system of equations which completely determine the geodesics in our universes. In section 3, we make a qualitative analysis of the motion in the coordinate r via the effective potential graphs, while, in section 4, we integrate exactly the system of equations established in section 2. In section 5, we sketch the graphs of the trajectories on the 2-plane (r,φ) and investigate some of their features. We conclude with section 6, where the most important new results are compiled and compared with those holding in Gödel's space-time.

2. THE EQUATIONS OF GEODESIC MOTION

The equations of geodesics for the line element (1.1) have the four straightforward first integrals

$$p_t = \dot{t} + (4\Omega/\ell^2)\sinh^2(\ell r/2)\dot{\varphi}, \tag{2.1}$$

$$p_\varphi = (4\Omega/\ell^2)\sinh^2(\ell r/2)\dot{t} + (4/\ell^2)\sinh^2(\ell r/2) \times \times [(4\Omega^2/\ell^2 - 1)\sinh^2(\ell r/2) - 1]\dot{\varphi}, \tag{2.2}$$

$$p_z = -\dot{z}, \tag{2.3}$$

$$\varepsilon = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{2.4}$$

where a superposed dot stands for a derivative with respect to the affine parameter τ associated to the geodesic,[25] and $\varepsilon = 1$ or 0, for timelike or null geodesics, respectively. The first three integrals (2.1)-(2.3) are due to the existence of the Killing vector fields $\partial/\partial t$, $\partial/\partial\varphi$ and $\partial/\partial z$, respectively; the fourth one, (2.4), is related to the invariance of the time-like or null character of a given geodesic. We recognize the constants of motion p_t, p_φ, p_z and ε as the energy, azymuthal angular momentum, axial linear momentum and rest energy per unit mass for massive particles (timelike curves, $\varepsilon = 1$), and as the energy, azymuthal angular momentum, axial linear momentum and rest energy for massless particles (null curves, $\varepsilon = 0$), traveling along the given geodesics.[22,23]

We can express all $\dot{x}^\mu$ as functions of r only

$$\dot{t} = p_t\left[1 + \frac{\Omega\gamma - (4\Omega^2/\ell^2)\sinh^2(\ell r/2)}{\cosh^2(\ell r/2)} \right], \tag{2.5}$$

$$\dot{\varphi} = p_t\left[\frac{\Omega}{\cosh^2(\ell r/2)} - \frac{\ell^2\gamma}{4\sinh^2(\ell r/2)\cosh^2(\ell r/2)} \right], \tag{2.6}$$

$$\dot{z} = - p_z, \tag{2.7}$$

$$\dot{r}^2 = p_t^2\left\{1 - \beta^2 - \left[\frac{(2\Omega/\ell)\sinh(\ell r/2)}{\cosh(\ell r/2)} - \frac{\ell\gamma}{2\sinh(\ell r/2)\cosh(\ell r/2)}\right]^2\right\}, \tag{2.8}$$

where the new parameters are defined by

$$\gamma := p_\varphi/p_t, \tag{2.9}$$

$$\beta^2 = (p_z^2 + \varepsilon)/p_t^2. \tag{2.10}$$

From (2.8), it is easy to show that, for the trajectories of physical particles, $0 \le \beta^2 \le 1$. This very equation can be rewritten as

$$\dot{r}^2 = p_t^2 - V(r), \tag{2.11}$$

where we have defined the *effective potential*

$$V(r) := p_t^2\left[\frac{(2\Omega/\ell)\sinh(\ell r/2)}{\cosh(\ell r/2)} - \frac{\ell\gamma}{2\sinh(\ell r/2)\cosh(\ell r/2)}\right]^2 + \beta^2 p_t^2. \quad (2.12)$$

Equation (2.11) is in the form of the equation of a one-dimensional problem for a particle in a potential field V(r). We remark that all equations in this section are well defined for $\ell^2 \leq 0$. The expressions for the case $\ell^2 = 0$ are explicitly given in the Appendix, where the geodesic motion in the Som-Raychaudhuri space-time is separately examined (cf. Ref. 24).

Using equation (2.11), we can now accomplish a complete characterization of the motion without explicitly integrating equations (2.5)-(2.8), a task that we shall postpone to section 4. As we shall see, this characterization depends essentially on the parameters β, γ, $2\Omega/|\ell|$.

3. GENERAL PROPERTIES OF THE TRAJECTORIES

Since the behavior of the effective potential depends decisively on the family of space-times considered, it is expedient to adhere to the following conventions: we will affix to the number of our formulae a letter H, S, or C, according as they apply to the hyperbolic, Som-Raychaudhuri, or circular family, respectively; when the equations hold for the three families, as all previous ones, we will simply omit any letter.

In conformity with this, the behavior of the potential at the extremes of the domain is given by

$$\lim_{r\to 0} V(r) = \begin{cases} +\infty, & \text{if } \gamma \neq 0; & (3.1) \\ \beta^2 p_t^2, & \text{if } \gamma = 0; & (3.2) \end{cases}$$

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For the other families

$$r_{min} = \sqrt{\gamma/\Omega}, \quad V_{min} = \beta^2 p_t^2, \qquad \text{if} \quad \gamma \geq 0; \qquad (3.12.S)$$

$$r_{min} = \sqrt{-\gamma/\Omega}, \quad V_{min} = (\beta^2 - 4\Omega\gamma)p_t^2, \qquad \text{if} \quad \gamma < 0; \qquad (3.13.S)$$

$$\sin^2 R_{min} = |\ell|^2\gamma/4\Omega, \quad V_{min} = \beta^2 p_t^2, \qquad \text{if} \quad 0 \leq \gamma \leq 4\Omega/|\ell|^2; \qquad (3.14.C)$$

$$\sin^2 R_{min} = -\gamma(4\Omega/|\ell|^2 - 2\gamma), \quad V_{min} = p_t^2(\beta^2 - 4\Omega\gamma + |\ell|^2\gamma^2),$$
$$\text{if} \quad \gamma \leq 0 \quad \text{or} \quad \gamma \geq 4\Omega/|\ell|^2. \qquad (3.15.C)$$

The physically admissible motion proceeds along those points for which $V(r) \leq p_t^2$ [cf. (2.11)] and, particularly, the existence of physical motion requires that the absolute minimum must satisfy

$$V_{min} \leq p_t^2. \qquad (3.16)$$

For the hyperbolic family this entails

$$(-2\Omega/|\ell| + \sqrt{\alpha})/|\ell| =: \overline{\gamma}_< \leq \gamma < +\infty, \qquad \text{if} \quad \alpha > 0; \qquad (3.17.H)$$

$$-2\Omega/|\ell|^2 < \gamma < +\infty, \qquad \text{if} \quad \alpha = 0; \qquad (3.18.H)$$

$$-\infty < \gamma < +\infty, \qquad \text{if} \quad \alpha < 0; \qquad (3.19.H)$$

where

$$\alpha := \lambda^2 + \beta^2 - 1$$
$$= 4\Omega^2/|\ell|^2 + \beta^2 - 1. \qquad (3.20)$$

For the Som-Raychaudhuri space-time we have, from (3.15),

$$-(1-\beta^2)/4\Omega \le \gamma < +\infty, \tag{3.21.S}$$

whereas for the circular family it implies

$$(2\Omega/|\ell| - \sqrt{\sigma})/|\ell| =: \tilde{\gamma}_< \le \gamma \le \tilde{\gamma}_> := (2\Omega/|\ell| + \sqrt{\sigma})/|\ell|, \tag{3.22.C}$$

where

$$\sigma := \lambda^2 - \beta^2 + 1 > 0. \tag{3.23}$$

The turning points in the coordinate r are those at which

$$\dot{r}^2 = p_t^2 - V(r) = 0. \tag{3.24}$$

Again, for the hyperbolic family, we will have three possibilities, according to the sign of α:

$$\sinh^2 R_{>,<} = \frac{1-\beta^2+2\Omega\gamma \pm \sqrt{1-\beta^2}\,\sqrt{\Delta}}{2\alpha}, \quad \text{if} \quad \alpha > 0, \tag{3.25.H}$$

with $>,<$ corresponding to $+,-$, respectively;

$$\sinh^2 R_< = \frac{|\ell|^2\gamma^2}{4(\lambda^2+2\Omega\gamma)}, \quad \text{if } \alpha = 0; \tag{3.26.H}$$

$$\sinh^2 R_< = \frac{1-\beta^2+2\Omega\gamma - \sqrt{1-\beta^2}\,\sqrt{\Delta}}{2\alpha}, \quad \text{if} \quad \alpha < 0; \tag{3.27.H}$$

for the Som-Raychaudhuri space-time we get

$$r^2_{\rangle,\langle} = \frac{1 - \beta^2 + 2\Omega\gamma \pm \sqrt{1 - \beta^2}\,\sqrt{1 - \beta^2 + 4\Omega\gamma}}{2\Omega^2}, \qquad (3.28.S)$$

whereas for the circular family it holds

$$\sin^2 R_{\rangle,\langle} = \frac{1 - \beta^2 + 2\Omega\gamma \pm \sqrt{1 - \beta^2}\,\sqrt{\Delta}}{2\sigma}. \qquad (3.29.C)$$

Here we have introduced the parameter

$$\begin{aligned} \Delta &:= \ell^2(\gamma + 2\Omega/\ell^2)^2 - \eta \\ &= \frac{(1 - \beta^2 + 2\Omega\gamma)^2 - \ell^2\gamma^2\eta}{1 - \beta^2} \geq 0, \end{aligned} \qquad (3.30)$$

with

$$\eta := 4\Omega^2/\ell^2 + \beta^2 - 1. \qquad (3.31)$$

Notice that η and α are different [cf. (3.20)], coinciding only for the hyperbolic family.

