Triality of Majorana-Weyl Spacetimes
with Different Signatures

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Abstract

Higher dimensional Majorana-Weyl spacetimes present space-time dualities which are induced by the $Spin(8)$ triality automorphisms. This corresponds to a very fundamental property of the supersymmetry in higher dimensions, i.e. that any given theory can be formulated in different signatures all interconnected by the S_3 permutation group.

Key-Words: Supersymmetry; Triality; Clifford Algebras.

1   Introduction.

Physical theories formulated in different-than-usual spacetimes signatures have recently found increased attention. One of the reasons can be traced to the conjectured F-theory [1] which supposedly lives in $(2 + 10)$ dimensions [2]. The current interest in AdS theories motivated by the AdS/CFT correspondence furnishes another motivation. Two-time physics e.g. has started been explored by Bars and collaborators in a series of papers [3]. For another motivation we can also recall that a fundamental theory is expected to explain not only the spacetime dimensionality, but even its signature (see [4]). Quite recently Hull and Hull-Khuri [5] pointed out the existence of dualities relating different compactifications of theories formulated in different signatures. Such a result provides new insights to the whole question of spacetime signatures. Other papers (e.g.[6]) have remarked the existence of space-time dualities.

Majorana-Weyl spacetimes (i.e. those supporting Majorana-Weyl spinors) are at the very core of the present knowledge of the unification via supersymmetry, being at the basis of ten-dimensional superstrings, superYang-Mills and supergravity theories (and perhaps the already mentioned F-theory). A well-established feature of Majorana-Weyl spacetimes is that they are endorsed of a rich structure. A legitimate question that could be asked is whether they are affected, and how, by space-time dualities. The answer can be stated as follows, all different Majorana-Weyl spacetimes which are possibly present in any given dimension are each-other related by duality transformations which are induced by the $Spin(8)$ triality automorphisms. The action of the triality automorphisms is quite non-trivial and has far richer consequences than the $\mathbf{Z}_2$-duality (its most trivial representative) associated to the space-time $(s, t) \leftrightarrow (t, s)$ exchange discussed in [4]. It corresponds to S_3, the six-element group of permutations of three letters, identified with the group of congruences of the triangle and generated by two reflections. The lowest-dimension in which the triality action is non-trivial is 8 (not quite a coincidence), where the spacetimes $(8 + 0) - (4 + 4) - (0 + 8)$ are all interrelated. They correspond to the transverse coordinates of the $(9 + 1) - (5 + 5) - (1 + 9)$ spacetimes respectively, where the triality action can also be lifted. Triality relates as well the 12-dimensional Majorana-Weyl spacetimes $(10 + 2) - (6 + 6) - (2 + 10)$, i.e. the potentially interesting cases for the F-theory. Triality allows explaining the presence of points (read theories) in the brane-molecule table of ref. [4], corresponding to the different versions of e.g. superstrings, 11-dimensional supermembranes, fivebranes.

As a consequence of triality, supersymmetric theories formulated with Majora-na-Weyl spinors in a given dimension but with different signatures, are all dually mapped one into another. A three-language dictionary can be furnished with the exact translations among, e.g., the different versions of such supersymmetric theories.

It should be stressed the fact that, unlike [5], the dualities here discussed are already present for the *uncompactified* theories and in this respect look more fundamental The reason why the triality of the $d = 8$-dimension plays a role even for Majorana-Weyl spacetimes in $d > 8$ is in consequence of the representation properties of Γ-matrices in higher dimensions.

2   The set of data for Majorana-Weyl supersymmetric theories.

At first we present the set of data needed to specify a supersymmetric theory involving Majorana-Weyl spinors. The notations here introduced follow [7] and [8].

The most suitable basis is the Majorana-Weyl basis (MWR), where all spinors are either real or imaginary. In such a representation the following set of data underlines any given theory:

i) the vector fields (or, in the string/brane picture, the bosonic coordinates of the target x_m), specified by a vector index denoted by m;

ii) the spinor fields (or, in the string/brane picture, the fermionic coordinates of the target ψ_a, $\chi_{\dot{a}}$), specified by chiral and antichiral indices a, $\dot{a}$ respectively;

iii) the diagonal (pseudo-)orthogonal spacetime metric $(g^{-1})^{mn}$, g_{mn} which we will assume to be flat;