The graphs of $V(r)$ are sketched in Figs. 1, 2, 3, where the continuous horizontal lines represent typical values the energy p_t^2 may assume. According to the sign of the parameter α defined in (3.20), the energy p_t^2 may be greater than the asymptotic value $(\lambda^2 + \beta^2)p_t^2$ of the potential V, for the *subclass* $\ell^2 > 4\Omega^2$ *of the hyperbolic family*. Thus, in the case $\alpha \leq 0$, the trajectories have only one turning point. We recall that, for all cases, the parameter p_t^2

determines the admissible domain of r for the physical motion [cf. (2.11)].

From these data we can infer particularly that our geodesic motion is always bounded in the coordinate r, except for those free particles in the hyperbolic family of space-times which have $\alpha \leq 0$. These radially unbounded orbits only exist for the globally causal Gödel-type space-times:[26] $\ell^2 \geq 4\Omega^2$.

4. ANALYTICAL INTEGRATION

To accomplish the exact determination of the geodesic motion we have to integrate (2.5)-(2.8).

A. The coordinate z

From (2.7) we have at once

$$z = - p_z \tau + z_o. \tag{4.1}$$

Along any geodesic the axial coordinate varies uniformly with respect to its affine parameter τ.

B. The coordinate r

It is convenient to introduce the new variable[27]

$$\xi := \sinh^2 (\ell r/2). \tag{4.2}$$

Then equation (2.8) becomes

$$\dot{\xi}^2 = \ell^2 p_t^2 \left[- \eta \xi^2 + (1 - \beta^2 + 2\Omega\gamma)\xi - \ell^2 \gamma^2/4 \right], \tag{4.3}$$

with η defined by (3.30).

The general solution of (4.3) is

$$\xi = \frac{1 - \beta^2 + 2\Omega\gamma - \sqrt{1-\beta^2}\,\sqrt{\Delta}\,\cos \ell p_t \sqrt{\eta}\,(\tau - \tau_0)}{2\eta}, \tag{4.4}$$

with Δ defined by (3.29). Here we have chosen the integration constant τ_0 so that $\xi(\tau_0)$ corresponds to the point nearest to the origin ("pericenter") and thus also $\dot{\xi}(\tau_0) > 0$. From this result, we may recover the expressions (3.25)-(3.29) for the turning points and (3.17)-(3.22) for the admissible ranges of the parameter γ. Also, we confirm the information obtained from Figs. 1, 2, 3 that, for the globally causal space-times ($\ell^2 \geq 4\Omega^2$), we may have, with a suitable choice of kinematic initial conditions (p_t, β, γ), either radially unbounded or bounded trajectories, whereas for the causality-violating space-times ($\ell^2 < 4\Omega^2$) the latter are necessarily bounded in the coordinate r.

C. The coordinate φ

Equation (2.6) may be recast, by means of (4.2), as

$$\dot{\varphi} = -\,p_t\left[\frac{\ell^2\gamma/4}{\xi} - \frac{\ell^2\gamma/4 + \Omega}{\xi + 1}\right]. \tag{4.5}$$

Now we will find φ as a function of ξ, that is, we will find the equation of the projection of the geodesics onto the 2-surface t,z = const. To that end, we need the expression of $\dot{\xi}$ as a function of ξ; from the two roots furnished by (4.3), we will choose

$$\dot{\xi} = +\,\ell p_t\sqrt{-\,\eta\xi^2 + (1 - \beta^2 + 2\Omega\gamma)\xi - \ell^2\gamma^2/4}\,. \tag{4.6}$$

Dividing (4.5) by (4.6) and solving the corresponding differential equation we produce[8]

$$\cos(\varphi - \varphi_o) = \frac{\ell(\gamma + 2\Omega/\ell^2)\xi + \ell\gamma/2}{\sqrt{\Delta}\,\sqrt{\xi(\xi + 1)}}. \tag{4.7}$$

Notice that the factor $\ell/\sqrt{\xi}$ is always real, and that this last formula does not hold when ξ is a constant. From this very formula and also (4.4) we see that, for the case $\alpha > 0$, since ξ is then a periodic function, the projections of the trajectories onto the surface (r,φ) are closed simple curves; this is not the case for $\alpha \leq 0$ (cf. subsection 4.B). We call attention to the fact that these projections always have the axis of symmetry $\varphi = \varphi_o$.

D. The coordinate t

We may rewrite (2.5), using (4.2) as

$$\dot{t} = p_t \frac{(1 - 4\Omega^2/\ell^2)\xi + \Omega\gamma + 1}{\xi + 1}. \tag{4.8}$$

After the substitution of (4.4) into (4.8), we obtain a differential equation for $t(\tau)$ whose general solution is

$$\tan\left\{ (\ell^2/4\Omega)\left[t - t_o + p_t(4\Omega^2/\ell^2 - 1)(\tau - \tau_o)\right] \right\} =$$
$$= \frac{\ell\sqrt{\eta}\,(\gamma + 4\Omega/\ell^2)}{\eta + 4\Omega^2/\ell^2 + 2\Omega\gamma - \sqrt{1 - \beta^2}\,\sqrt{\Delta}} \tan\left[\ell p_t\sqrt{\eta}(\tau - \tau_o)/2 \right]. \tag{4.9}$$

An alternative expression to calculate $t(\tau)$ is obtained from the obvious integral of (2.4)

$$p_t t(\tau) - F(r(\tau)) + p_\varphi\varphi(\tau) + p_z z(\tau) - \varepsilon(\tau - \tau_o) = 0, \tag{4.10}$$

where

$$F(r) = \int dr \left[p_t^2 - V(r)\right]^{1/2}. \tag{4.11}$$

5. PROJECTIONS ONTO THE 2-SURFACE (r,φ)

With the aid of the preceding results we are now in a position to sketch the projections of the geodesics onto the 2-surface (r,φ). From (4.7) it follows that[28]

$$\cos^2 (\varphi - \varphi_o) = 1, \tag{5.1}$$

for $R = R_{>,<}$. Furhermore, equation (4.5) defines the value

$$\xi_\varphi = \frac{\ell^2 \gamma}{4\Omega}, \tag{5.2}$$

where $\dot{\varphi} = 0$; therefore, we see that, for $\gamma > 0$, $\dot{\varphi}$ changes sign along the projection, whereas, for $\gamma < 0$, it remains always positive. In the last instance ($\gamma < 0$), this implies, in particular, that the trajectories with closed projection onto the surface (r,φ) *do* enclose the origin. More specifically, for the hyperbolic family

a) if $\alpha > 0$ and

. $\gamma > 0$:	$\varphi = \varphi_o$,	for $R = R_{>,<}$;	(5.2.H)
. $\gamma = 0$:	$\varphi = \varphi_o$,	for $R = R_{>}$;	(5.3.H)
. $\gamma < 0$:	$\varphi = \varphi_o$,	for $R = R_{>}$,	(5.4.H)
	$\varphi = \varphi_o + \pi$,	for $R = R_{<}$.	(5.5.H)

b) if $\alpha \leq 0$ and

$$. \ \gamma > 0: \quad \varphi = \varphi_0, \quad \text{for} \quad R = R_<; \tag{5.6.H}$$

$$. \ \gamma < 0: \quad \varphi = \varphi_0 + \pi, \quad \text{for} \quad R = R_<, \tag{5.7.H}$$

with $R_>$, $R_<$ given by (3.24)-(3.26).

For the Som-Raychaudhuri space-time

$$. \ \gamma > 0: \quad \varphi = \varphi_0, \quad \text{for} \quad r = r_{>,<}; \tag{5.8.S}$$

$$. \ \gamma = 0: \quad \varphi = \varphi_0, \quad \text{for} \quad r = r_>; \tag{5.9.S}$$

$$. \ \gamma < 0: \quad \varphi = \varphi_0, \quad \text{for} \quad r = r_>, \tag{5.10.S}$$

$$\varphi = \varphi_0 + \pi \quad \text{for} \quad r = r_<, \tag{5.11.S}$$

with $r_>$, $r_<$ given by (3.27).

For the circular family

$$. \ \gamma > 0: \quad \varphi = \varphi_0, \quad \text{for} \quad R = R_{>,<}; \tag{5.12.C}$$

$$. \ \gamma = 0: \quad \varphi = \varphi_0, \quad \text{for} \quad R = R_>; \tag{5.13.C}$$

$$. \ \gamma < 0: \quad \varphi = \varphi_0, \quad \text{for} \quad R = R_>, \tag{5.14.C}$$

$$\varphi = \varphi_0 + \pi \quad \text{for} \quad R = R_<, \tag{5.15.C}$$

with $R_>$, $R_<$ given by (3.28). As remarked before, the coordinate R is defined on a sphere, with $R = 0, \pi/2$ corresponding to the north and south poles, respectively.

For the open trajectories $(\alpha < 0)$, we may determine, by means of (4.7), the asymptotic values of φ as ξ approaches infinity:

$$\lim_{\xi \to +\infty} \cos(\varphi - \varphi_0) = \frac{\ell(\gamma + 2\Omega/\ell^2)}{\sqrt{\ell^2(\gamma + 2\Omega/\ell^2)^2 - \alpha}}. \tag{5.15.H}$$

From this we find that (i) for $\alpha = 0$, $\lim_{\xi \to +\infty} \varphi = \varphi_o$, (ii) for $\alpha < 0$, the asymptotes start from $\varphi = \varphi_o$ when $\gamma \to +\infty$, gradually open to $\varphi = \varphi_o \pm \pi/2$ when $\gamma = -2\Omega/\ell^2$, and then gradually close to $\varphi = \varphi_o + \pi$ when $\gamma \to -\infty$.