iv) the $\mathcal{A}$ matrix, used to define barred spinors, coinciding with the Γ^0-matrix in the Minkowski case; in a Majorana-Weyl basis is decomposed in an equal-size block diagonal form such as $\mathcal{A} = A \oplus \tilde{A}$, with structure of indices $(A)_a{}^b$ and $(\tilde{A})_{\dot{a}}{}^{\dot{b}}$ respectively;

v) the charge-conjugation matrix $\mathcal{C}$ which also appears in an equal-size block diagonal form $\mathcal{C} = C^{-1} \oplus \tilde{C}^{-1}$. It is invariant under bispinorial transformations and it can be promoted to be a metric in the space of chiral (and respectively antichiral) spinors, used to raise and lower spinorial indices. Indeed we can set $(C^{-1})^{ab}$, $(C)_{ab}$, and $(\tilde{C}^{-1})^{\dot{a}\dot{b}}$, $(\tilde{C})_{\dot{a}\dot{b}}$;

vi) the Γ-matrices, which are decomposed in equal-size blocks, σ^m's the upper-right blocks and $\tilde{\sigma}^m$'s the lower-left blocks having structure of indices $(\sigma^m)_a{}^{\dot{b}}$ and $(\tilde{\sigma}^m)_{\dot{a}}{}^b$ respectively;

vii) the $\eta = \pm 1$ sign, labeling the two inequivalent choices for $\mathcal{C}$.

The above structures are common in any theory involving Majorana-Weyl spinors. An explicit dictionary relating Majorana-Weyl spacetimes having the same dimensionality but different signature is presented in [9]. The structures i)-vii) are related via triality transformations which close the S_3 permutation group. They constitute the "words" in the three-language dictionary.

Majorana-Weyl spacetimes exist for different signatures of a given dimension d if $d \geq 8$. The special $d = 8$ case is the fundamental one. Indeed, we are able to express higher-dimensional Γ matrices (and in consequence all the above-mentioned structures which define a Majorana-Weyl theory) in terms of lower-dimensional ones according to the recursive formula

$$
\begin{aligned}
\Gamma_d{}^{i=1,\dots,s+1} &= \sigma_x \otimes \mathbf{1}_L \otimes \Gamma_s{}^{i=1,\dots,s+1} \\
\Gamma_d{}^{s+1+j=s+2,\dots,d} &= \sigma_y \otimes \Gamma_r{}^{j=1,\dots,r+1} \otimes \mathbf{1}_R
\end{aligned}
\tag{1}
$$

where $\mathbf{1}_{L,R}$ are the unit-matrices in the respective spaces, while $\sigma_x = e_{12} + e_{21}$ and $\sigma_y = -ie_{12} + ie_{21}$ are the 2-dimensional Pauli matrices. $\Gamma_r{}^{r+1}$ corresponds to the "generalized Γ^5-matrix" in $r+1$ dimensions. In the above formula the values $r, s = 0$ are allowed. The corresponding $\Gamma_0{}^1$ is just 1.

With the help of this formula we are able to reduce the analysis of different-signatures Majorana-Weyl spacetimes to the 8-dimensional case. In this particular dimension the

three indices, vector (m), chiral (a) and antichiral $(\dot{a})$ take values $m, a, \dot{a} \in \{1, ..., 8\}$.

The three Majorana-Weyl solutions, for signatures $(4 + 4)$, $(8 + 0)$, $(0 + 8)$ find a representation in a Majorana-Weyl basis with definite (anti-)symmetry property of the Γ matrices. In particular for the $(4+4)$-signature the $(4_S + 4_A)$-representation (see [9]) of the Γ-matrices has to be employed for both values of η in order to provide a Majorana-Weyl basis. In the $(t = 8, s = 0)$ signature the $(8_S + 0_A)$-representation offers a MW basis for $\eta = +1$, while the $(0_S + 8_A)$ offers it for $\eta = -1$. The converse is true in the $(t = 0, s = 8)$-signature.

3   Trialities.

The S_3 outer automorphisms of the D_4 Lie algebra is responsible for the triality property among the 8-dimensional vector, chiral and antichiral spinor representations of $SO(8)$ and $SO(4,4)$ which has been first discussed by Cartan [10]. However, besides such Cartan's V-C-A triality, other triality related properties follow as a consequence. For purpose of clarity it will be convenient to represent them symbolically with triangle diagrams.