Again from equation (4.5), we have that, for $\gamma \geq 0$, $\dot{\varphi} > 0$ if $r > r_\varphi$. These results are readily summarized in Figs. 4, 5, 6.

6. CONCLUSIONS

We extended the investigation of the geodesics of Gödel's space-time to the family of homogeneous Gödel-type space-times, which possess some unusual properties such as acausality and violation of Mach's principle. These geometries have been studied extensively and presented as solutions to several theories of gravitation: general relativity theory, Einstein-Cartan theory and the higher derivative theory. We first examined the qualitative behavior of those curves by the introduction of an effective potential which governed the motion in the coordinate r. Thereby we were able to determine whether the motion was bounded in that coordinate, the turning points, the admissible ranges for the ratio γ of the azymuthal angular momentum to the energy, the existence of r = const solutions. Only after this previous qualitative investigation did we integrate the equations of geodesics. The new results, contrasted to those prevailing in Gödel's space-time, are [cf. Ref. (8)]:

i) for the circular family of space-times ($\ell^2 < 0$), the allowable range for the parameter γ is bounded both above and below: $\tilde{\gamma}_< \leq \gamma \leq \tilde{\gamma}_>$ [cf. (3.22)].

ii) for the space-times of the hyperbolic family with $0 \leq \ell^2 < 4\Omega^2$, the allowable range for γ is bounded only below [cf. (3.17), (3.18)] and all projections onto the surface (r,φ) are closed simple curves (cf. Fig. 4). This is the typical behavior of the geodesics in Gödel's universe.

iii) the space-times of the hyperbolic family with $\ell^2 \geq 4\Omega^2$ admit closed or open projections, according as $\beta^2 > 1 - 4\Omega^2/\ell^2$ $(\alpha > 0)$ or $\beta^2 \leq 1 - 4\Omega^2/\ell^2$ $(\alpha \leq 0)$, respectively (cf. Figs. 5, 6). Whereas for Gödel's universe all projections with $\gamma < 0$ enclose the origin, now for $\beta^2 < 1 - 4\Omega^2/\ell^2$ we may have projections $(\gamma < -2\Omega/\ell^2)$ which do not enclose the origin (cf. Fig. 6).

It is worth mentioning a quantum-mechanical analogue of the qualitative behavior of the classical solutions of the equations of geodesic motion presented here. Indeed, it can be shown that to the classical solutions with bounded (unbounded) projections there correspond positive energy Klein-Gordon solutions with discrete (continuous) spectra of energy, as might be expected. This will be the subject of a forthcoming paper.[29] We also remark that, in spite of the geometries with $\ell^2 > 4\Omega^2$ being solutions to Einstein-Cartan theory, the above results are still legitimate for this class of universes, since spinless test particles will in fact follow the (metric) geodesics investigated here.[30]

ACKNOWLEDGEMENT

One of us (M. O. C.) would like to acknowledge the financial support received from Conselho Nacional de Pesquisas Científicas (CNPq).

APPENDIX

For convenience we present here the relevant explicit formulae for the Som-Raychaudhuri space-time, which should be compared with those presented in Ref. 24.

The equations corresponding to (2.5)-(2.8) are

$$\dot{t} = p_t[(1 - \Omega^2 r^2) + \Omega\gamma], \tag{A.1.S}$$

$$\dot{\varphi} = p_t(\Omega - \gamma/r^2), \tag{A.2.S}$$

$$\dot{z} = - p_z, \tag{A.3.S}$$

$$\dot{r}^2 = p_t^2 - V(r), \tag{A.4.S}$$

with

$$V(r) := p_t^2(\Omega r - \gamma/r)^2 + \beta^2 p_t^2. \tag{A.5.S}$$

The coordinate r varies along the geodesic according to

$$r^2 = \frac{1 - \beta^2 + 2\Omega\gamma - \sqrt{1 - \beta^2}\,\sqrt{1 - \beta^2 + 4\Omega\gamma}\,\cos 2\Omega p_t(\tau - \tau_0)}{2\Omega^2}, \tag{A.6.S}$$

The equation for the projection onto the 2-plane (r,φ) is

$$\cos (\varphi - \varphi_0) = \frac{\Omega r^2 + \gamma}{\sqrt{1 - \beta^2 + 4\Omega\gamma}\; r}. \tag{A.7.S}$$

Finally, the equation which governs the behavior of the coordinate t is given by

$$t - t_o = \frac{\sqrt{1 - \beta^2}\ \sqrt{1 - \beta^2 + 4\Omega\gamma}}{4\Omega}\ \text{sen}\ [2\Omega p_t(\tau - \tau_o)] + \\ + (1/2)p_t(1 + \beta^2)(\tau - \tau_o). \qquad \text{(A.8.S)}$$

FIGURE CAPTIONS

FIG. 1. Graphs of the effective potential for the hyperbolic family ($\ell^2 > 0$). The continuous horizontal lines correspond to typical values the energy p_t^2 may ssume, according to the sign of α.

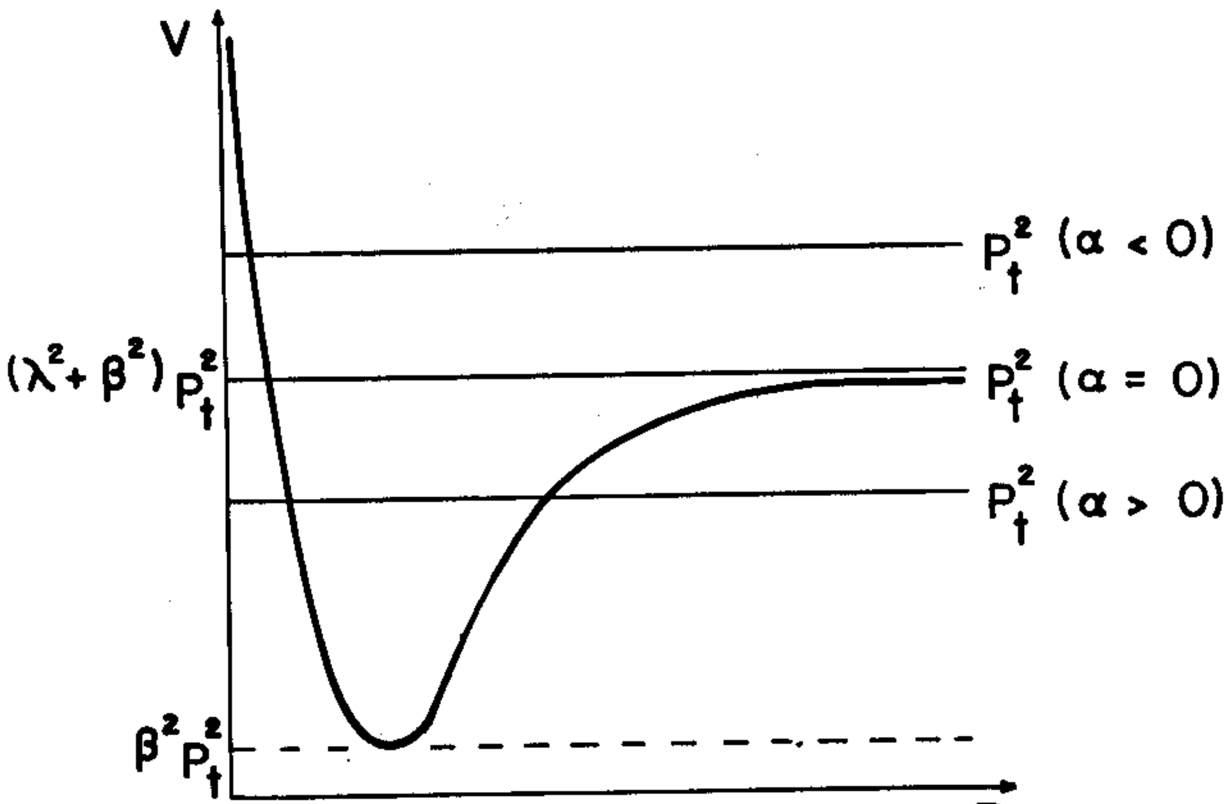
FIG. 2. Graphs of the effective potential for the Som-Raychaudhuri space-time ($\ell^2 \to 0$).

FIG. 3. Graphs of the effective potential for the circular family ($\ell^2 < 0$).

FIG. 4. Projections of the geodesics onto the 2-plane (r,φ) for the hyperbolic family ($\ell^2 > 0$). Typical curves associated to the several values the angular momentum parameter γ may assume are represented.

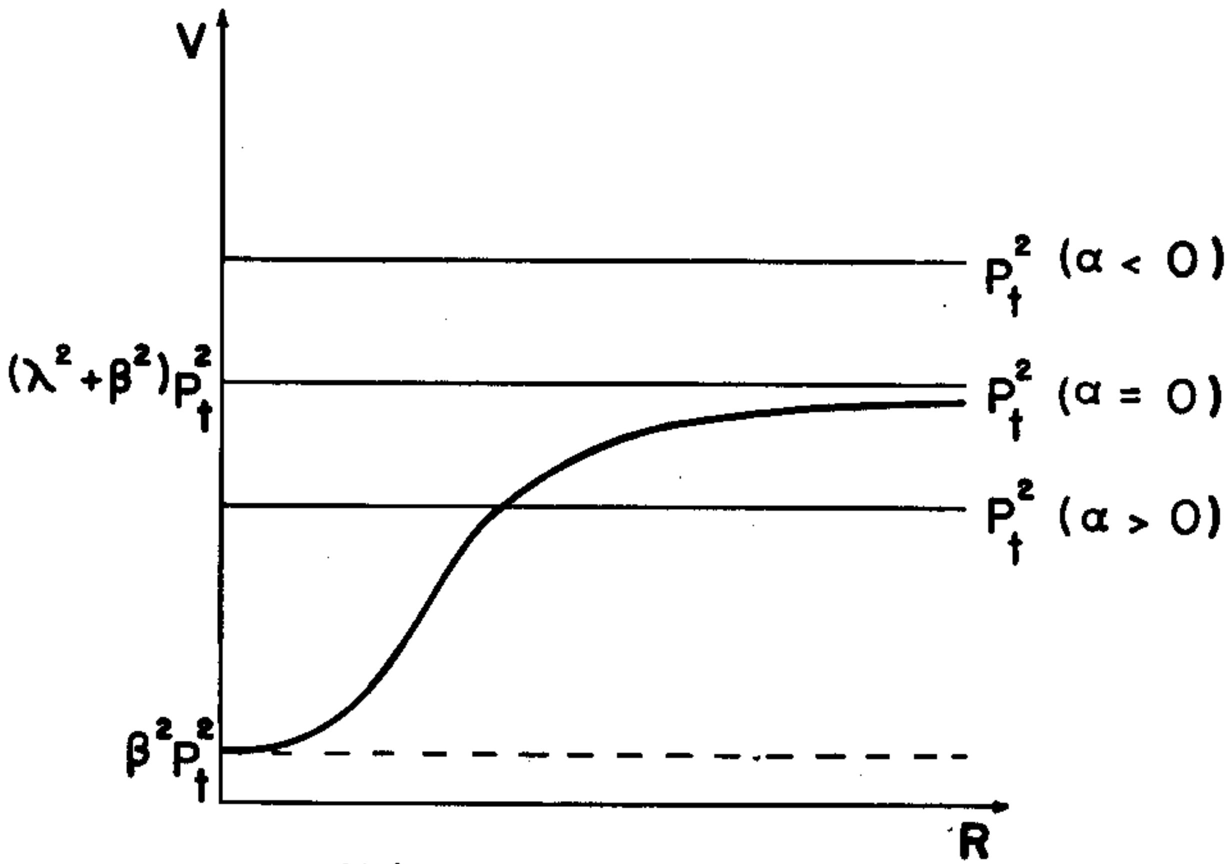
FIG. 5. Projections of the geodesics onto the 2-plane (r,φ) for the Som-Raychaudhuri space-time ($\ell^2 \to 0$).

FIG. 6. Projections of the geodesics onto the 2-plane (r,φ) for the circular family $(\ell^2 < 0)$. This planar representation is not suitable for curves corresponding to the effective potential of Fig. 3(b), which attain the south pole, $R = \pi/2$. In a planar representation with the point $R = \pi/2$ taken as the origin, these curves are similar to those corresponding to the case $\gamma = 0$ above.