A first extra-consequence of triality appears at the level of Majorana-Weyl type of representations for Clifford's Γ-matrices. Such representations can be defined as those where all Γ's matrices exhibit a well-defined (anti-)symmetry property. In dimension $d = 8$ three different representations of this kind exist (they have been mentioned in the previous section). Such different eight-dimensional representations can be accomodated into the triangle diagram

$$
\begin{matrix}
 & (4_S + 4_A) & \\
\circ & & \circ \\
(8_S + 0_A) & \circ & (0_S + 8_A)
\end{matrix}
\tag{2}
$$

exhibiting the triality operating at the level of Γ-matrices. The S_3 transformations relating the three above representations are realized by similarity transformations. They depend on the concrete choice of the Γ-matrix representatives and will not been furnished here (see however [9]).
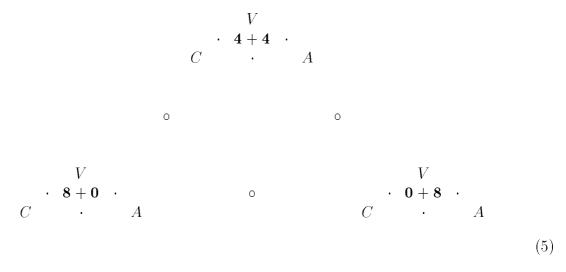
We have already recalled that such MW-representations are associated with the space-time signature, due to the fact that the introduction of a Majorana-Weyl basis for spinors requires the use of the corresponding Majorana-Weyl representation for Clifford's Γ matrices. As a consequence the triality can be lifted to operate on the whole set of data introduced in the previous section; it can therefore be regarded as operating on the different space-times signatures which support Majorana-Weyl spinors in a given dimensionality, according to the triangles

$$
\begin{pmatrix}
 & 5 + 5 & \\
\circ & & \circ \\
9 + 1 & \circ & 1 + 9
\end{pmatrix}
\mapsto
\begin{pmatrix}
 & 4 + 4 & \\
\circ & & \circ \\
8 + 0 & \circ & 0 + 8
\end{pmatrix}
\tag{3}
$$

The arrow has been inserted to recall that such triality can be lifted to higher dimensions or, conversely, that the 8-dimensional spacetimes arise as transverse coordinates spaces in some physical theories (the most natural example is the 10-dimensional superstring).

The Cartan's V-C-A triality, schematically represented as

$$
\begin{pmatrix}
 & & V & & \\
 & \circ & & \circ & \\
 C & & \circ & & A
\end{pmatrix}
\tag{4}
$$

and the signature triality can also be combined and symbolically represented by a sort of fractal-like double-triality diagram as follows

$$
\begin{array}{ccc}
 & V & \\
\cdot\ \mathbf{4+4}\ \cdot & & \\
C\quad\quad\cdot\quad\quad A & &
\end{array}
$$

$$
\circ \qquad\qquad\qquad\qquad\qquad \circ
$$

$$
\begin{array}{ccccc}
 & V & & & V \\
\cdot\ \mathbf{8+0}\ \cdot & & \circ & & \cdot\ \mathbf{0+8}\ \cdot \\
C\quad\cdot\quad A & & & & C\quad\cdot\quad A
\end{array}
\tag{5}
$$

The bigger triangle illustrates the signature triality, while the smaller triangles visualize the trialities for vectors, chiral and antichiral spinors which can be accomodated in each space-time.

It is worth stressing the fact that the arising of the S_3 permutation group as a signature-duality group for Majorana-Weyl spacetimes in a given dimension *is not* a completely straightforward consequence of the existence of Majorana-Weyl spacetimes in three different signatures. Some extra-requirements have to be fulfilled in order to reach this result. As an example we just mention that a necessary condition for the presence of S_3 requires that each given couple of the three different spacetimes must differ by an *even* number of signatures (in [9] this point is discussed in full detail); the flipping of an *odd* number of signatures, like the Wick rotation from Minkowski to the Euclidean space, cannot be achieved with a $\mathbf{Z}_2$ group when spinors are involved. An example is provided by the fact that the change of signature e.g. from $(++) \mapsto (--)$ can be realized on Γ-matrices through similarity trasformations expressed in terms of the σ_y Pauli matrix $\sigma_y = -ie_{12} + ie_{21}$, through

$$
\sigma_y \cdot \mathbf{1}_2 \cdot \sigma_y{}^T \;=\; -\mathbf{1}_2
\tag{6}
$$

Of course σ_y satisfies $\sigma_y{}^2 = \mathbf{1}_2$ and therefore it closes a $\mathbf{Z}_2$ group. On the contrary, a standard Wick rotations from the Minkowski to the Euclidean space leads to a $\mathbf{Z}_4$ group when represented on Γ matrices.