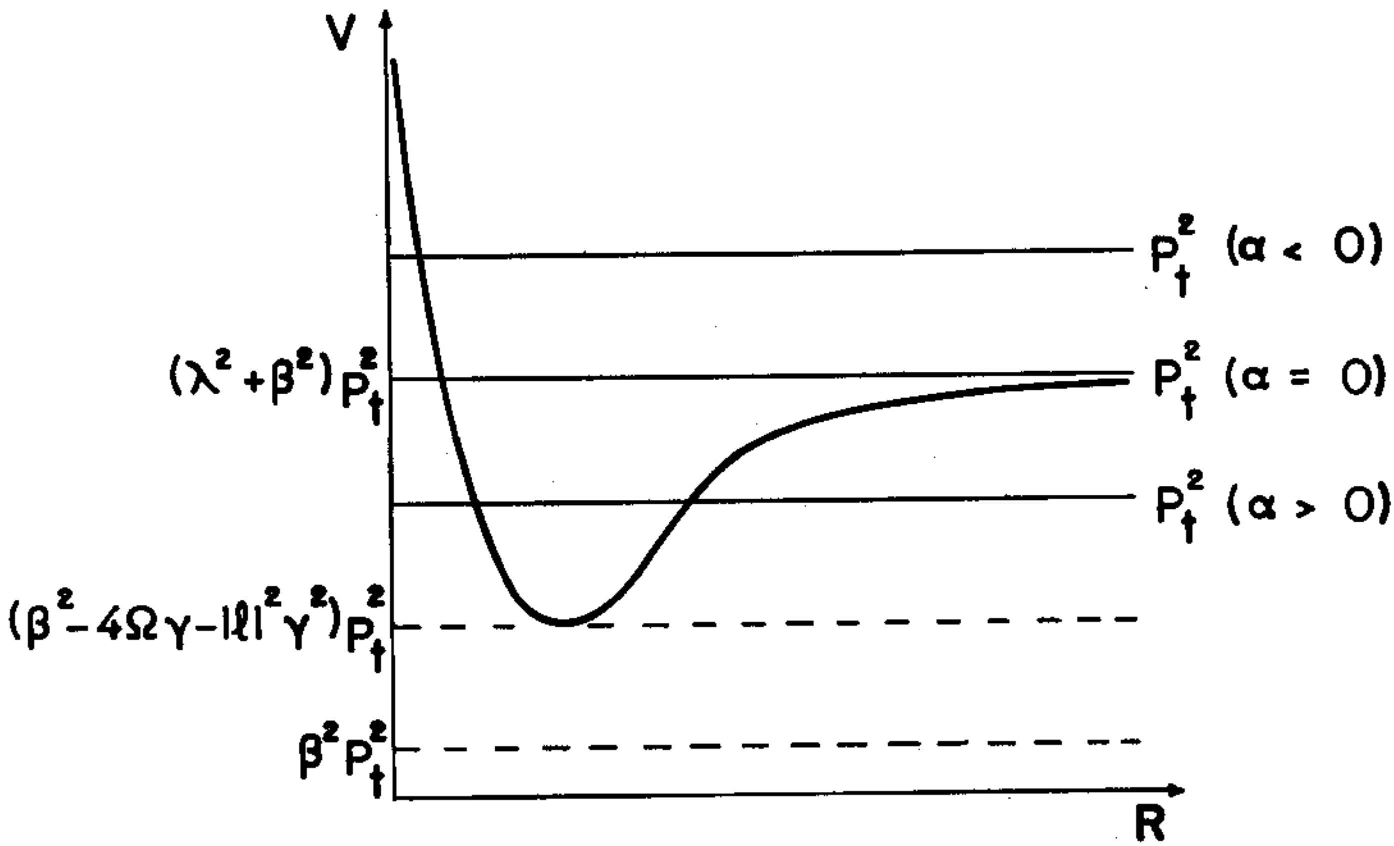
(a) $0 < \gamma < +\infty$

Fig. 1



(b) $\gamma = 0$

Fig. 1



(c) $-2\Omega/|\ell|^2 < \bar{\gamma} \leq \gamma < 0$

Fig. 1

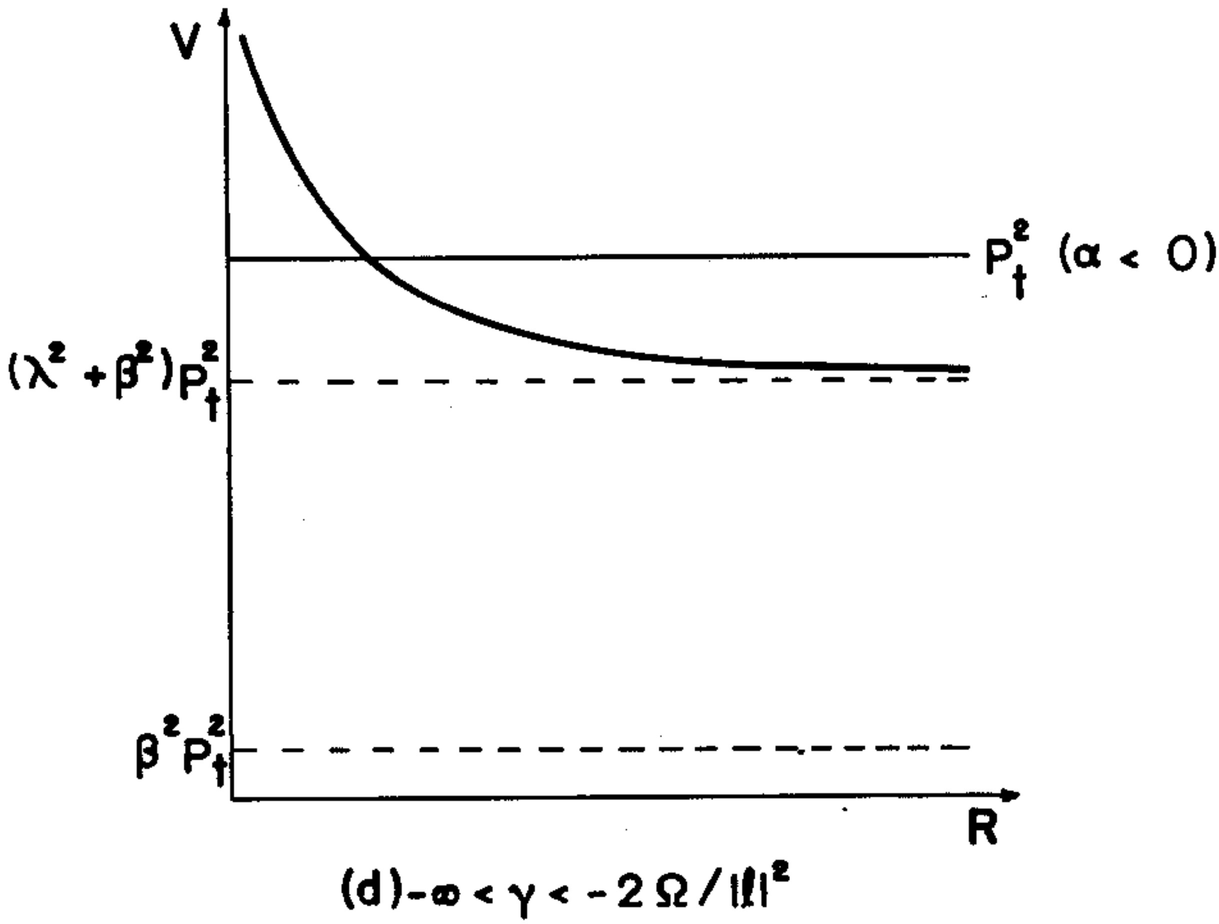
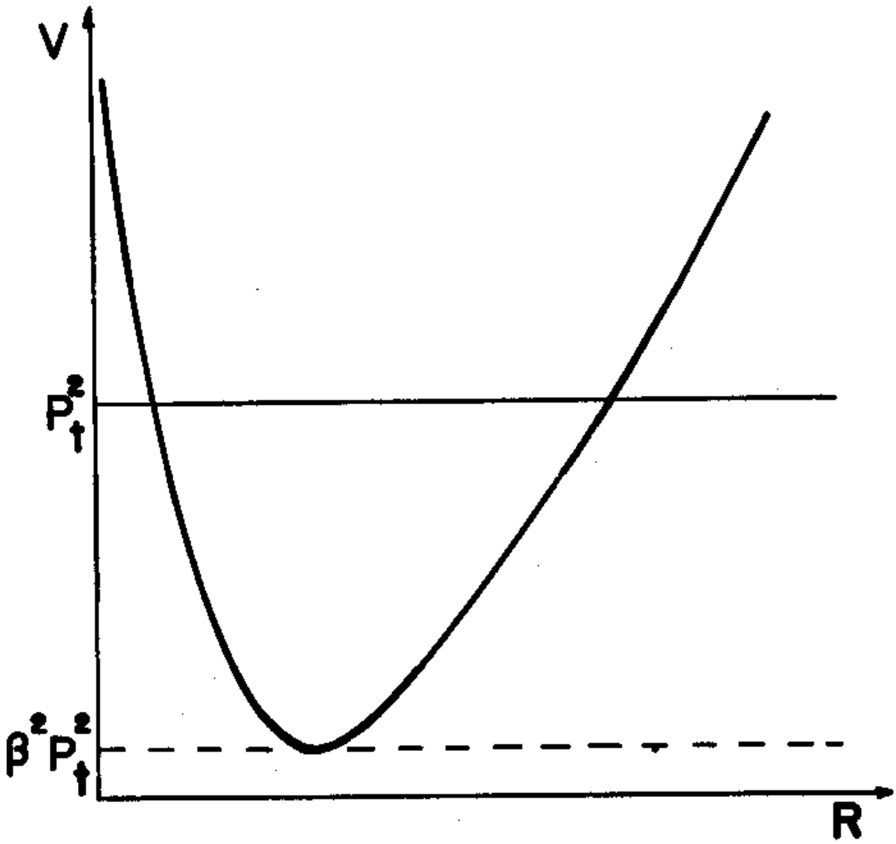
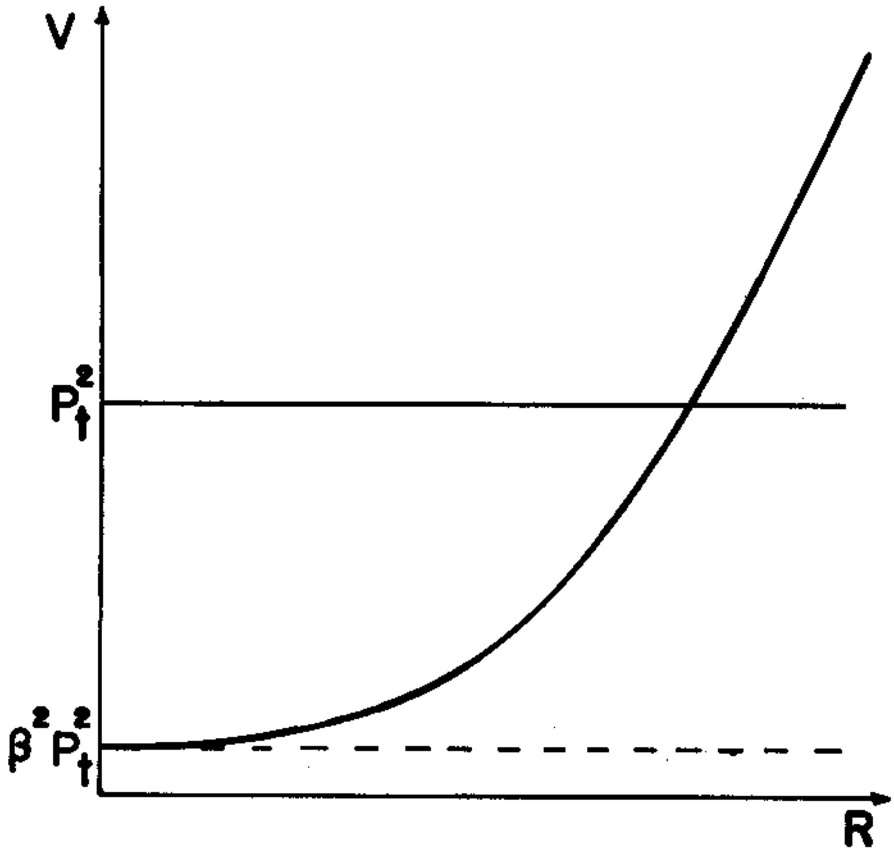

V
P_t^2 $(\alpha < 0)$
$(\lambda^2 + \beta^2)P_t^2$
$\beta^2 P_t^2$
R
(d) $-\infty < \gamma < -2\Omega/\|\ell\|^2$

Fig. 1



(a) $0 < \gamma < +\infty$

Fig. 2



(b) $\gamma = 0$

Fig. 2

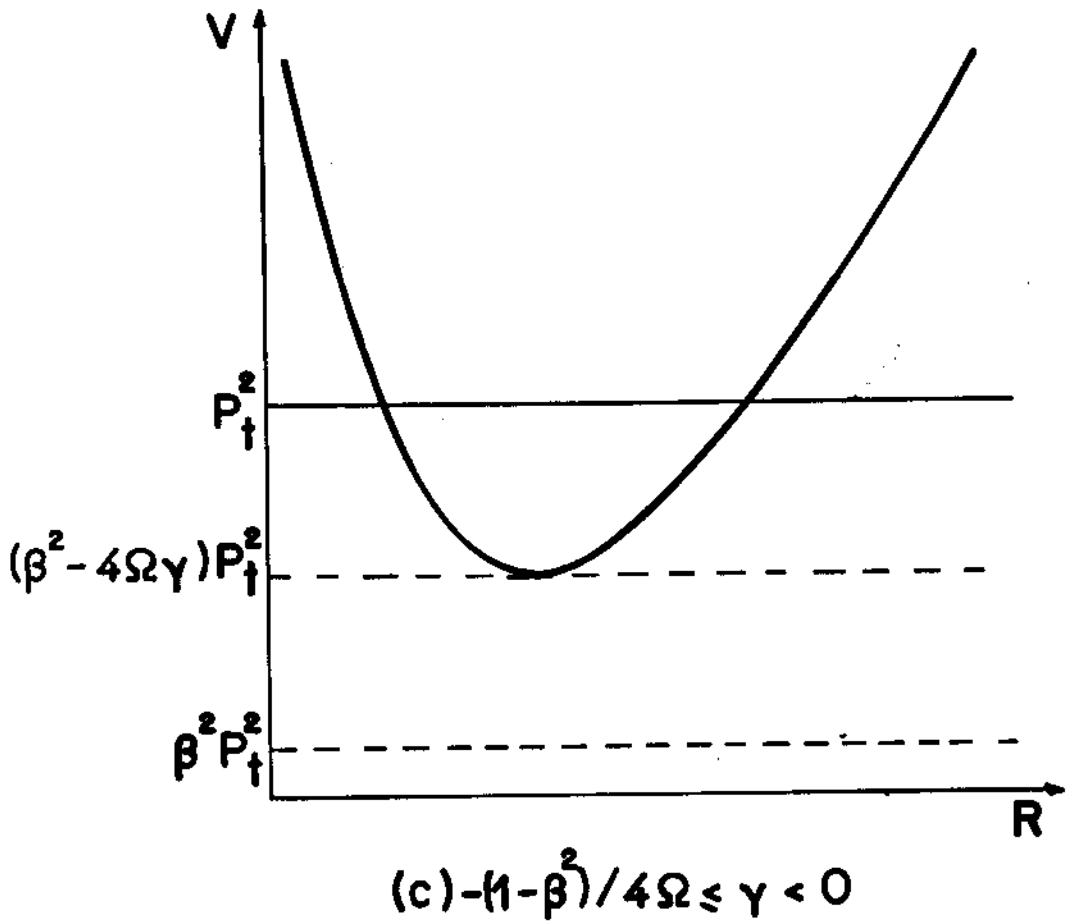
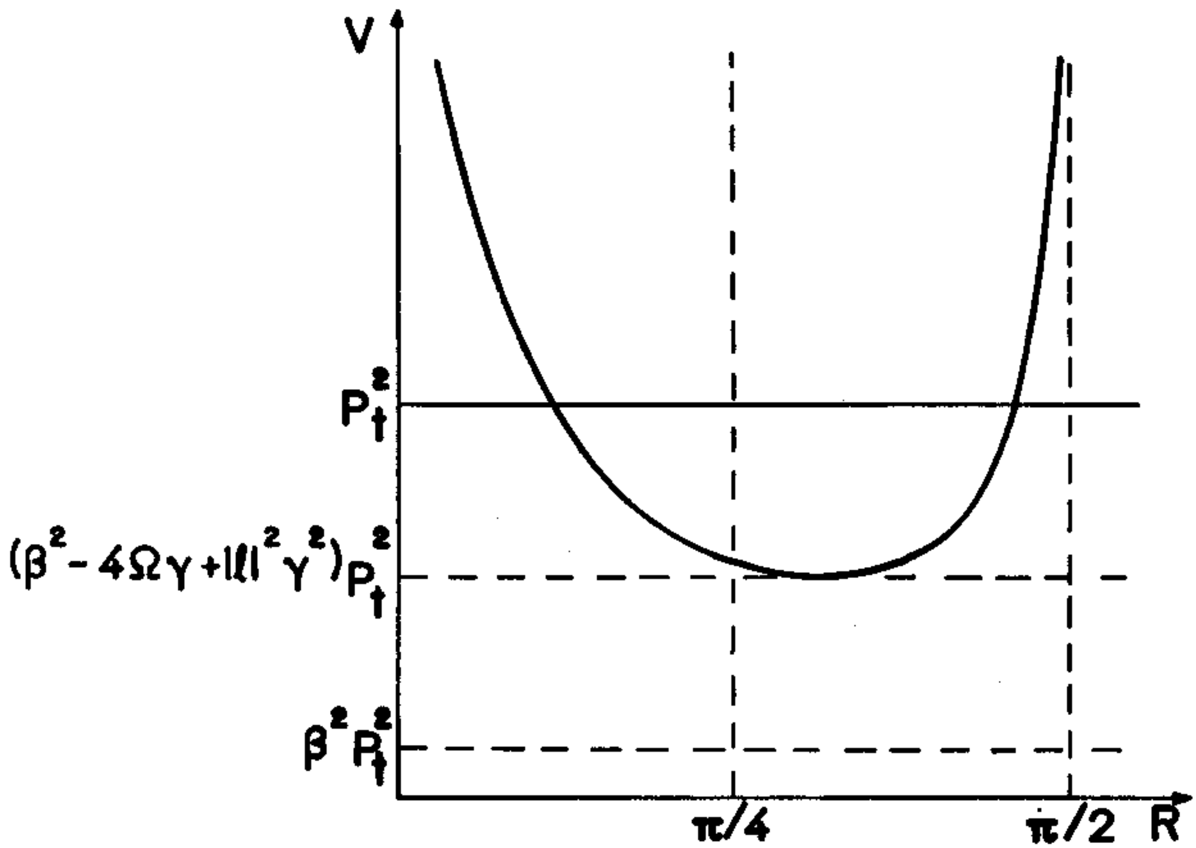
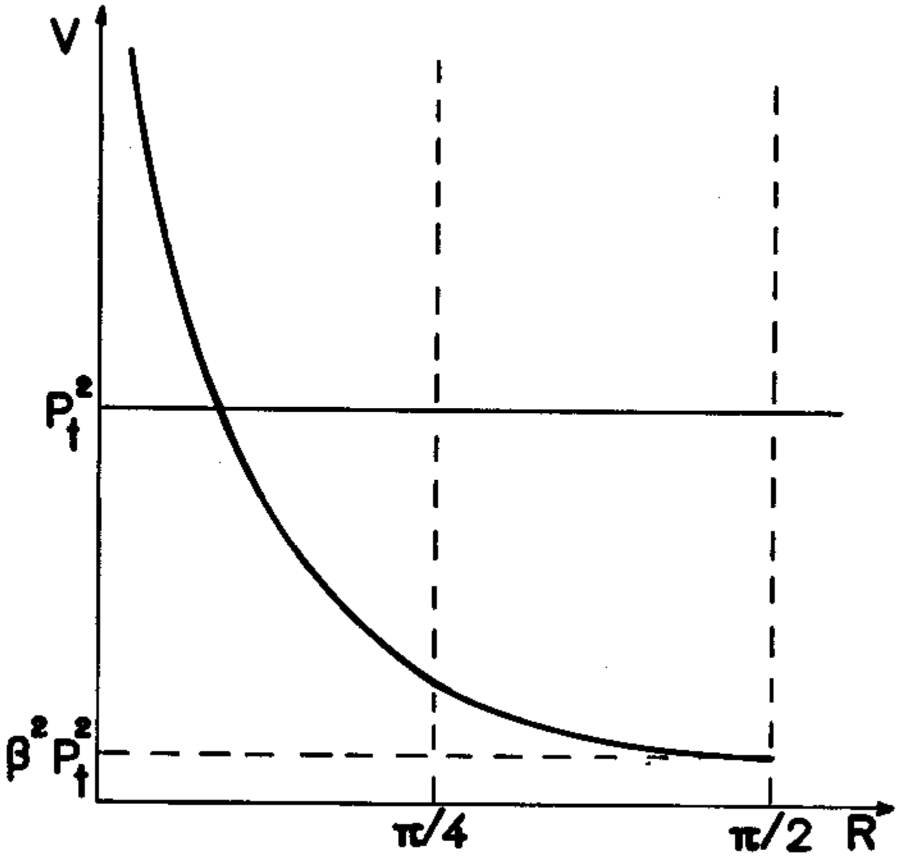

V
P_t^2
$(\beta^2-4\Omega\gamma)P_t^2$
$\beta^2 P_t^2$
R
(c) $-(1-\beta^2)/4\Omega \leq \gamma < 0$

Fig. 2



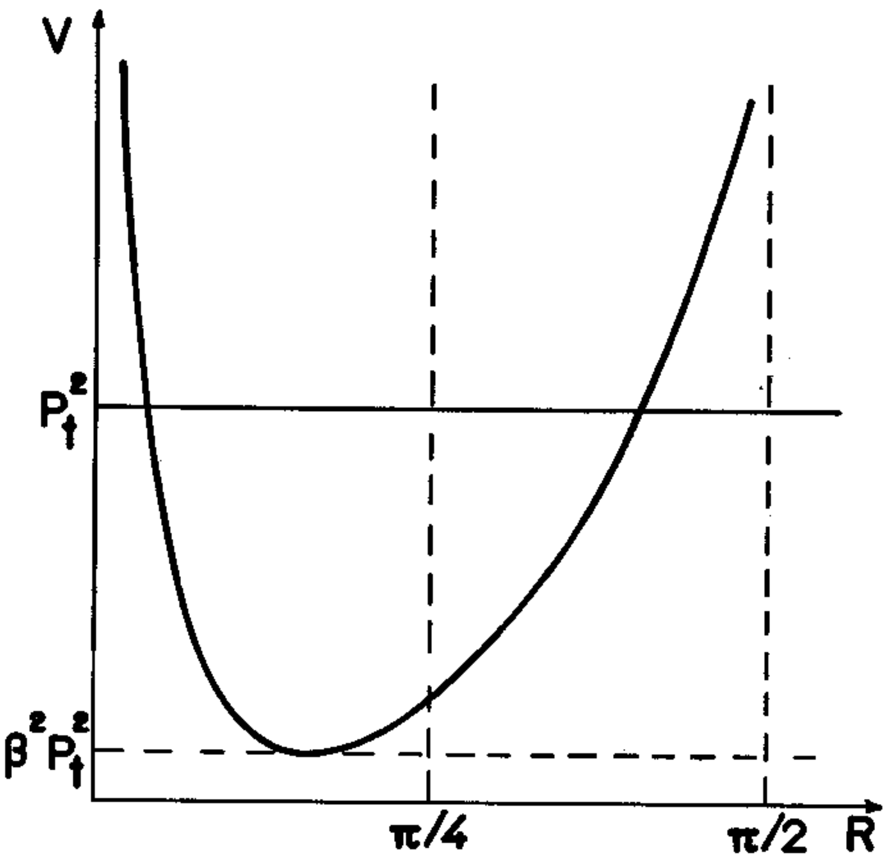
(a) $4\Omega/|\ell|^2 < \gamma \leq \tilde{\gamma}_>$