Similarity transformations realized by σ_y Pauli matrices are among the building blocks for constructing the S_3 duality transformations for different signature Majorana-Weyl spacetimes. The formulas will not be reproduced here (they are furnished in [9], together with the details of the construction).

Let us conclude this section by mentioning that triality can be seen not only as a source of duality-mappings, but as an invariance property. In the original Cartan's formulation [10] this is seen as follows. At first a group $\mathcal{G}$ of invariance is introduced as the group of linear homogeneous transformations acting on the $8 \times 3 = 24$ dimensional space leaving invariant, separately, the bilinears $\mathcal{B}_V$, $\mathcal{B}_C$, $\mathcal{B}_A$ for vectors, chiral and antichiral spinors respctively (the spinors are assumed commuting in this case) plus a trilinear term $\mathcal{T}$. Next, the triality group $\mathcal{G}_{Tr}$ is defined by relaxing one condition, as the group of linear homogeneous transformations leaving invariant $\mathcal{T}$ and the total bilinear $\mathcal{B}_{Sum}$,

$$\mathcal{B}_{Sum} \quad = \quad \mathcal{B}_V + \mathcal{B}_C + \mathcal{B}_A \tag{7}$$

It can be proven that $\mathcal{G}_{Tr}$ is given by the semidirect product of $\mathcal{G}$ and S_3:

$$\mathcal{G}_{Tr} \quad = \quad \mathcal{G} \otimes_S S_3$$

This feature can be extended to the other aspects of triality. It follows the possibility to look at formulations of higher dimensional supersymmetric theories presenting an S_3 group of symmetry under the exchange of space and time coordinates.

It should be mentioned that the higher-dimensional supersymmetry strongly restricts the class of finite groups which can provide "unification between space and time" or, otherwise stated, symmetry under time-versus-space coordinates exchange. In the bosonic case such class of groups is quite large, while if we consider e.g. the 10-dimensional supersymmetric case only three possibilities are left, namely **i)** the identity **1**, corresponding to a theory formulated in the single spacetime $(5,5)$, **ii)** the $\mathbf{Z}_2$ group for a theory which is formulated by using two spacetimes copies $(1,9)$ and $(9,1)$, **iii)** the S_3 group; whose corresponding "space-time unified" theory requires the introduction of the whole set of three 10-dimensional Majorana-Weyl spacetimes $(1,9)$, $(5,5)$, $(9,1)$.

4   Conclusions.

In this paper we have shown that the triality automorphisms of $Spin(8)$, besides its consequences on the representation properties of the 8-dimensional vectors, chiral and antichiral spinors (the usual Cartan's notion of triality), can be realized on classes of Γ-matrices representations which furnish a Majorana-Weyl basis for Majorana-Weyl spinors. Next, triality transformations can be lifted to connect spacetimes supporting Majorana-Weyl spinors sharing the same dimensionality, but different signatures. Recursive formulas for Γ-matrix representations allow to extend the 8-dimensional properties to higher-dimensional cases as well. Dualities induced by triality are found connecting even-dimensional Majorana-Weyl spacetimes (and odd-dimensional Majorana ones). The presence of different formulations of e.g. brane theories, as shown in the brane-scan molecule table of ref. [4] arises as a consequence.

Indeed higher dimensional supersymmetric theories admits formulations in different signatures which are all interrelated by triality induced transformations.

Besides this action of triality as a source of duality mappings between different versions of supersymmetric theories, triality can provide a setting to discuss formulation of theories invariant under space-versus-time coordinates exchange. This would amount to

investigate the formulation of supersymmetric theories exhibiting a manifest S_3-invariance under signature-triality transformations.

The range of possible applications for the methods and the ideas here discussed is vast. Let us just mention that are currently investigated the web of dualities connecting the six different versions of the 12-dimensional Majorana-Weyl spacetimes which should support the F-theory (the number $6 = 3 \times 2$ is due to the three different signatures of Majorana-Weyl spacetimes and the two values of the η sign), with the 6 versions of the 11-dimensional Majorana spacetimes (for the M-theory) in $(10 + 1)$, $(9 + 2)$, $(6 + 5)$, $(5 + 6)$, $(2 + 9)$, $(1 + 10)$ signatures and with the different (again 3×2) versions of the 10-dimensional Majorana-Weyl spacetimes.

Acknowledgments

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