Fig. 3



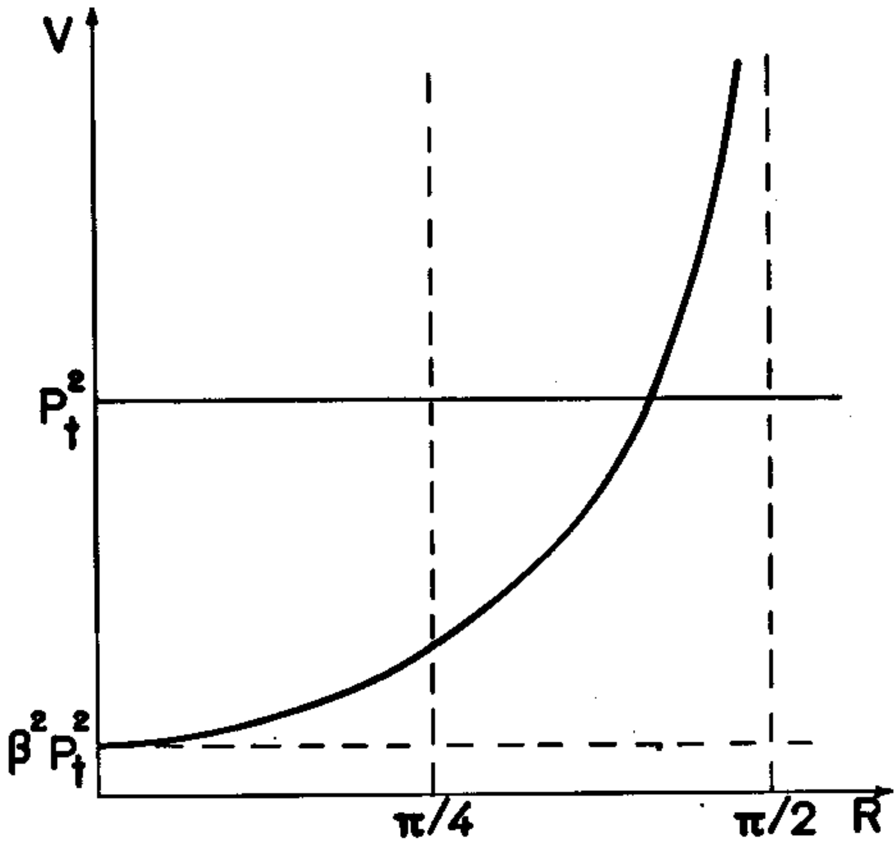
(b) $\gamma = 4\Omega/|\ell|^2$

Fig. 3



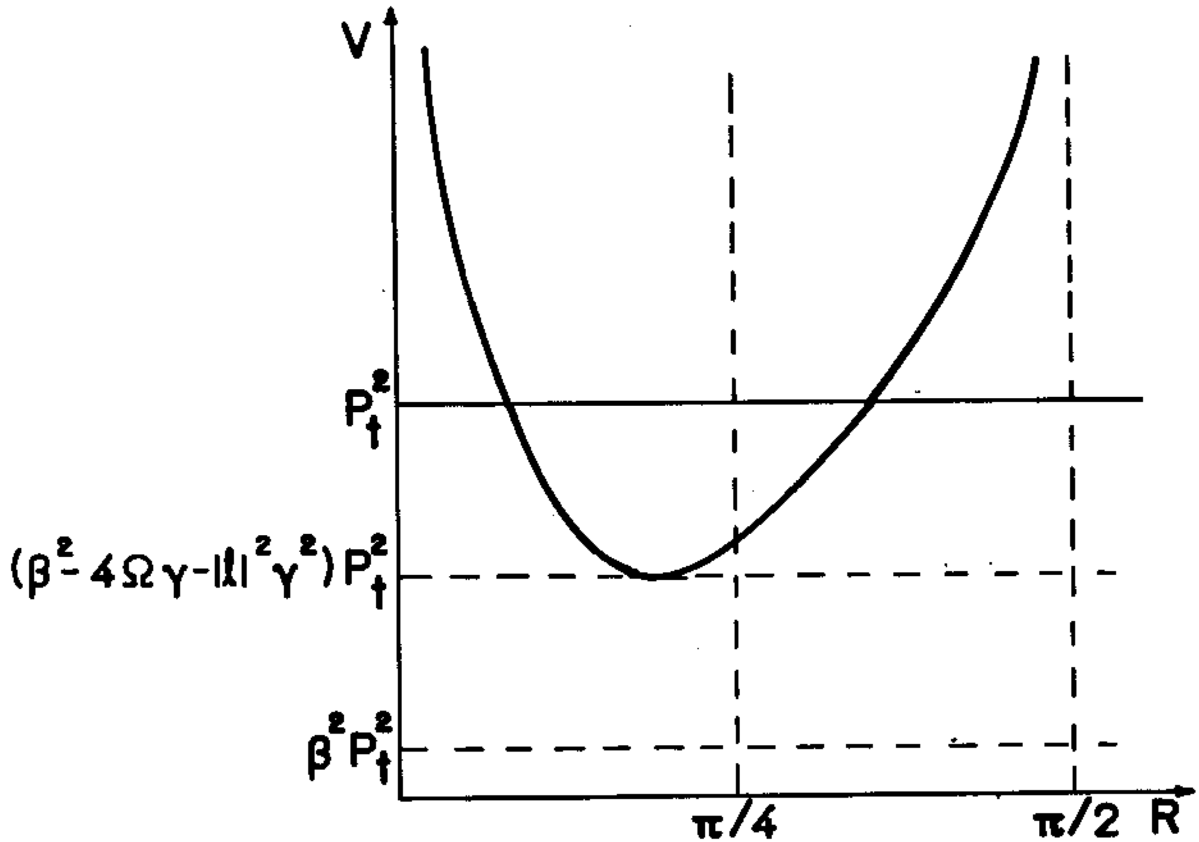
(c) $0 < \gamma < 4\Omega/|\ell|^2$

Fig. 3



(d) γ = 0

Fig. 3



(e) $\tilde{\gamma}_< \leq \gamma < 0$

Fig. 3

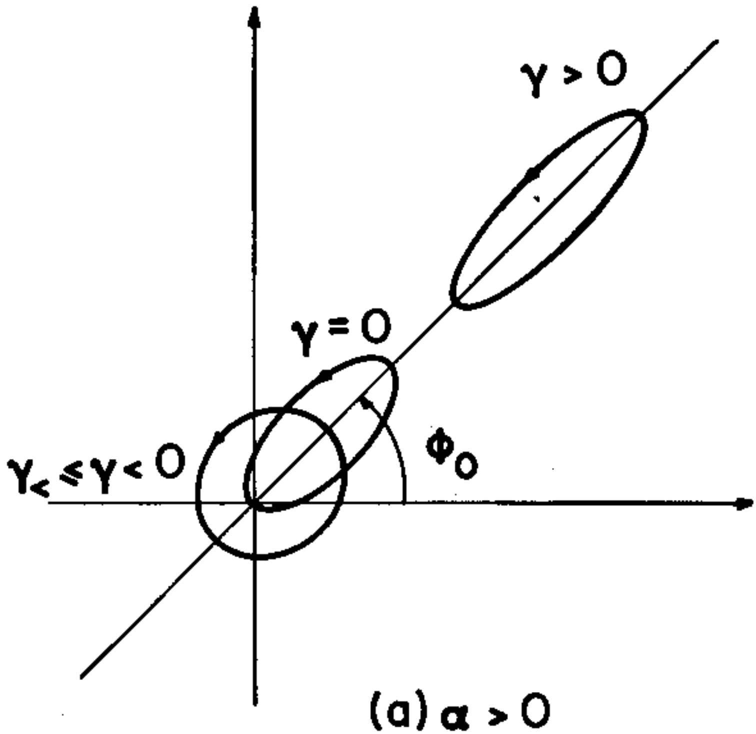


Fig. 4

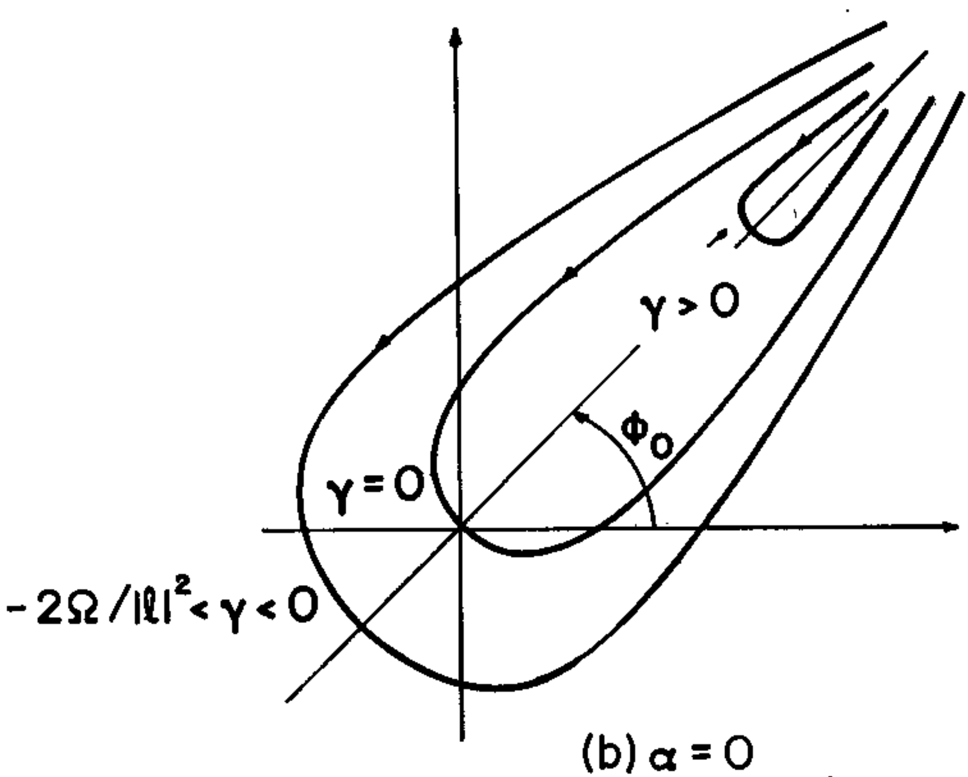


Fig. 4

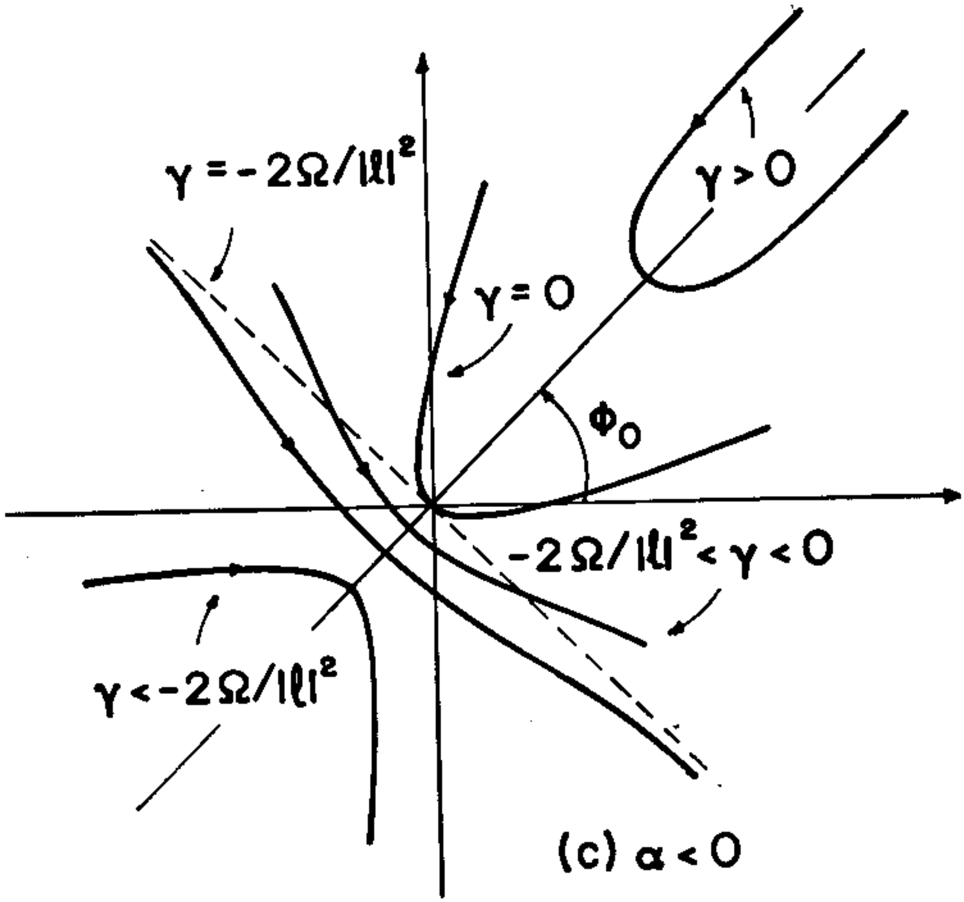


Fig. 4

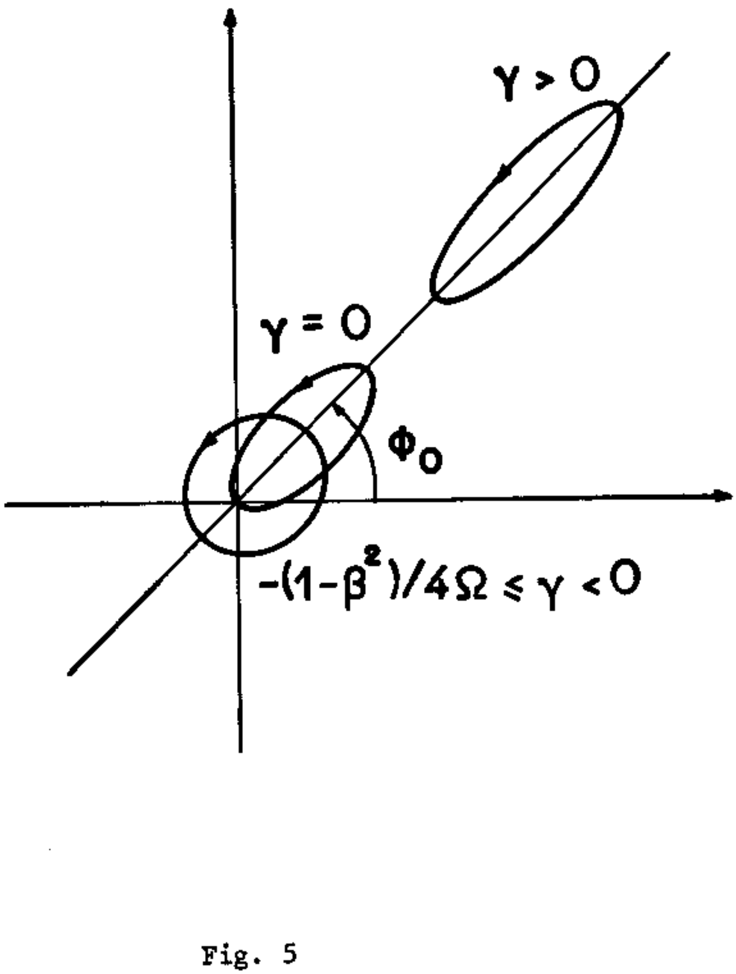


Fig. 5

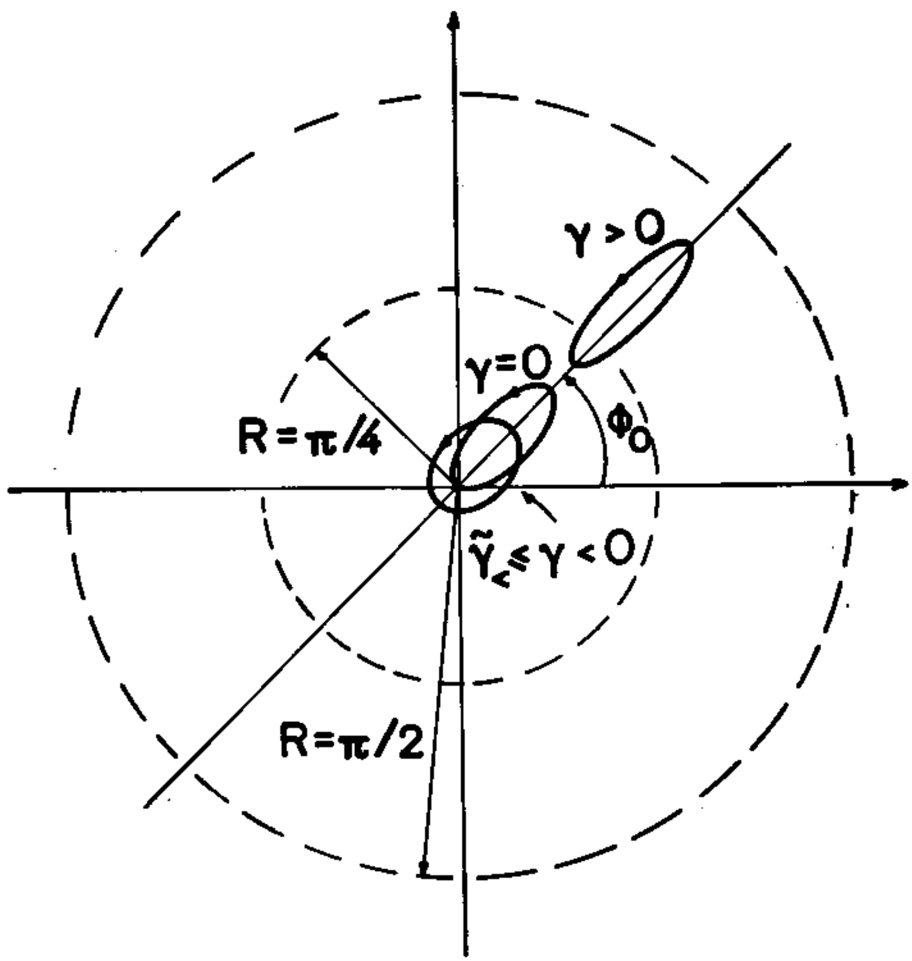


Fig. 6

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27) For the Som-Raychaudhuri space-time the appropriate variable is r^2. In any case, all the subsequent formulae furnish the correct results for this geometry if the limit as $\ell \to 0$ is adequately evaluated, by retaining terms of the smallest order in ℓ only. For the explicit expressions, we refer the reader to the Appendix. Cf. also Ref. 24.

28) It is convenient to use the identity $[\ \ell(\gamma + 2\Omega/\ell^2)\xi + \ell\gamma/2\ ]^2 =$

$$= \frac{\Gamma}{4}\left\{ 4\xi(\xi + 1) - \frac{1-\beta^2}{\eta} + \frac{1-\beta^2}{\eta}\left[\frac{\xi - P}{Q}\right]^2 \right\},$$

where P and Q are given from equation (4.5) rewritten as $\xi = P +$ $+ Q\cos \ell p_t\sqrt{\eta}\,(\tau - \tau_o)$.

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